

# Hamiltonian action on generalized complex manifolds and the equivariant $\bar{\partial}\partial$ -lemma

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# Outline

- ▶ **Review of generalized complex geometry**
  - ▶ N. Hitchin, "Generalized Calabi-Yau", Q.J.Math, 54 (3): 281-308, 2003
  - ▶ M. Gualtieri, "Generalized complex geometry", Ph.D thesis, Oxford University, 2003, math.DG/0401221
- ▶ **Hamiltonian actions in generalized complex geometry**
- ▶ **Equivariant  $\bar{\partial}\partial$ -lemma**
  - ▶ L, " Generalized geometry, equivariant  $\bar{\partial}\partial$ -lemma, and torus actions", math.DG/0607401

# Basic Scenario

- ▶  $TM \oplus T^*M$
- ▶ **Canonical pairing** on  $TM \oplus T^*M$ :

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2} (\eta(X) + \xi(Y)).$$

By definition, this pairing is symmetric, non-degenerate and of signature  $(m, m)$ , where  $m = \dim M$ .

- ▶  **$H$ -twisted Courant bracket**:

$$[X + \xi, Y + \eta] = [X, Y] + L_X \eta - L_Y \xi - \frac{1}{2} d(\eta(X) - \xi(Y)) + \iota_Y \iota_X H,$$

where  $H$  is a closed three form.

- ▶ **Symmetry group** of  $(TM \oplus T^*M, \langle \cdot, \cdot \rangle, [\cdot, \cdot], H)$ :

$$\text{Diff}(M) \times \mathcal{C}_H = \{(\varphi, B) \mid \varphi^* H - H = dB\}.$$

# Generalized complex structures

## Definition

- ▶ A **generalized almost complex** structure is an orthogonal bundle map  $\mathcal{J} : TM \oplus T^*M \rightarrow TM \oplus T^*M$  such that  $\mathcal{J}^2 = -1$ .
- ▶ A generalized almost complex structure  $\mathcal{J}$  is an  $H$ -twisted **generalized complex structure** if the  $\sqrt{-1}$ -eigenbundle of  $\mathcal{J}$ ,  $L \subset T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M$ , is closed under the  $H$ -twisted Courant bracket.

## Example

- a) Let  $(M, I)$  be a  $2n$  dimensional complex manifold. Then

$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}$$

is a generalized complex structure whose  $\sqrt{-1}$  eigenbundle of  $\mathcal{J}_I$  is  $L_{\mathcal{J}} = T_{0,1} \oplus T_{1,0}^*$ , where  $T_{1,0}$  is the  $\sqrt{-1}$  eigenbundle of  $J$ .

- b) Let  $(M, \omega)$  be a symplectic manifold. Then

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}$$

is a generalized complex structure whose  $\sqrt{-1}$  eigenbundle of  $\mathcal{J}_\omega$  is  $L_\omega = \{X + \sqrt{-1}\iota_X\omega \mid X \in T_{\mathbb{C}}M\}$ .

## Digression on Clifford action

Let  $V$  be a real vector space with a generalized complex structure  $\mathcal{J} : V \oplus V^* \rightarrow V \oplus V^*$ .

- ▶ **Clifford action** of  $CL(V \oplus V^*)$  on  $\wedge^* V^*$ :

$$(X + \xi) \cdot \alpha = \iota_X \alpha + \xi \wedge \alpha, \quad (X + \xi)^2 \cdot \alpha = \langle X + \xi, X + \xi \rangle \alpha.$$

- ▶ **Clifford action** of  $\mathcal{J}$  on  $\wedge^* V^*$ :

Since  $\langle \mathcal{J}x, \mathcal{J}y \rangle = \langle x, y \rangle$  and  $\mathcal{J}^2 = -1$ ,

$$\langle \mathcal{J}x, y \rangle + \langle x, \mathcal{J}y \rangle = 0, \text{ for any } x, y \in V \oplus V^*.$$

Since  $\mathfrak{so}(V \oplus V^*) \cong \wedge^2(V \oplus V^*) \subset CL(V \oplus V^*)$ , there is a natural Clifford action of  $\mathcal{J}$  on  $\wedge V^*$ .

