ON THE RENORMALIZATION GROUP APPROACH TO PERTURBATION THEORY FOR PDES

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Abstract. We investigate the rigorous application of the renormalization group method to (singular) perturbation theory for nonlinear partial differential equations. As a paradigm, we consider the concrete example of the nonlinear Schrödinger equation with quadratic nonlinearity in three spatial dimensions. We obtain an approximate solution using the RG method together with an estimate of the difference between the true and approximate solutions. Our analysis applies to cases where (space-time) resonances are present.

1. Introduction

The Chen-Goldenfeld-Oono renormalization group (RG) approach to singular perturbation theory, introduced in [1, 2], has proven to be very versatile in handling different classes of perturbations in a unified manner. Moreover, it is conceptually simple and elegent: a naive perturbative expansion together with the RG condition are generally sufficient to give all the relevant scales in the problem. The method has been put on a rigorous footing for a wide class of ODEs in [3, 4], and for some PDEs defined on bounded intervals with periodic boundary conditions in [5, 6, 7]. Here, we show that the method is rigorously applicable to nonlinear PDEs in the continuum and in the presence of (space-time) resonances. For the sake of concreteness, we consider as an example the nonlinear Schrödinger equation with quadratic nonlinearity in three spatial dimensions. We obtain a long-time approximation for the true solution using the RG method. Our analysis in handling resonances relies on the application of the Coifman-Meyer theorem, [8], together with fractional integration. A similar approach to resonances has been used in studying the well-posedness of the quadratic nonlinear Schrödinger equation in three dimensions with small data, see [9], and the Gross-Pitaevski equation, [10]. We hope that the conceptual simplicity and elegance of the RG method will motivate further rigorous applications.

The organization of this note is as follows. In Sect. 2, we recall the RG approach to singular perturbation theory in the setting of PDEs. We then discuss our model and state the main result (Theorem 1) in Sect. 3. After discussing some properties of the approximate solution in Sect. 4, we prove the main result in Sect. 5. In order to simplify the presentation, we defer to the Appendix the proof of a technical estimate that relies on the Coifman-Meyer theorem and fractional integration.

2. The RG method

We start with a general discussion about the RG approach to singular perturbation theory. Suppose u belongs to some Banach space X, and suppose that A is a generator of a semigroup on X. We consider the following equation

$$\partial_t u = Au + \epsilon f(u), \tag{1}$$

with initial condition $u_{t_0} \in X$, where $f(u)$ is a polynomial nonlinearity, and $\epsilon \ll 1$. We are interested in finding approximate solutions of the above initial value problem in the limit $\epsilon \to 0$. [1] The weak solution of the above equation is given by the Duhamel formula

$$u(t) = e^{A(t-t_0)}u_{t_0} + \epsilon e^{At}\int_{t_0}^{t} ds\ e^{-As} f(u(s)).$$

Performing a naive perturbative expansion in ϵ, we have

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots,$$

where

$$\begin{aligned} u_0(t) &= e^{A(t-t_0)}u_{t_0} \\ u_1(t) &= e^{At}\int_{t_0}^{t} ds\ e^{-As} f(u_0(s)) \\ u_2(t) &= e^{At}\int_{t_0}^{t} ds\ e^{-As}\nabla_u f(u_0(s)) \cdot u_1(s) \\ &\cdots \end{aligned}$$

Depending on the nonlinearity, we have

$$e^{-As} f(e^{As}v) = f_{res}(v) + f_{osc}(s, v)$$

[1] This is equivalent to studying the asymptotics of

$$\partial_\tau u = \frac{1}{\epsilon}Au + f(u)$$

where $\tau = \epsilon t$.

where f_{res} stands for the resonance part, and f_{osc} stands for the oscillatory part. Formally, the solution is given by

$$u(t) = e^{A(t-t_0)}u_{t_0} + \epsilon(t-t_0)e^{At}f_{res}(u_{t_0}) + \epsilon e^{At}F_{osc}(t,u_{t_0}) + O(\epsilon^2),$$

where $F_{osc}(t,v) := \int_{t_0}^{t} ds f_{osc}(t,v)$. Note that the second term in the expansion, the so called *secular term*, grows with t, and the naive perturbative expansion will break down with time. The purpose of the CGO-method is to renormalize this term so that the main contribution coming from the secular term is taken into account.

