MEAN-FIELD DYNAMICS OF ROTATING BOSONS IN CONFINING TRAPS

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ABSTRACT. The mean-field dynamics of rotating many-body bosons in a confining trap is rigorously studied. It is shown that the magnetic Hartree equation describes the dynamics of an initially factorized bosonic state in the mean-field limit. Explicit estimates of the convergence rate are given.

1. INTRODUCTION

1.1. **Earlier results and heuristic discussion.** There has been substantial developments in the study of the mean-field dynamics of many-body bosonic systems, in the absence of magnetic potentials. Early results using coherent states were given in [18] for bounded pair interaction, and extended in [14, 15] to more singular potentials. This approach is considerably extended in [20] to the case where the initial condition corresponds to a factorized state. Another approach based on the reduced density matrix was developed in [21] and was extended to more general potentials and to the derivation of the Gross-Pitaevskii equation in [8], [4], [5], [9], [10]. Furthermore, a new approach was developed in [12] that treats the quantum many-body dynamics as a deformation of the classical Hamiltonian dynamics, see also [13, 2, 1]. Recently, estimates of the rate of convergence of the many-body dynamics to the Hartree (or the Gross-Pitaevskii) one was found using different techniques in [20], [11], [17] and [3].

In this note, we study the mean-field dynamics of rotating bosons in confining traps, which is relevant to concrete experiments on Bose-Einstein condensates. We show that the magnetic Hartree equation describes the dynamics of an initially factorized bosonic state in the mean-field limit. The magnetic vector potential corresponding to a constant magnetic field appears due to splitting the Coriolis and centrifugal forces in the co-moving reference frame.

Although the approach introduced in [12] is conceptually simple and beautiful, we opt to extend the coherent state approach that has been used in [18, 15, 20] to the case at hand, simply because the latter approach gives stronger estimates of the rate of convergence in the mean-field limit. We note that the analysis below is simpler than the one in [20].

1.2. **The model and statement of the main result.** We consider a quantum system of N bosons ($N \in \mathbb{N}$) in a confining potential V and in a moving reference frame with angular velocity Ω. The Hilbert space of the system is $L^2_S(\mathbb{R}^{3N})$, the symmetrized (bosonic) space of L^2 functions. The dynamics of the system is generated by the (N-body) quantum Hamiltonian

$$H_N = \sum_{j=1}^{N}(-\nabla_j^2 + V(x_j) - L_j \cdot \Omega) + \frac{1}{N}\sum_{1\le i<j\le N} w(x_i - x_j), \tag{1}$$

where $x \in \mathbb{R}^3$ is a point is configuration space, w is the (rescaled) pair interaction, and $L = -ix \wedge \nabla$ is the angular momentm. Note that the rescaling is such that the kinetic energy term and the pair interaction scale similarly in the large N limit. Splitting the rotational effects into Coriolis and centrifugal forces, the Hamiltonian can equivalently be written as

$$H_N = \sum_{j=1}^{N}[(-i\nabla_j - A(x_j))^2 + V_{eff}(x_j)) + \frac{1}{N}\sum_{1\le i<j\le N} w(x_i - x_j),$$

where

$$V_{eff}(x) = V - \frac{1}{4}\Omega^2 r^2,$$

r being the distance from the rotation axis, and

$$A(x) = \frac{1}{2}\Omega \wedge x \tag{2}$$

the *magnetic vector potential* corresponding to a constant magnetic field. We assume that there exists a positive constant K such that pair interaction satisfies the operator inequality

$$|w(x)| \le K\sqrt{1-\Delta}, \tag{3}$$

where $\Delta = \nabla^2$. [1] For the potential to be confining, we want

$$V(x) - \frac{1}{4}\Omega^2 r^2 \to \infty \;\; as \;\; |x| \to \infty.$$

For simplicity, we assume that the confining potential is homogeneous with

$$V(\lambda x) = \lambda^s V(x), \quad s > 2, \quad \lambda > 0. \tag{4}$$

Note that $A \in L^2_{loc}(\mathbb{R}^3)$, which, together with the above assumptions on V and w, implies the self-adjointness of H_N, with domain

$$\mathcal{D}(H_N) = \{\psi \in L^2(\mathbb{R}^{3N}),\; \sum_{j=1}^{N} V_{eff}(x_j)\psi \in L^2(\mathbb{R}^{3N}),\; \sum_{j=1}^{N}(\nabla_j - iA(x_j))^2\psi \in L^2(\mathbb{R}^{3N})\}.$$

In the Schrödinger picture, the dynamics of the N-body system is given by

$$i\partial_t \Psi_N = H_N \Psi_N. \tag{5}$$

[1] It follows from Kato's inequality that the Coulomb interaction $w(x) = \pm\frac{1}{|x|}$ satisfies this assumption in $\mathbb{R}^3$.