# Decomposition of differential forms

- ▶ **Decomposition of differential forms** (Gualtieri): Let  $U^k$  be the  $-k\sqrt{-1}$  eigenspace of the Clifford action of  $\mathcal{J}$  on  $\wedge^* V^*$ . Then we have

$$\wedge^* V^* \cong \bigoplus_k U^k.$$

- ▶ (Gualtieri) Let  $V$  be a real vector space with a complex structure  $I$ . Then  $U^k = \bigoplus_{q-p=k} (\wedge^{p,q} V_{\mathbb{C}}^*)$ .
- ▶ (Cavalcanti) Let  $(V, \omega)$  be a  $2n$  dimensional symplectic vector space. Then there is an isomorphism

$$\wedge^{n+k} \rightarrow U^k, \alpha \mapsto e^{i\omega} e^{\frac{\pi}{2i}} \alpha,$$

where  $\pi = \omega^{-1}$  is the canonical Poisson bi-vector associated to the symplectic form  $\omega$ .

## $\bar{\partial}, \partial$ -operators

- ▶ Let  $\mathcal{J}$  be an almost generalized complex structure on  $M$ . It induces a grading

$$\Omega(M) = \bigoplus_k U^k.$$

- ▶ For  $\alpha \in U^k$ ,  $\bar{\partial}, \partial$  are defined by:

$\bar{\partial}\alpha =$  the  $U^{k+1}$  component of  $d\alpha$ ,

$\partial\alpha =$  the  $U^{k-1}$  component of  $d\alpha$ .

- ▶ (**Gualtieri**)  $\mathcal{J}$  is integrable if and only if  $d = \bar{\partial} + \partial$ .

# $\bar{\partial}\partial$ -lemma

- ▶  $\bar{\partial}\partial$ -lemma:

$$\text{im}\bar{\partial} \cap \ker\partial = \ker\bar{\partial} \cap \text{im}\partial = \text{im}\bar{\partial}\partial.$$

- ▶ When a generalized complex structure is induced by a symplectic structure, the  $\bar{\partial}\partial$ -lemma is equivalent to the symplectic  $d\delta$ -lemma due to Guillemin and Merkulov.
- ▶ When a generalized complex structure is induced by a complex structure, the  $\bar{\partial}\partial$ -lemma coincides with the usual  $\bar{\partial}\partial$ -lemma in complex geometry.
- ▶ Recently, Gualtieri proved the  $\bar{\partial}\partial$ -lemma for compact twisted generalized Kähler manifolds.

- ▶ Bursztyn, Cavalcanti and Gualtieri, math.DG/0509064;
- ▶ Vaisman, Izu math.DG/0511013.
- ▶ L., S. Tolman, CMP 268(2006) 199-222 ,  
math.DG/0509069, math.DG/0510010;
- ▶ S. Hu, math.DG/0509060;
- ▶ Stiénon and Xu, math.DG/0509393;

# Hamiltonian actions on symplectic manifolds

- ▶ Consider the action of  $G$  on a symplectic manifold  $(M, \omega)$ . An equivariant map  $f : M \rightarrow \mathfrak{g}^*$  is said to be a moment map if there is a Lie algebra homomorphism  $\mathfrak{g} \rightarrow C^\infty(M)$ ,  $\xi \mapsto f^\xi$ , such that the following diagram commute

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & C^\infty(M) \\ & \searrow & \downarrow \\ & & C^\infty(TM) \end{array}$$

where the vertical map is defined by  $f \mapsto \omega^{-1}(df)$ , and the diagonal map is given by  $\xi \mapsto \xi_M$ .