Let W satisfy

$$\partial_t W = \epsilon f_{res}(W), \tag{2}$$

with initial condition $W(t_0) = e^{-At_0}u_{t_0}$. Eq. (2) is called the renormalization group condition. The approximate solution, to first order, is given by [2]

$$\overline{u}(t) := e^{At}\{W(t) + \epsilon F_{osc}(t, W(t))\}. \tag{3}$$

Differentiating (3) with respect to t gives

$$\partial_t\overline{u} = A\overline{u} + \epsilon f(\overline{u}) + R_\epsilon(\overline{u}, W),$$

where the remainder term R_ϵ is given by

(4)

$$R_\epsilon(\overline{u}, W) = \epsilon(f(e^{At}W(t)) - f(\overline{u})) + \epsilon^2 e^{At}\nabla_W F_{osc}(t, W(t))f_{res}(W(t)).$$

Note that the initial condition $\overline{u}(t_0) = u_{t_0}$.

In what follows, we apply the above abstract analysis to the concrete example of a nonlinear Schrödinger equation with quadratic nonlinearity, for which the resonant term is nontrivial. We will show that indeed, $\overline{u}$ is an approximation of u in a suitable Banach space, up to time of order $O(\frac{|\log\epsilon|}{\epsilon})$.

3. The model and statement of the main result

We consider the nonlinear Schrödinger equation with quadratic nonlinearity in three spatial dimensions,

$$i\partial_t u = -\Delta u - \epsilon u^2, \tag{5}$$

with initial condition u_{t_0}. Here, $A = i\Delta$, where Δ is the spatial Laplacian in $\mathbb{R}^3$, and $f(u) = iu^2$.

[2]The RG method can be applied recursively to obtain an approximate solution to higher orders (see, for example, [4]), but we restrict the discussion to first order in order to simplify the presentation.

For $\tau > 0$, we let $\mathcal{B}_\tau$ be the Banach space with norm

$$\|u\|_{\mathcal{B}_\tau} = \|u\|_{L^\infty([t_0,t_0+\tau];L^2(\mathbb{R}^3))} + \|\widehat{e^{-i\Delta t}u}\|_{L^\infty([t_0,t_0+\tau];L^\infty(\mathbb{R}^3))} + \|\frac{x}{\log t}e^{-i\Delta t}u\|_{L^\infty([t_0,t_0+\tau];L^2(\mathbb{R}^3))} + \|\frac{x^2}{\sqrt{t}}e^{-i\Delta t}u\|_{L^\infty([t_0,t_0+\tau];L^2(\mathbb{R}^3))} + \|t^{3/2}u\|_{L^\infty([t_0,t_0+\tau];L^\infty(\mathbb{R}^3))},$$

where the hat stands for the Fourier transform. Without loss of generality, we assume that $t_0 = 2$ (alternatively, replace t appearing in the denominator in the $\mathcal{B}$-norm with $\langle t\rangle = \sqrt{1+t^2}$). The well-posedness of the nonlinear Schrödinger equation with quadratic nonlinearity in $\mathcal{B}_\infty$ for small initial data has been proven in [9]. Small initial data corresponds to small ϵ in (5) after a simple rescaling of u. Therefore, if $\|e^{i\Delta(t-t_0)}u_{t_0}\|_{\mathcal{B}_\infty} < \infty$, there exists $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$, (5) is well-posed in $\mathcal{B}_\infty$. The following is the main result of this note.

Theorem 1. *Consider (5) with initial condition u_{t_0} such that*

$$\phi_0 := \|e^{i\Delta(t-t_0)}u_{t_0}\|_{\mathcal{B}_\infty} < \infty.$$

Then there exists $\epsilon_0 > 0$, that depends on ϕ_0, such that, for all $|\epsilon| < \epsilon_0$ and $\delta \in (0,1)$,

$$\|u - \overline{u}\|_{L^\infty([t_0,t_0+\delta\frac{|\log\epsilon|}{e\phi_0\epsilon})];L^2(\mathbb{R}^3))} < C\epsilon^{1-\delta}$$

for some positive constant C that is independent of ϵ and δ.

4. Approximate solution

In this section, we investigate some properties of W and $\overline{u}$, which are sufficient to prove Theorem 1. Since

$$e^{-i\Delta s}f(e^{i\Delta s}u) = i\mathcal{F}^{-1}\int d\xi e^{is(k^2-\xi^2-(k-\xi)^2)}\widehat{u}(\xi)\widehat{u}(k-\xi)$$

where $\mathcal{F}^{-1}$ stands for the inverse Fourier transform, we have

$$f_{res}(u) = i\widehat{u}(0)u, \tag{6}$$

and

$$f_{osc}(u) = i\mathcal{F}^{-1}\int_{\xi\neq k} d\xi\ e^{is(k^2-\xi^2-(k-\xi)^2)}\widehat{u}(\xi)\widehat{u}(k-\xi). \tag{7}$$

The RG equation corresponding to (5) is given by

$$\partial_t W(t,x) = i\epsilon\widehat{W}(t,0)W(t,x) \tag{8}$$

with initial condition $W(t_0) = e^{-i\Delta t_0}u_{t_0}$. We have the following straightforward result, which follows from Gronwall lemma and a boot-strap argument.