Given an initial condition $\Psi_{N,0}$ at time $t_0 = 0$, the solution of (5) at time t is

$$\Psi_{N,t} = e^{-itH_N}\Psi_{N,0}.$$

In what follows, we normalize the initial condition, $\|\Psi_{N,0}\|_{L^2} = 1$. The density matrix associated with $\Psi_{N,t}$ is

$$\Gamma_{N,t} = |\Psi_{N,t}\rangle\langle\Psi_{N,t}|,$$

which is an orthogonal projection (in L^2) onto the state $\Psi_{N,t}$. It is a positive trace-class operator on $L^2(\mathbb{R}^{3N})$ with trace 1 and kernel

$$\Gamma_{N,t}(\mathbf{x}_N; \mathbf{x}'_N) = \Psi_{N,t}(\mathbf{x}_N)\overline{\Psi_{N,t}(\mathbf{x}'_N)}.$$

Here, $\mathbf{x}_m = (x_1, \cdots, x_m) \in \mathbb{R}^{3m}$, and $\bar{\cdot}$ denotes complex conjugation. The k^{th} *marginal density*, $\Gamma^{(k)}_{N,t}$, is defined by taking the partial trace of $\Gamma_{N,t}$ with respect to the last $N-k$ entries. Its kernel is given by

$$\Gamma^{(k)}_{N,t}(\mathbf{x}_k; \mathbf{x}'_k) := \int d\mathbf{x}_{N-k}\Gamma_{N,t}(\mathbf{x}_k, \mathbf{x}_{N-k}; \mathbf{x}'_k, \mathbf{x}_{N-k}), \quad k = 1, \cdots, N.$$

Note that $\Gamma^{(k)}_{N,t}$ is a positive trace-class operator in $L^2(\mathbb{R}^{3k})$ with trace 1. We will show that for an initial condition

$$\Psi_{N,0}(\mathbf{x}_N) = \prod_{j=1}^{N}\phi(x_j), \quad \|\phi\|_{L^2} = 1,$$

the marginal density matrix

$$\Gamma^{(k)}_{N,t} \to |\phi_t\rangle\langle\phi_t|^{\otimes k}$$

as $N \to \infty$, in trace-norm topology, where ϕ_t satisfies the magnetic Hartree equation

$$i\partial_t\phi_t = h_A\phi_t + w \star |\phi_t|^2\phi_t, \quad \phi_{t=0} = \phi, \tag{6}$$

with

$$h_A := -\nabla^2 + V(x) - L \cdot \Omega = (-i\nabla - A)^2 + V - \frac{1}{4}\Omega^2 r^2. \tag{7}$$

We introduce the quadratic form

$$Q_A(u, v) := \langle u, h_A v\rangle.$$

It follows from (4) that $Q_A(u, u)$ is bounded from below. More precisely, there exists a positive constant M such that

$$Q_A(u, u) \geq -M\|u\|^2_{H^1}.$$

We also introduce the Banach space $X_A \subset L^2(\mathbb{R}^3)$ with norm

$$\|u\|_{X_A} := \sqrt{(M+1)\|u\|_{H^1} + Q_A(u, u)}. \tag{8}$$

In Subsect. 2.1, we show that (6) is globally well-posed in X_A.

We are in a position to precisely state our main result, which gives an estimate of the difference between the true quantum evolution given by the marginal density matrix and the magnetic Hartree dynamics.

Theorem 1. *Consider $\Psi_{N,t}$ the solution of the N-body Schrödinger equation (5) with initial condition*

$$\Psi_{N,0}(\mathbf{x}_N) = \prod_{j=1}^{N} \phi(x_j), \quad \phi \in X_A, \quad \|\phi\|_{L^2} = 1.$$

Suppose that the pair interaction satisfies (3) and that the confining potential satisfies (4). Then there exists positive constants C and α that depend only on $\|\phi\|_{X_A}$ and the constant K appearing in (3), such that

$$Tr\Big|\Gamma^{(1)}_{N,t} - |\phi_t\rangle\langle\phi_t|\Big| \leq C\frac{e^{\alpha t}}{\sqrt{N}}, \quad t > 0,$$

where ϕ_t satisfies the magnetic Hartree equation (6) with initial condition $\phi_{t=0} = \phi$.