# Hamiltonian actions on generalized complex manifolds

- ▶ (L., Tolman, 2006) Consider the action of  $G$  on a generalized complex manifold  $(M, \mathcal{J})$ . An equivariant map  $f : M \rightarrow \mathfrak{g}^*$  is said to be a **generalized moment map** if there is a Lie algebra homomorphism  $\mathfrak{g} \rightarrow C^\infty(M)$ ,  $\xi \mapsto f^\xi$ , such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\quad} & C^\infty(M) \\ & \searrow & \downarrow \\ & & C^\infty(TM \oplus T^*M) \end{array}$$

Here the vertical map is given by  $f \mapsto -\mathcal{J}df$ . The diagonal map is given by  $\xi \mapsto \xi_M + \alpha^\xi$ , where  $\alpha \in \Omega(M, \mathfrak{g}^*)$  is a  $\mathfrak{g}^*$ -valued one form, called the **moment one form**.

- ▶ When the generalized complex structure is induced by a symplectic structure, our definition of moment maps coincides with the usual definition of moment maps in symplectic geometry.

# Motivation

- ▶ Consider the Hamiltonian action on a symplectic manifold  $(M, \omega)$  which satisfies the symplectic  $d\delta$ -lemma. There is an equivariant extension of the symplectic  $d\delta$ -lemma, i.e., the equivariant  $d_G\delta$ -lemma. In particular, it yields a simple proof of (an improved version of) the Kirwan-Ginzburg equivariant formality theorem.
- ▶ Hamiltonian actions on generalized complex manifolds generalize the notion of Hamiltonian action on symplectic manifold.
- ▶ The  $\bar{\partial}\partial$ -lemma generalizes the symplectic  $d\delta$ -lemma.
- ▶ **Question:** Does  $\bar{\partial}\partial$ -lemma have an equivariant extension as well? Does it imply some sort of equivariant formality results?

## Cartan model

- ▶ The Cartan complex  $\Omega_G(M)$  of the  $G$ -manifold  $M$  is defined to be the complex of equivariant differential forms, i.e., equivariant polynomial mappings

$$\alpha : \mathfrak{g} \rightarrow \Omega(M)$$

with the coboundary operator given by

$$(d_G \alpha)(\xi) = d(\alpha(\xi)) - \iota_{\xi_M} \alpha(\xi).$$

- ▶ It has a bi-grading induced by the usual grading of polynomial rings and that of differential forms.
- ▶ It has a vertical differential  $d$  and a horizontal differential  $d'$  given by

$$(d' \alpha)(\xi) = \iota_{\xi_M} \alpha(\xi) = \xi_M \cdot (\alpha),$$

where  $\cdot$  denotes the Clifford algebra action.

## Equivariant $d_G\delta$ -lemma

- ▶ (L, Sjamaar, 2003) Consider the Hamiltonian action of a compact Lie group  $G$  on a symplectic manifold  $(M, \omega)$  which satisfies the  $d\delta$ -lemma. Then there is an equivariant symplectic  $d\delta$ -lemma:

$$\text{im}d_G \cap \ker\delta = \ker d_G \cap \text{im}\delta = \text{im}d_G\delta,$$

where  $d_G$  is the coboundary operator on the Cartan complex,  $\delta$  is induced by the symplectic adjoint of  $d$  on the equivariant Cartan double complex.

- ▶ Consequence: an improved version of the Kirwan-Ginzburg equivariant formality theorem which asserts that every closed differential form has an equivariantly closed extension which is canonical up to coboundaries.

# Generalized equivariant cohomology

- ▶ Recall that the Hamiltonian action of a compact Lie group  $G$  on a generalized complex manifold involves a  $\mathfrak{g}^*$  valued 1-form. The usual Cartan model does not encode any information of this data.
- ▶ Remedy:
  - ▶ Let  $\Omega_G$  be the space of equivariant polynomial mappings from  $\mathfrak{g}$  to  $\Omega^*(M)$ . Extend  $d'$  to a new operator  $\mathcal{A}$  by

$$\begin{aligned}(\mathcal{A}\alpha)(\xi) &= (-\xi_M - \sqrt{-1}(d\mathfrak{f}^\xi + \sqrt{-1}\alpha^\xi)) \cdot \alpha \\ &= -\iota_{\xi_M}\alpha + \sqrt{-1}(d\mathfrak{f}^\xi + \sqrt{-1}\eta^\xi) \wedge \alpha.\end{aligned}$$