Lemma 1. *Suppose that $\phi_0 < \infty$. Then*

$$\sup_{s\in[t_0,t_0+\frac{1}{e\phi_0\epsilon}]} \|e^{i\Delta t}W(s)\|_{\mathcal{B}_\infty} \le e\phi_0.$$

Proof. Let T be the maximal time such that

$$\sup_{s\in[t_0,T]} \|e^{i\Delta t}W(s)\|_{\mathcal{B}_\infty} \le e\phi_0.$$

It follows from the RG equation (8) and Gronwall lemma that [3]

$$\begin{aligned}
&\|e^{i\Delta t}W(s)\|_{\mathcal{B}_\infty} \le e^{\epsilon\int_{t_0}^s ds'\|\widehat{W}\|_{L^\infty([t_0,T'];L^\infty)}}\|e^{i\Delta(t-t_0)}u_{t_0}\|_{\mathcal{B}_\infty}\\
&\le e^{\epsilon(s-t_0)\sup_{s'\in[t_0,s]}\|e^{i\Delta t}W(s')\|_{\mathcal{B}_\infty}}\phi_0\\
&\le e^{\epsilon(s-t_0)e\phi_0}\phi_0 \le e\phi_0
\end{aligned}$$

for all $s \le t_0 + \frac{1}{e\phi_0\epsilon}$. Hence, the maximal time $T \ge t_0 + \frac{1}{e\phi_0\epsilon}$. ☐

We now introduce the Banach space $\mathcal{C}_\tau$ whose norm is given by

(9) $$\|u\|_{\mathcal{C}_\tau} := \|u\|_{L^\infty([t_0,t_0+\tau];L^2(\mathbb{R}^3))} + \|t^{3/2}u\|_{L^\infty([t_0,t_0+\tau];L^\infty(\mathbb{R}^3))}.$$

The approximate solution $\overline{u}$ defined in (3) is explicitly given by

(10)
$$\overline{u}(t) = e^{i\Delta t}(W(t)+\epsilon\underbrace{i\int_{t_0}^t ds\mathcal{F}^{-1}(\int_{\xi\neq k} d\xi e^{i(k^2-\xi^2-(k-\xi)^2)s}\widehat{W}(t,\xi)\widehat{W}(t,k-\xi))}_{F_{osc}(t,W(t))}),$$

where $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. It follows from Lemma 1 that

(11) $$\|e^{i\Delta t}W(t)\|_{\mathcal{C}_{\frac{1}{e\phi_0\epsilon}}} \le \sup_{s\in[t_0,t_0+\frac{1}{e\phi_0\epsilon}]} \|e^{i\Delta t}W(s)\|_{\mathcal{B}_\infty} < C$$

for some constant C that is independent of ϵ. We also have the following estimate, which follows from the Coifman-Meyer theorem and fractional integration, and which we prove in the Appendix,

(12) $$\|e^{i\Delta t}F_{osc}(t,W(t))\|_{\mathcal{C}_{\frac{1}{e\phi_0\epsilon}}} \le C,$$

for some constant C that is independent of ϵ, but that depends on ϕ_0. The following result follows from (10) - (12).

Lemma 2. *Suppose that $\phi_0 < \infty$. Then $\overline{u} \in \mathcal{C}_{\frac{1}{e\phi_0\epsilon}}$.*

[3]The (weak) solution of (8) is given by $W(t,x) = e^{i\epsilon\int_{t_0}^t ds\ \widehat{W}(s,0)}e^{-i\Delta t_0}u_{t_0}(x)$.

5. Proof of Theorem 1

We start by estimating the L^2-norm of the difference between the approximate solution $\overline{u}$ and the true solution u up to times of order $O(\epsilon^{-1})$, and then we reiterate our estimates to longer times. We have the following proposition.

Proposition 1. *Suppose that $\phi_0 < \infty$. Then there exists $\epsilon_0 > 0$ such that, for all $|\epsilon| < \epsilon_0$,*

$$\|u(t) - \overline{u}(t)\|_{L^\infty([t_0, t_0 + \frac{1}{e\phi_0\epsilon}]; L^2(\mathbb{R}^3))} < C\epsilon,$$

for some positive constant C that is independent of ϵ.