An analysis similar to the proof of Theorem 1 can be used to prove that

$$Tr(\Big|\Gamma^{(m)}_{N,t} - |\phi_t\rangle\langle\phi_t|^{\otimes m}\Big|) \leq C_m\frac{e^{\alpha_m t}}{\sqrt{N}}, \quad m < \infty.$$

This implies that for a bounded operator $A^{(m)}$ on $L^2(\mathbb{R}^{3m})$,

$$\langle\Psi_{N,t}, (A^{(m)} \otimes 1^{N-m})\Psi_{N,t}\rangle \to \langle\phi_t^{\otimes m}, A^{(m)}\phi_t^{\otimes m}\rangle$$

as $N \to \infty$.

The organization of this note is as follows. In Sect. 2, we discuss the well-posedness of the magnetic Hartree equation, and, for the sake of the convenience of the reader, recall basic results about the Fock space representation and coherent states. In Sect. 3, we prove Theorem 1.

2. Mathematical preliminary

2.1. **Well-posedness of the magnetic Hartree equation.** To set the stage, we discuss the well-posedness of (6). We have the following theorem.

Theorem 2. *Consider the magnetic Hartree equation (6). Suppose that (3) and (4) hold. Then (6) is globally well-posed in X_A.*

Proof. The proof relies on the following equivalence, which follows from (2), (4) and (8),

$$(9) \quad \|u\|_{H^1} + \||x|u\|_{L^2} + \|V^{1/2}u\|_{L^2} \lesssim \|u\|_{X_A} \lesssim \|u\|_{H^1} + \||x|u\|_{L^2} + \|V^{1/2}u\|_{L^2}.$$

Step 1. Lipschitz continuity.

We introduce the Banach space Σ with norm

$$\|u\|_\Sigma = \|u\|_{H^1} + \||x|u\|_{L^2} + \|V^{1/2}u\|_{L^2}.$$

It follows from (3) that

$$\|w \star |u|^2 u - w \star |v|^2 v\|_\Sigma \lesssim (\|u\|_\Sigma^2 + \|v\|_\Sigma^2)\|u-v\|_\Sigma.$$

Together with (9), this implies the Hartree nonlinearity is locally Lipschitz in X_A,

$$\|w \star |u|^2 u - w \star |v|^2 v\|_{X_A} \lesssim (\|u\|_{X_A}^2 + \|v\|_{X_A}^2)\|u-v\|_{X_A}. \tag{10}$$

Step 2. Local well-posedness.

The one particle Hamiltonian h_A defined in (7) gives rise to a self-adjoint operator on L^2. Its extension to X_A generates a C^0-group of isometries $\{e^{-ih_A t}\}_{t\geq 0}$ on X_A. Using standard arguments for evolution equations with locally Lipschitz nonlinearities (contraction argument for short time), the solution of the initial value problem

$$\phi_t = e^{-ih_A t}\phi_0 - ie^{-ih_A t}\int_0^t ds\ e^{ih_A s} w \star |\phi_s|^2 \phi_s$$

is the unique solution

$$\phi_t \in C^0([0,T); X_A) \cap C^1([0,T); X_A^*)$$

where X_A^* is the dual space of X_A. Here, the maximal time T is either ∞, or $T < \infty$ with $\lim_{t \nearrow T} \|\phi_t\|_{X_A} = \infty$. Furthermore, ϕ_t depends continuously on ϕ_0, see for example [7].

Step 3. Conservation of charge and energy.

It follows from (6) and Step 2 that $\frac{d}{dt}\langle \phi_t, \phi_t\rangle = 0$, and hence

$$\|\phi_t\|_{L^2} = \|\phi_0\|_{L^2}, \quad t \in [0,T). \tag{11}$$

Furthermore, using a standard regularization argument, it follows from (6) that the energy functional

$$\mathcal{E}(\phi) := \frac{1}{2}\langle \phi, h_A \phi\rangle + \frac{1}{4}\langle \phi, w \star |\phi|^2 \phi\rangle \tag{12}$$

is conserved, i.e.

$$\mathcal{E}(\phi_t) = \mathcal{E}(\phi_0), \quad t \in [0,T). \tag{13}$$

Step 4. Global well-posedness.