- ▶ Replace the usual derivative  $d$  by the twisted one  $d_H = d - H \wedge$ . Since  $H$  is an invariant form, the twisted exterior derivative  $d_H$  is  $G$ -equivariant and induces a well defined operator on  $(S\mathfrak{g}^* \otimes \Omega(M))^G$ , which we still denote by  $d_H$ . We have

$$d_H \mathcal{A} = -\mathcal{A} d_H, \text{ and } (d_H + \mathcal{A})^2 = 0.$$

# Generalized equivariant cohomology

(L,math.DG/0607401 and independently shengda, Hu,and Uribe, Bernardo, math.DG/0608319)

## Definition

Let  $\Omega_G = (Sg^* \otimes \Omega(M))^G$  be  $Z_2$  graded. Then  $D_G = d_H + \mathcal{A}$  is a differential of degree 1. And the ( $Z_2$  graded) **generalized equivariant cohomology** is defined to be

$$H^{\text{even/odd}}(\Omega_G, D_G) = \frac{\ker \left( \Omega_G^{\text{even/odd}} \xrightarrow{D_G} \Omega_G^{\text{odd/even}} \right)}{\text{im} \left( \Omega_G^{\text{odd/even}} \xrightarrow{D_G} \Omega_G^{\text{even/odd}} \right)}.$$

# Generalized equivariant Dolbeault cohomology

## Definition

Define  $U_G = (Sg^* \otimes \Omega(M))^G$  to be the double complex with the bigrading

$$U_G^{i,j} = (S^i g^* \otimes U^{j-i}(M))^G,$$

where  $U^{j-i}$  is the  $(i-j)\sqrt{-1}$  eigenspace of the Clifford action of  $\mathcal{J}$  on the space of differential forms. It is equipped with the vertical differential  $\bar{\partial}$  and the horizontal differential  $\mathcal{A}$ . As a total complex,  $U_G$  has the grading  $U_G^k = \bigoplus_{i+j=k} U_G^{i,j}$  and the total differential  $\bar{\partial}_G = \bar{\partial} + \mathcal{A}$ . The cohomology of the total complex  $(U_G, \bar{\partial}_G)$  is defined to be the **generalized equivariant Dolbeault cohomology**.

## Equivariant $\bar{\partial}\partial$ -lemma

- ▶ The Hamiltonian generalized complex manifold  $M$  is said to satisfy the  $\bar{\partial}_G\partial$ -lemma if and only if

$$\ker\partial \cap \text{im}\bar{\partial}_G = \ker\bar{\partial}_G \cap \text{im}\partial = \text{im}\bar{\partial}_G\partial.$$

- ▶ (L, 2006) Assume that the generalized complex manifold  $M$  satisfies the  $\bar{\partial}\partial$ -lemma. Assume that the action of compact Lie group  $G$  on  $M$  is Hamiltonian. Then  $M$  satisfies the  $\bar{\partial}_G\partial$ -lemma.
- ▶ When  $\mathcal{J}$  is induced by a symplectic structure  $\omega$ , we have

$$e^{i\omega} e^{\frac{\pi}{2i}} d_G\alpha = \bar{\partial}_G e^{i\omega} e^{\frac{\pi}{2i}} \alpha, \quad \partial e^{i\omega} e^{\frac{\pi}{2i}} \alpha = e^{i\omega} e^{\frac{\pi}{2i}} \delta\alpha.$$

So the  $\bar{\partial}_G\partial$ -lemma specializes to the symplectic  $d_G\delta$ -lemma.

# Consequences



$$H(U_G, \bar{\partial}_G) \cong (\mathcal{S}g^*)^G \otimes H_H(M).$$



$$H^{\text{even/odd}}(\Omega_G, D_G) \cong (\mathcal{S}g^*)^G \otimes H_H^{\text{even/odd}}(M).$$