Proof. Let

$$v(t) := u(t) - \overline{u}(t).$$

The Duhamel formula yields

$$v(t) = \underbrace{e^{i\Delta(t-t_0)}v(t_0)}_{I} + \underbrace{i\epsilon e^{i\Delta t}\int_{t_0}^{t} ds\; e^{-i\Delta s}(f(u(s)) - f(\overline{u}(s)))}_{II}$$

$$+ \underbrace{i\epsilon e^{i\Delta t}\int_{t_0}^{t} ds\; e^{-i\Delta s}R_\epsilon(\overline{u}(s), W(s))}_{III}. \tag{13}$$

It follows from the unitarity of the *free* evolution that

$$\|I\|_{L^2} = \|v(t_0)\|_{L^2}. \tag{14}$$

Furthermore, it follows from an energy estimate that

$$\begin{aligned}
\|II\|_{L^2} &= \epsilon\| \int_{t_0}^{t} ds\; e^{-i\Delta s}(u(s) + \overline{u}(s))(u(s) - \overline{u}(s))\|_{L^2} \\
&\leq \epsilon \int_{t_0}^{t} ds \|(u(s) + \overline{u}(s))(u(s) - \overline{u}(s))\|_{L^2} \\
&\leq \epsilon \int_{t_0}^{t} \frac{ds}{s^{3/2}}(\|u\|_{\mathcal{C}_t} + \|\overline{u}\|_{\mathcal{C}_t})\|u - \overline{u}\|_{L^\infty([t_0,t]; L^2(\mathbb{R}^3))} \\
&\leq C\epsilon(\|u\|_{\mathcal{B}_t} + \|\overline{u}\|_{\mathcal{C}_t})\|u - \overline{u}\|_{L^\infty([t_0,t]; L^2(\mathbb{R}^3))}
\end{aligned} \tag{15}$$

where C is a positive constant that is independent of ϵ and t. We also have

$$\|III\|_{L^2} = \epsilon^2 \| \int_{t_0}^{t} ds\ e^{-i\Delta s} \nabla_W F_{osc}(s, W(s)) f_{res}(W(s)) \|_{L^2}$$
$$\leq 2\epsilon^2 \int_{t_0}^{t} ds \int_{t_0}^{s} ds'\ \|\widehat{W}(s,0) e^{-i\Delta s} (e^{i\Delta s'} W(s))^2 \|_{L^2}$$
$$\leq 2\epsilon^2 \|\widehat{W}(s,0)\|_{L^\infty([t_0,t])} \int_{t_0}^{t} ds \int_{t_0}^{s} \frac{ds'}{s'^{3/2}} \|s'^{3/2} e^{i\Delta s'} W(s)\|_{L^\infty(\mathbb{R}^3)} \|e^{i\Delta s'} W(s)\|_{L^2(\mathbb{R}^3)}.$$

Now, applying the stationary phase estimate

$$\|e^{i\Delta t} g\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{t^{3/2}} \|\widehat{g}\|_{L^\infty(\mathbb{R}^3)} + \frac{1}{t^{7/4}} \|x^2 g\|_{L^2(\mathbb{R}^3)},$$

we have

$$\|III\|_{L^2} \leq C\epsilon^2 \int_{t_0}^{t} ds \left(\frac{1}{t_0^{1/2}} - \frac{1}{s^{1/2}}\right) \sup_{s' \in [t_0,t]} \|e^{i\Delta t'} W(s')\|^2_{\mathcal{B}_\infty}$$
$$\leq C\epsilon, \tag{16}$$

where C is a constant that is independent of ϵ and t. Estimates (14)-(16) and Lemmata 1 and 2 yield, for ϵ small enough and depending on ϕ_0,

$$\|u(t) - \overline{u}(t)\|_{L^2} \leq C(\|u(t_0) - \overline{u}(t_0)\|_{L^2} + \epsilon)$$

uniformly in $t \in [t_0, t_0 + \frac{1}{e\phi_0\epsilon}]$, where $C > 1$ is a constant that is independent of ϵ. □

We now reiterate Proposition 1 in order to prove the main result of this note.

Proof of Theorem 1. Let $n \in \mathbb{N}$ to be choosen below. It follows from applying the result of Proposition 1 n times that

$$\|u(t) - \overline{u}(t)\|_{L^\infty([t_0, t_0 + \frac{n}{e\phi_0\epsilon}]; L^2(\mathbb{R}^3))} \leq \sum_{j=1}^{n} C^j (\epsilon + \underbrace{\|u_2 - \overline{u}(2)\|_{L^2}}_{=0})$$
$$\leq C^{n+1}\epsilon. \tag{17}$$

Now, for $\delta \in (0,1)$, choose n such that $C^{n+1} < \epsilon^{-\delta}$, i.e., $n < \delta|\log\epsilon| - 1$. For this choice, it follows from (17) that

$$\|u(t) - \overline{u}(t)\|_{L^\infty([t_0, t_0 + \frac{\delta|\log\epsilon|}{e\phi_0\epsilon}]; L^2(\mathbb{R}^3))} \leq C\epsilon^{1-\delta}$$

which is the claim of the theorem. □

6. Appendix

For the convenience of the reader, we recall in this Appendix certain standard results, such as the Coifman-Meyer theorem and fractional integration, see [8], before proving estimate (12).