Note that (3) implies

$$\sup_{x\in\mathbb{R}^3} \int dy\ w(x-y)|\phi(y)|^2 \leq \epsilon \|\nabla \phi\|_{L^2}^2 + \epsilon^{-1}\|\phi\|_{L^2}^2, \quad \forall \epsilon \tag{14}$$

Now (4), (12) and (14) imply

$$\begin{aligned} \mathcal{E}(\phi) &\lesssim \|\phi\|_\Sigma^2 (1 + \|\phi\|_{L^2}^2) \\ &\lesssim \|\phi\|_{X_A}^2 (1 + \|\phi\|_{L^2}^2), \end{aligned} \tag{15}$$

where we have used (9) in the last inequality. Furthermore, they yield

$$\|\phi\|_{X_A}^2 \lesssim \mathcal{E}(\phi) + \|\phi\|_{L^2}^2 + \|\phi\|_{L^2}^4. \tag{16}$$

Conservation of charge, (11), and of energy, (13), together with estimates (15) and (16) imply global well-posedness, since the blow-up alternative can never be satisfied for finite maximal time T. □

2.2. **Bosonic Fock space and second quantization.** The bosonic Fock space over $L^2(\mathbb{R}^3)$ is defined as the Hilbert space

$$\mathcal{F} = \bigoplus_{n\geq 0} L^2(\mathbb{R}^3)^{\otimes_s n} = \mathbb{C} \oplus \bigoplus_{n\geq 1} L^2_s(\mathbb{R}^{3n}).$$

For $\psi \in \mathcal{F}$, $\psi = \{\psi^{(n)}\}_{n\geq 0}$, which is a sequence of n-particle wave functions $\psi^{(n)} \in L^2_s(\mathbb{R}^{3n})$. The scalar product on $\mathcal{F}$ is given by

$$\begin{aligned}\langle \psi_1, \psi_2 \rangle &= \sum_{n\geq 0} \langle \psi_1^{(n)}, \psi_2^{(n)} \rangle_{L^2(\mathbb{R}^{3n})} \\ &= \overline{\psi_1^{(0)}} \psi_2^{(0)} + \sum_{n\geq 1} \int dx_1 \dots dx_n \, \overline{\psi_1^{(n)}}(x_1, \dots, x_n) \psi_2^{(n)}(x_1, \dots, x_n).\end{aligned}$$

The vacuum vector in $\mathcal{F}$ is given by $\Omega = \{1, 0, 0, \dots\}$, while the N-particle wavefunction ψ_N is given by $\{\psi^{(n)}\}_{n\geq 0}$ where $\psi^{(N)} = \psi_N$ and $\psi^{(n)} = 0$ for all $n \neq N$.

On $\mathcal{F}$, we introduce the creation operator $a^*(f)$ and the annihilation operator $a(f)$, which are defined through their action

$$\begin{aligned}\left(a^*(f)\psi\right)^{(n)}(x_1, \dots, x_n) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \psi^{(n-1)}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ \left(a(f)\psi\right)^{(n)}(x_1, \dots, x_n) &= \sqrt{n+1} \int dx \, \overline{f}(x) \, \psi^{(n+1)}(x, x_1, \dots, x_n) \, .\end{aligned}$$

Note that $a^*(f)$ and $a(f)$ are unbounded, densely defined, closed operators, and they satisfy the canonical commutation relations (CCR)

$$[a(f), a^*(g)] = \langle f, g \rangle_{L^2(\mathbb{R}^3)}, \qquad [a^\#(f), a^\#(g)] = 0, \tag{17}$$

where $a^\#$ is either a or a^*. The operator valued distributions a^*_x and a_x ($x \in \mathbb{R}^3$) are defined by

$$a^*(f) = \int dx \; f(x) \, a^*_x, \quad a(f) = \int dx \; \overline{f}(x) \, a_x,$$

for every $f \in L^2(\mathbb{R}^3)$. The CCR in this case are

$$[a_x, a^*_y] = \delta(x - y) \qquad [a^\#_x, a^\#_y] = 0.$$

We define the particle number operator $\widehat{N}$ by

$$(\widehat{N}\psi)^{(n)} = n\psi^{(n)}.$$

Expressed through the distributions a_x and a_x^*, it is given by

$$\widehat{N} = \int dx\, a_x^* a_x.$$

The following are standard estimates for the creation and annihilation operators in terms of the number of particle operator. They follow from a direct application of the CCR, see for example [6].

$$\begin{aligned} \|a(f)\psi\| &\leq \|f\|_{L^2}\, \|\widehat{N}^{1/2}\psi\| \\ \|a^*(f)\psi\| &\leq \|f\|_{L^2}\, \| \left(\widehat{N}+1\right)^{1/2} \psi\|. \end{aligned} \tag{18}$$

For $\psi \in \mathcal{F}$, the one-particle density $\widehat{\Gamma}^{(1)}_\psi$ associated with ψ is the positive trace-class operator on $L^2(\mathbb{R}^3)$ with kernel

$$\widehat{\Gamma}^{(1)}_\psi(x;y) = \frac{1}{\langle \psi, \widehat{N}\psi\rangle}\, \langle \psi, a_y^* a_x \psi\rangle \tag{19}$$

and trace 1.