Theorem 2 (Coifman-Meyer Theorem)**.** *Consider the Fourier multiplier $m(k,\xi)$ and the associated operator T_m defined by*

$$\mathsf{T}_m(f,g) := \mathcal{F}^{-1}\int d\xi\ m(k,\xi)\widehat{f}(\xi)\widehat{g}(k-\xi)$$

where $\mathcal{F}$ is the inverse Fourier transform. Suppose that the Fourier multiplier m satisfies

$$|\partial_k^\alpha \partial_\xi^\beta m(k,\xi)| \le \frac{C}{(|k|+|\xi|)^{|\alpha|+|\beta|}}$$

for sufficiently many multi-indices (α,β). Then the operator

$$\mathsf{T}_m : L^p \times L^q \to L^r$$

is bounded for

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}, \quad 1<p,q\le\infty, \quad 0<r<\infty.$$

We also have the following standard result on fractional integration.

Lemma 3 (Fractional integration)**.** *Let $\Lambda^\alpha := (-\Delta)^{\frac{\alpha}{2}}$ and $\Lambda_t^\alpha := (\frac{1}{t}-\Delta)^{\frac{\alpha}{2}}$, $t>0$. Then the following holds.*

(i) *If $\alpha\ge 0$ and $1<p,q<\infty$, $\frac{1}{q}-\frac{1}{p}=\frac{\alpha}{3}$, then*

$$\|\Lambda^{-\alpha} f\|_{L^p} \le C\|f\|_{L^q}$$

for some constant $C>0$.

(ii) *If $\alpha\ge 0$ then*

$$\|\Lambda^{-\alpha} f\|_{L^\infty} \le C\|f\|_{L^{\frac{3}{\alpha},1}}$$

for some constant $C>0$, where $L^{p,q}$ is the standard Lorenz space.

(iii) *If $\alpha\ge 0$ and $1\le p,q\le\infty$, $0\le\frac{1}{q}-\frac{1}{p}<\frac{\alpha}{3}$, then*

$$\|\Lambda_t^{-\alpha} f\|_{L^p} \le C t^{\frac{\alpha}{2}+\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^q}$$

for some constant $C>0$.

6.1. **Proof of the estimate (12).** We start by estimating

$$\|e^{i\Delta t}F_{osc}(t,W(t))\|_{L^\infty([t_0,t_0+\frac{1}{e\phi_0\epsilon}];L^2)}.$$

Using an energy estimate together with Hölder's inequality, we have

$$\begin{aligned}
&\|e^{i\Delta t}F_{osc}(t,W(t))\|_{L^2} = \|F_{osc}(t,W(t))\|_{L^2}\\
&\leq \int_{t_0}^{t} ds \|(e^{i\Delta s}W(t))^2\|_{L^2}\\
&\leq \int_{t_0}^{t} ds \|e^{i\Delta s}W(t)\|_{L^\infty}\|e^{i\Delta s}W(t)\|_{L^2}\\
&\leq \|e^{i\Delta s}W(t)\|_{L^\infty([t_0,t];L^2)} \int_{t_0}^{t}\frac{ds}{s^{3/2}}\|s^{3/2}e^{i\Delta s}W(t)\|_{L^\infty}\\
&\leq C \sup_{t'\in[t_0,t_0+\frac{1}{e\phi_0\epsilon}]} \|e^{i\Delta t}W(t')\|^2_{\mathcal{B}_\infty}\\
&\leq C\phi_0^2, \qquad (18)
\end{aligned}$$

uniformly in $t\in[t_0,t_0+\frac{1}{e\phi_0\epsilon}]$ and $|\epsilon|<\epsilon_0$. Here we used Lemma 1 in the last inequality.

Estimating $\|t^{3/2}e^{i\Delta t}F_{osc}(t,W(t))\|_{L^\infty([t_0,t_0+\frac{1}{e\phi_0\epsilon}];L^\infty)}$ is some more work. It involves integration by parts and the application of the Coifman-Meyer theorem. In what follows, C denotes a positive constant that is independent of ϵ, and that might change from line to line. Let

$$\begin{aligned}
\theta &= k^2-\xi^2-(k^2-\xi)^2\\
P &= -\xi+\frac{1}{2}k\\
Z &= \theta + P\cdot\partial_\xi\theta.
\end{aligned}$$

Then

$$\frac{1}{\frac{1}{t}+iZ}\left(\frac{1+P\cdot\partial_\xi}{t}+\partial_t\right)e^{i\theta t} = e^{i\theta t}. \tag{19}$$

Integrating by parts in time, we have

$$\begin{aligned}
\widehat{F}_{osc}(t,W(t)) &= \int_{t_0}^{t} ds \int_{\xi\neq k} d\xi \frac{1}{\frac{1}{s}+iZ}\left(\frac{1+P\cdot\partial_\xi}{s}+\partial_s\right)e^{is\theta}\widehat{W}(t,\xi)\widehat{W}(t,k-\xi)\\
&= \widehat{g}(k)+\widehat{h}(k)
\end{aligned}$$

where

$$\widehat{g}(k) = \int_{t_0}^{t} \int_{\xi \neq k} \frac{\frac{1}{s} + P\partial_\xi \theta}{\frac{1}{s} + iZ} e^{is\theta} \widehat{W}(t,\xi)\widehat{W}(t,k-\xi)$$
$$- i \int_{t_0}^{t} \int_{\xi \neq k} s^{-2} \frac{1}{(\frac{1}{s} + iZ)^2} e^{is\theta} \widehat{W}(t,\xi)\widehat{W}(t,k-\xi)$$

and the boundary term

$$\widehat{h}(k) = \underbrace{\int_{\xi \neq k} \frac{1}{\frac{1}{t} + iZ} e^{it\theta} \widehat{W}(t,\xi)\widehat{W}(t,k-\xi)}_{h_1}$$
$$- \underbrace{\int_{\xi \neq k} \frac{1}{\frac{1}{t_0} + iZ} e^{it_0\theta} \widehat{W}(t,\xi)\widehat{W}(t,k-\xi)}_{h_2}.$$

We start by estimating the boundary terms. By the Coifman-Meyer theorem and fractional integration, we have

$$\|e^{it\Delta} h_1\|_{L^\infty} = \|\mathcal{F}^{-1} \frac{1}{\frac{1}{t} + \xi^2} \int_{\xi \neq k} \frac{\frac{1}{t} + \xi^2}{\frac{1}{t} + iZ} \widehat{e^{i\Delta t} W(t)}(\xi) \widehat{e^{i\Delta t} W(t)}(k-\xi)\|_{L^\infty}$$
$$= \|\Lambda_t^{-2} \mathsf{T}_{\frac{\frac{1}{t}+\xi^2}{\frac{1}{t}+iZ}} (e^{i\Delta t} W(t), e^{i\Delta t} W(t))\|_{L^\infty}$$
$$\leq C t^{3/4} \|\mathsf{T}_{\frac{\frac{1}{t}+\xi^2}{\frac{1}{t}+iZ}} (e^{i\Delta t} W(t), e^{i\Delta t} W(t))\|_{L^6}$$
$$\leq C t^{3/4} \|e^{i\Delta t} W(t)\|_{L^6} \|e^{i\Delta t} W(t)\|_{L^\infty}$$
(20)
$$\leq C t^{-7/4} \phi_0^2$$

where we have used Lemma 1 in the last inequality. Similarly,

$$\|e^{it\Delta} h_2\|_{L^\infty} \leq C t^{-7/4} \phi_0^2. \tag{21}$$

In order to estimate $\|e^{i\Delta t} g\|_{L^\infty}$, we use the following three estimates uniformly in $t \in [t_0, t_0 + \frac{1}{e\phi_0 \epsilon}]$, which we prove below.

$$\|\widehat{h}\|_{L^\infty} \leq C\phi_0^2, \tag{22}$$
$$\|\widehat{F}_{osc}(t, W(t))\|_{L^\infty} \leq C\phi_0^2, \tag{23}$$
$$\|x^2 g(t)\|_{L^2} \leq C t^\gamma \tag{24}$$

for arbitrary $\gamma > 0$. It follows from the stationary phase estimate that

$$
\begin{aligned}
&\|e^{i\Delta t}g\|_{L^\infty} \leq \frac{1}{t^{3/2}}\|\widehat{g}\|_{L^\infty} + \frac{1}{t^{7/4}}\|x^2 g\|_{L^2}\\
&\leq \frac{1}{t^{3/2}}(\|\widehat{F}_{osc}(t, W(t)))\|_{L^\infty} + \|\widehat{h}\|_{L^\infty}) + \frac{1}{t^{7/4}}\|x^2 g\|_{L^2}\\
&\leq C\frac{1}{t^{3/2}}\phi_0^2. \qquad (25)
\end{aligned}
$$

Estimate (12) follows from (18), (20), (21) and (25).

We now prove estimates (22) - (24).