We introduce the *second quantized* Hamiltonian $\widehat{H}_N$ acting on $\mathcal{F}$, which is given by

$$(\widehat{H}_N\psi)^{(n)} = \widehat{H}_N^{(n)}\psi^{(n)},$$
$$\widehat{H}_N^{(n)} = -\sum_{j=1}^n h_{A_j} + \frac{1}{N}\sum_{i<j}^n w(x_i - x_j).$$

Equivalently,

$$\widehat{H}_N = \int dx\; a_x^* h_A a_x + \frac{1}{2N}\int dxdy\, w(x-y) a_x^* a_y^* a_y a_x. \tag{20}$$

Note that when restricted to the N-particle sector, $\widehat{H}_N$ coincides with the Hamiltonian H_N defined in (1). Furthermore, $[\widehat{H}_N, \widehat{N}] = 0$.

We now introduce the notion of *coherent states*, which will be useful in the subsequent analysis. For $f \in L^2(\mathbb{R}^3)$, we define the Weyl-operator

$$W(f) := \exp\left(a^*(f) - a(f)\right).$$

The coherent state $\psi(f) \in \mathcal{F}$ with one-particle wave function f is given by

$$\psi(f) := W(f)\Omega.$$

It follows from the identity

$$\exp(a^*(f) - a(f)) = e^{-\|f\|_{L^2}^2/2} \exp(a^*(f)) \exp(-a(f))$$

that

$$\psi(f) = W(f)\Omega = e^{-\|f\|_{L^2}^2/2} \sum_{n\geq 0} \frac{(a^*(f))^n}{n!}\Omega = e^{-\|f\|_{L^2}^2/2} \sum_{n\geq 0} \frac{1}{\sqrt{n!}}\, f^{\otimes n}\,, \tag{21}$$

where $f^{\otimes n}$ corresponds to $\{0, \dots, 0, f^{\otimes n}, 0, \dots\} \in \mathcal{F}$.

We recall some useful properties of Weyl-operators and coherent states, whose proof follows from the definitions. For $f, g \in L^2(\mathbb{R}^3)$, the following holds:

$$
\begin{aligned}
&W(f)W(g) = W(g)W(f)e^{-2i\,\Im\langle f,g\rangle} = W(f+g)e^{-i\,\Im\langle f,g\rangle},\\
&W(f)^* = W(f)^{-1} = W(-f),\\
&W^*(f)a_xW(f) = a_x + f(x),\\
&W^*(f)a^*_xW(f) = a^*_x + \overline{f}(x),\\
&a(g)\psi(f) = \langle g, f\rangle\psi(f),\\
&\langle\psi(f), \widehat{N}\psi(f)\rangle = \|f\|^2_{L^2}.
\end{aligned} \tag{22}
$$

3. Proof of the main result

We first express the initially factorized state in terms of coherent states. This representation has been used in [20]. Since

$$
\begin{aligned}
\int_0^{2\pi}\frac{d\theta}{2\pi}e^{iN\theta}W(e^{-i\theta}\sqrt{N}\phi)\Omega &= \int_0^{2\pi}\frac{d\theta}{2\pi}e^{iN\theta}\sum_{n\geq 0}e^{-N\|\phi\|^2_{L^2}/2}\frac{e^{-in\theta}N^{n/2}(a^*(\phi))^n}{n!}\Omega\\
&= \sum_{n\geq 0}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{-N/2}e^{iN\theta}\frac{e^{-in\theta}N^{n/2}}{\sqrt{n!}}\phi^{\otimes n}\\
&= \frac{N^{N/2}e^{-N/2}}{\sqrt{N!}}\phi^{\otimes N},
\end{aligned}
$$

we have

$$
\phi^{\otimes N} = a_N\int_0^{2\pi}\frac{d\theta}{2\pi}e^{iN\theta}W(e^{-i\theta}\sqrt{N}\phi)\Omega \tag{23}
$$

with $a_N = \frac{e^{N/2}\sqrt{N!}}{N^{N/2}}$. Using Stirling's formula, $a_N \sim N^{1/4}$ for $N \gg 1$.