• *Estimate (22).*

$$
\begin{aligned}
&\|\widehat{h}\|_{L^\infty} \leq 2\int \frac{1}{\xi^2}|\widehat{W}(t,\xi)||\widehat{W}(t,k-\xi)|\\
&\leq C\|\widehat{e^{i\Delta t}W(t)}\|^2_{L^2\cap L^\infty}\\
&\leq C\phi_0^2.
\end{aligned}
$$

• *Estimate (23).*

$$
\begin{aligned}
&|\widehat{F}_{osc}(t, W(t))(k)| = |\int_{t_0}^t dse^{isk^2/2}\int_{\xi\neq k} d\xi e^{-2is\xi^2}\widehat{W}(t,k/2-\xi)\widehat{W}(t,k/2+\xi)|\\
&\leq C\int_{t_0}^t \{\frac{1}{s^{3/2}}|\widehat{W}(t,k/2)|^2 + \frac{1}{s^{7/4}}\|\partial_\xi^2(\widehat{W}(t,k/2-\xi)\widehat{W}(t,k/2+\xi))\|_{L^2}\}\\
&\leq C\int_{t_0}^t \{\frac{1}{s^{3/2}}|\widehat{W}(t,k/2)|^2 + \frac{1}{s^{7/4}}(\|\partial_\xi^2\widehat{W}(t)\|_{L^2}\|\widehat{W}(t)\|_{L^\infty} + \|\partial_\xi\widehat{W}(t)\|^2_{L^4})\}\\
&\leq C\int_{t_0}^t \{\frac{1}{s^{3/2}}|\widehat{W}(t,k/2)|^2 + \frac{1}{s^{7/4}}\|\partial_\xi^2\widehat{W}(t)\|_{L^2}\|\widehat{W}(t)\|_{L^\infty}\}\\
&\leq C(\phi_0^2 + \phi_0^2\int_{t_0}^t ds\frac{s^{1/2}}{s^{7/4}}) \leq C\phi_0^2,
\end{aligned}
$$

where we have used the Gagliardo-Nirenberg inequality in the fourth line and Lemma 1 in the last line.

• *Estimate (24).* In what follows, P_n denotes a homogeneous polynomial in (k,ξ) of degree n. Applying ∂_k^2 to $\widehat{g}$ gives terms of the form

$$(a): \int_{t_0}^t \int_{\xi\neq k} s^{-j}\frac{P_{2k-2j-2}}{(\frac{1}{s}+iZ)^k} e^{is\theta}\widehat{W}(t,\xi)\widehat{W}(t,k-\xi)$$
$$k \geq 0, k-1 \geq j \geq 0.$$

$$(b): \int_{t_0}^t \int_{\xi\neq k} s^{-j}\frac{P_{2k-2j-1}}{(\frac{1}{s}+iZ)^k} e^{is\theta}\widehat{W}(t,\xi)\partial_k\widehat{W}(t,k-\xi)$$
$$k \geq 0, k-\frac{1}{2} \geq j \geq 0.$$

$$(c): \int_{t_0}^t \int_{\xi\neq k} s^{-j}\frac{P_{2k-2j}}{(\frac{1}{s}+iZ)^k} e^{is\theta}\widehat{W}(t,\xi)\partial_k^2\widehat{W}(t,k-\xi)$$
$$k \geq 0, k-1 \geq j \geq 0.$$

$$(d): \int_{t_0}^t \int_{\xi\neq k} s\frac{P_{2k}}{(\frac{1}{s}+iZ)^k} e^{is\theta}\widehat{W}(t,\xi)\widehat{W}(t,k-\xi)$$

$$(e): \int_{t_0}^t \int_{\xi\neq k} s\frac{P_{2k+1}}{(\frac{1}{s}+iZ)^k} e^{is\theta}\widehat{W}(t,\xi)\partial_k\widehat{W}(t,k-\xi)$$

$$(f): \int_{t_0}^t \int_{\xi\neq k} s^2\frac{\partial_k^2\theta P\cdot\partial_\xi\theta}{(\frac{1}{s}+iZ)} e^{is\theta}\widehat{W}(t,\xi)\widehat{W}(t,k-\xi).$$

We estimate (a) using the Coifman-Meyer theorem and fractional integration. We have

$$\begin{aligned}\|(a)\|_{L^2} &\leq \int_{t_0}^t dss^{-j}\|e^{-i\Delta s}\mathsf{T}_{\frac{P_{2k-2-2j}}{(\frac{1}{s}+iZ)^k}}(e^{is\Delta}W(t),e^{is\Delta}W(t))\|_{L^2}\\
&\leq C\int_{t_0}^t dss^{-j}\|\Lambda_s^{-2-2j}e^{i\Delta s}W(t)\|_{L^2}\|e^{i\Delta s}W(t)\|_{L^\infty}\\
&\leq C\int_{t_0}^t dss^{-j}s^{j+\frac{1}{2}}\|e^{i\Delta s}W(t)\|_{L^{6/5}}\|e^{is\Delta}W(t)\|_{L^\infty}.\end{aligned}$$