We now substitute representation (23) in the expression of the one-particle marginal (19) corresponding to the state $\widehat{\Psi}_{N,t} = e^{-i\widehat{H}_Nt}\phi^{\otimes N}$. We have

$$
\begin{aligned}
\widehat{\Gamma}^{(1)}_{N,t}(x;y) &= \frac{1}{\langle\psi, \widehat{N}\psi\rangle}\langle\widehat{\Psi}_{N,t}, a^*_ya_x\widehat{\Psi}_{N,t}\rangle\\
&= \frac{1}{N}\langle\phi^{\otimes N}, a^*_y(t)a_x(t)\phi^{\otimes N}\rangle\\
&= \frac{a_N^2}{N}\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}e^{iN(\theta_2-\theta_1)}\langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega, a^*_y(t)a_x(t)W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle,
\end{aligned}
$$

where we have used the fact that the time evolution commutes with $\widehat{N}$ in the denominator. Here, $a^{\#}(t) = e^{i\widehat{H}_Nt}a^{\#}e^{-i\widehat{H}_Nt}$, the time evolved operator in the Heisenberg picture. We are interested in comparing the true dynamics to the

Hartree one in the large N limit. This motivates adding and subtracting $\sqrt{N}\phi_t$. We have

(24)
$$\begin{aligned}\widehat{\Gamma}^{(1)}_{N,t}(x;y) =&\frac{a_N^2}{N}\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}e^{iN(\theta_2-\theta_1)}\langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega,(a_y^*(t)-e^{i\theta_1}\sqrt{N}\overline{\phi}_t(x))\times\\ &\times(a_x(t)-e^{-i\theta_2}\sqrt{N}\phi_t(y))W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle\\ &+\overline{\phi}_t(y)\phi_t(x)+\frac{\overline{\phi}_t(y)f_N(x)}{\sqrt{N}}+\frac{\overline{f}_N(y)\phi_t(x)}{\sqrt{N}},\end{aligned}$$

where

(25)
$$\begin{aligned}f_N(x) =&a_N^2\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}e^{iN\theta_2}e^{-i(N-1)\theta_1}\langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega,(a_x(t)-e^{-i\theta_2}\sqrt{N}\phi_t(x))\times\\ &\times W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle.\end{aligned}$$

The first term in (24) motivates introducing the unitary propagator

$$\widehat{U}_N(t):=W^*(\sqrt{N}\phi_t)e^{-i\widehat{H}_Nt}W(\sqrt{N}\phi).$$

This propagator appeared in [18], and it describes the dynamics of the fluctuations away from the Hartree evolution. Using the third and fourth relations in (22), one can verify that $\widehat{U}_N$ satisfies the initial value problem

$$\begin{aligned}&i\partial_t\widehat{U}_N(t)=\widehat{L}_N(t)\widehat{U}_N(t),\\ &\widehat{U}_N(0)=1,\end{aligned}\tag{26}$$

with

(27)
$$\begin{aligned}\widehat{L}_N(t)=&\int dx\ a_x^*h_Aa_x+\int dx\ w\star|\phi_t|^2(x)a_x^*a_x+\int dx\ dy\ w(x-y)\overline{\phi}_t(y)\phi_t(x)a_x^*a_y\\ &+\frac{1}{2}\int dx\ dy\ w(x-y)\phi_t(y)\phi_t(x)a_y^*a_x^*+\frac{1}{2}\int dx\ dy\ w(x-y)\overline{\phi}_t(y)\overline{\phi}_t(x)a_ya_x\\ &+\frac{1}{\sqrt{N}}\int dx\ dy\ w(x-y)a_x^*\phi_t(y)a_y^*a_x+\frac{1}{\sqrt{N}}\int dx\ dy\ w(x-y)a_x^*\overline{\phi}_t(y)a_ya_x\\ &+\frac{1}{2N}\int dx\ dy\ w(x-y)a_x^*a_y^*a_ya_x.\end{aligned}$$

Here, ϕ_t satisfies (6) with initial condition ϕ. Note that it follows directly from the defintion of $\widehat{U}_N$ that

$$W^*(\sqrt{N}\phi_t)e^{i\widehat{H}_Nt}(a_x-\sqrt{N}\phi_t)e^{-i\widehat{H}_Nt}W(\sqrt{N}\phi)=\widehat{U}_N(-t)a_x\widehat{U}_N(t).\tag{28}$$

We need the following estimate.

Lemma 1. *Suppose that (3) and (4) hold. Then there exist positive constants C and α that depend on $\|\phi\|_{X_A}$ and K only, such that*

$$\|\widehat{N}^{1/2}\widehat{U}_N(t)\Omega\|^2 \leq Ce^{\alpha t},$$

uniformly in $N \geq 1$.

Proof. Step 1. Auxiliary dynamics.
We introduce the parity preserving auxiliary dynamics

$$i\partial_t\tilde{U}_N(t) = \tilde{L}_N(t)\tilde{U}_N(t), \quad \tilde{U}_N(0) = 1, \tag{29}$$

with

$$\begin{aligned}\tilde{L}_N(t) = &\int dx\ a_x^* h_A a_x + \int dx\ w \star |\phi_t|^2(x) a_x^* a_x + \int dx\ dy\ w(x-y)\overline{\phi}_t(y)\phi_t(x)a_x^* a_y \\ &+ \frac{1}{2}\int dx\ dy\ w(x-y)\phi_t(y)\phi_t(x)a_y^* a_x^* + \frac{1}{2}\int dx\ dy\ w(x-y)\overline{\phi}_t(y)\overline{\phi}_t(x)a_y a_x \\ &+ \frac{1}{2N}\int dx\ dy\ w(x-y)a_x^* a_y^* a_y a_x.\end{aligned}$$

We now show the following estimates before comparing the true dynamics to the auxiliary one,

$$\langle\Omega, \tilde{U}_N(-t)\widehat{N}\tilde{U}_N(t)\Omega\rangle \lesssim e^{\alpha t} \tag{30}$$

$$\langle\Omega, \tilde{U}_N(-t)\widehat{N}^3\tilde{U}_N(t)\Omega\rangle \lesssim e^{\alpha t} \tag{31}$$

uniformly in $N \geq 1$.
Let

$$I_N^{(m)} := \langle\Omega, \tilde{U}_N(-t)\widehat{N}^m\tilde{U}_N(t)\Omega\rangle, \quad m = 1, 3.$$

It follows from (29) that

$$\begin{aligned}\frac{dI_N^{(1)}}{dt} &= \langle\Omega, \tilde{U}_N(-t)i[\tilde{L}_N(t), \widehat{N}]\tilde{U}_N(t)\Omega\rangle \\ &= i\Re\int dx\ dy\ w(x-y)\phi_t(x)\phi_t(y)\langle\Omega, \tilde{U}_N(-t)[a_x^* a_y^*, \widehat{N}]\tilde{U}_N(t)\Omega\rangle.\end{aligned} \tag{32}$$

Using the push-through relations

$$a_x\widehat{N} = (\widehat{N}+1)a_x, \quad a_x^*\widehat{N} = (\widehat{N}-1)a_x \tag{33}$$

we have

$$
\begin{aligned}
&|\int dx\ dy\ w(x-y)\phi_t(x)\phi_t(y)\langle\Omega, \tilde{U}_N(-t)[a_x^* a_y^*, \widehat{N}]\tilde{U}_N(t)\Omega\rangle| \\
&\leq 2\sup_x \|w(x-\cdot)\phi_t\|_{L^2}\|\phi_t\|_{L^2}\|(\widehat{N}+1)^{1/2}\tilde{U}_N(t)\Omega\|^2 \\
&\leq 2K\|\phi_t\|_{X_A}\|\phi_t\|_{L^2} I_N^{(1)} \\
&\leq C(\mathcal{E}(\phi)+2) I_N^{(1)} \\
&\leq C'\|\phi\|_{X_A} I_N^{(1)},
\end{aligned} \tag{34}
$$

for some constant C'.Here, we have used (18) and Hölder's inequality in the first line, (3) in the second line, and (11), (13), (15) and (16) in the last two lines. Similarly, using (33), we have

$$
\begin{aligned}
\frac{dI_N^{(3)}}{dt} &\leq C\|\phi\|_{X_A}\|(\widehat{N}+3)^{3/2}\tilde{U}_N(t)\Omega\|^2 \\
&\leq C'\|\phi\|_{X_A} I_N^{(3)}
\end{aligned} \tag{35}
$$

Estimates (30) and (31) follow from Gronwall lemma and (34) and (35), respectively.

Step 2. Comparison between auxiliary and true dynamics.

In order to complete the proof of the lemma, we need to estimate $\|(\tilde{U}_N(t) - \widehat{U}_N(t))\Omega\|$. We have

$$
\begin{aligned}
&|\langle\Omega, (1-\tilde{U}_N(-t)\widehat{U}_N(t))\Omega\rangle| = |\int_0^t ds\ \langle\Omega, \partial_s(\tilde{U}_N^*(s)\widehat{U}_N(s))\Omega\rangle| \\
&= |\int_0^t ds\ \langle\Omega, \tilde{U}_N^*(s)(\widehat{L}_N(s) - \tilde{L}_N(s))\widehat{U}_N(s))\Omega\rangle| \\
&= \frac{1}{\sqrt{N}}|\Im\int_0^t ds\ \int dx\ dy\ \langle\Omega, \tilde{U}_N^*(s)(w(x-y)a_x a_y^* a_x \phi_t(y))\widehat{U}_N(s))\Omega\rangle| \\
&\leq \frac{C}{\sqrt{N}}\|\phi\|_{X_A}\int_0^t \|(\widehat{N}+3)^{3/2}\tilde{U}_N(s)\Omega\|
\end{aligned}
$$

where we have used (3) in the last line. Together with (31), this implies that

$$
|\langle\Omega, (1-\tilde{U}_N(-t)\widehat{U}_N(t))\Omega\rangle| \leq \frac{C}{\sqrt{N}}e^{\alpha t}. \tag{36}
$$