Now, Lemma 1 and the interpolation between different norms in $\mathcal{B}$ gives

$$\|e^{i\Delta s}W(t)\|_{L^{6/5}} \leq Cs^\gamma$$

for arbitrary $\gamma > 0$, and hence

$$\begin{aligned}\|(a)\|_{L^2} &\leq C\int_{t_0}^t s^{1/2}s^\gamma s^{-3/2}\|s^{3/2}e^{i\Delta s}W(t)\|_{L^\infty}\\
&\leq Ct^\gamma \qquad (26)\end{aligned}$$

where we have used Lemma 1 in the last inequality. Similarly, using the Coifman-Meyer theorem and fractional integration, we have

$$\|(b)\|_{L^2} \le \int_{t_0}^t s^{-j} \|e^{-is\Delta} \mathsf{T}_{\frac{P_{2k-1-2j}}{(\frac{1}{s}+iZ)^k}} (e^{is\Delta}W(t), e^{is\Delta}xW(t))\|_{L^2}$$

$$\le C \int_{t_0}^t s^{-j} \{ \|\Lambda_s^{-2j-1} e^{is\Delta} xW(t)\|_{L^2} \|e^{is\Delta}W(t)\|_{L^\infty} + \|e^{is\Delta}xW(t)\|_{L^2} \|\Lambda_s^{-2j-1} e^{is\Delta}W(t)\|_{L^\infty}\}$$

$$\le C \int_{t_0}^t ds s^{-j} s^{j+\frac{1}{2}} \|xW(t)\|_{L^2} \|e^{i\Delta s}W(t)\|_{L^\infty}$$

(27)

$$\le C \int_{t_0}^t ds s^{\frac{1}{2}-\frac{3}{2}} \log s \le Ct^\gamma$$

and

$$\|(c)\|_{L^2} \le \int_{t_0}^t s^{-j} \|e^{-is\Delta} \mathsf{T}_{\frac{P_{2k-2j}}{(\frac{1}{s}+iZ)^k}} (e^{is\Delta}W(t), e^{is\Delta}x^2W(t))\|_{L^2}$$

$$\le C \int_{t_0}^t s^{-j} \{ \|\Lambda_s^{-2j} e^{is\Delta} x^2W(t)\|_{L^2} \|e^{is\Delta}W(t)\|_{L^\infty} + \|e^{is\Delta}x^2W(t)\|_{L^2} \|\Lambda_s^{-2j} e^{is\Delta}W(t)\|_{L^\infty}\}$$

$$\le C \int_{t_0}^t ds s^{-j} s^{j} \|x^2W(t)\|_{L^2} \|e^{i\Delta s}W(t)\|_{L^\infty}$$

(28)

$$\le C \int_{t_0}^t s^{\frac{1}{2}} \frac{1}{s^{\frac{3}{2}}} \le C \log t.$$

Using the identity (19), we have

$$(d) = \int_{t_0}^t \int_{\xi \neq k} s \frac{P_{2k}}{(\frac{1}{s}+iZ)^{k+1}} (\frac{1}{s} + \partial_s + \frac{P}{s}\partial_\xi) e^{is\theta} \widehat{W}(t,\xi)\widehat{W}(t,k-\xi),$$

which, after integrating by parts in s and ξ, gives terms of the form $(a) - (c)$ plus boundary terms that are easily controlled. Hence, we also have

$$\|(d)\|_{L^2} \le Ct^\gamma. \tag{29}$$

Similarly,

$$(e) = \int_{t_0}^t \int_{\xi \neq k} s \frac{P_{2k+1}}{(\frac{1}{s}+iZ)^{k+1}} (\frac{1}{s} + \partial_s + \frac{P}{s}\partial_\xi) e^{is\theta} \widehat{W}(t,\xi)\partial_k\widehat{W}(t,k-\xi),$$

which, after integrating by parts in s and ξ, gives terms of the form $(a)-(d)$, *except* for the term

$$\int_{t_0}^{t}\int_{\xi\neq k}\frac{P_{2k+1}}{(\frac{1}{s}+iZ)^{k+1}}Pe^{is\theta}\partial_\xi\widehat{W}(t,\xi)\partial_k\widehat{W}(t,k-\xi). \tag{30}$$

We control (30) using the Coifman-Meyer theorem, fractional integration, and the Gagliardo-Nirenberg inequality

$$\|e^{is\Delta}(xW(t))\|_{L^4}\leq\|e^{i\Delta s}W(t)\|_{L^\infty}\|e^{i\Delta s}x^2W(t)\|_{L^2},$$

see estimate of (c). Therefore,

$$\|(e)\|_{L^2}\leq Ct^\gamma. \tag{31}$$

To estimate (f), we integrate by parts in ξ, which gives terms of the form (d) and (e), yielding

$$\|(f)\|_{L^2}\leq Ct^\gamma. \tag{32}$$

Estimate (24) follows from (26) - (32).

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