It follows from (36) and the unitarity of $\tilde{U}_N$ that

$$
\|(\tilde{U}_N(t) - \widehat{U}_N(t))\Omega\| \leq \frac{C}{\sqrt{N}}e^{\alpha t} \tag{37}
$$

where C and α are positive constants that depend only on $\|\phi\|_{X_A}$ and K. The claim of the lemma follows directly from (30) and (37). □

We are now in a position to prove the main result.

Proof of Theorem 1. We want to estimate the Hilbert-Schmidt norm of the difference $\widehat{\Gamma}^{(1)}_{N,t}(x;y) - \overline{\phi}_t(y)\phi_t(x)$. It follows from (28) that

$$
\begin{aligned}
&|\frac{a_N^2}{N}\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}e^{iN(\theta_2-\theta_1)}\langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega,(a_y^*(t)-e^{i\theta_1}\sqrt{N}\overline{\phi}_t(y))\times\\
&\times(a_x(t)-e^{-i\theta_2}\sqrt{N}\phi_t(x))W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle|\\
&\le\frac{a_N^2}{N}\|a_y^*\widehat{U}_N^{\theta_1}(t)\Omega\|\|a_x\widehat{U}_N^{\theta_2}(t)\Omega\|
\end{aligned}
$$

where $\widehat{U}_N^{\theta}$ corresponds to $\widehat{U}_N$ with ϕ replaced by $e^{-i\theta}\phi$. Together with Lemma 1, this implies that

(38)
$$
\begin{aligned}
&\frac{a_N^4}{N^2}\int dxdy\,|\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}e^{iN(\theta_2-\theta_1)}\langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega,(a_y^*(t)-e^{i\theta_1}\sqrt{N}\overline{\phi}_t(y))\times\\
&\times(a_x(t)-e^{-i\theta_2}\sqrt{N}\phi_t(x))W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle|^2\\
&\le\frac{Ca_N^4}{N^2}\|\widehat{N}^{1/2}\widehat{U}_N^{(\theta_1)}(t)\Omega\|^2\|\widehat{N}^{1/2}\widehat{U}_N^{(\theta_2)}(t)\Omega\|^2\\
&\le\frac{C}{N}e^{\alpha t}.
\end{aligned}
$$

Estimating f_N, which appears in the third and fourth terms of (24), is somewhat more delicate. We make use of the fast oscillating phase of the integrand in (25). Integrating by parts in θ_1, we have

$$
\begin{aligned}
f_N(x)=&a_N^2\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}e^{iN\theta_2}[\frac{1}{i(N-1)}\frac{d}{d\theta_1}e^{-i\theta_1(N-1)}]\times\\
&\times\langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega,(a_x(t)-e^{-i\theta_2}\sqrt{N}\phi_t(x))W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle\\
&=\frac{a_N^2}{N-1}\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}e^{iN\theta_2}e^{-i(N-1)\theta_1}\langle(a^*(e^{-i\theta_1}\sqrt{N}\phi)-a(e^{-i\theta_1}\sqrt{N}\phi))\times\\
&\times W(e^{-i\theta_1}\sqrt{N}\phi)\Omega,(a_x(t)-e^{-i\theta_2}\sqrt{N}\phi_t(x))W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|f_N\|_{L^2}&\le\frac{4a_N^2}{N-1}\|\widehat{N}^{1/2}W(e^{-i\theta_1}\sqrt{N}\phi)\Omega\|\|\widehat{N}^{1/2}\widehat{U}_N^{\theta_2}(t)\Omega\|\\
&\le\frac{Ca_N^2}{\sqrt{N}}\|\widehat{N}^{1/2}\widehat{U}_N^{\theta_2}(t)\Omega\|,
\end{aligned}
$$

which, together with Lemma 1, yields the estimate

$$
\|f_N\|_{L^2}\le Ce^{\alpha t}. \tag{39}
$$

Eq. (24) together with estimates (38) and (39) imply that the Hilbert-Schmidt norm of the difference

$$\|\widehat{\Gamma}^{(1)}_{N,t} - |\phi_t\rangle\langle\phi_t|\|_{HS} \leq \frac{C}{\sqrt{N}} e^{\alpha t},$$

which gives the claim of the theorem. □

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