A NOTE ON THE DERIVATION OF THE TIME-DEPENDENT MAGNETIC HARTREE EQUATION

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Abstract. The mean-field dynamics of bosons in the presence of a (time-dependent) magnetic field is rigorously studied. It is shown that the magnetic Hartree equation describes the dynamics of an initially factorized bosonic state in the mean-field limit. Explicit estimates of the convergence rate are given. It is also remarked how to extend the analysis to arbitrary smooth Riemannian manifolds.

1. Introduction

1.1. **Heuristic discussion.** The magnetic Hartree equation appears naturally in several physical systems. For example, in three dimensions, the Maxwell-Schrödinger equation in the Coulomb gauge ($\nabla A = 0$) is given by

$$
\begin{aligned}
& i\partial_t \psi = -\Delta_A \psi + A_0 \psi \\
& \Delta \psi = -|\psi|^2 \\
& \Box A + \nabla(\partial_t A_0) = \Im \overline{\psi} \nabla_A \psi,
\end{aligned}
$$

where $\Delta_A = \nabla_A^2$, and $\nabla_A = (\nabla + iA)$. Eliminating (formally) A_0 using the second equation, one gets

$$
\begin{aligned}
& i\partial_t \psi = -\Delta_A \psi + w \star |\psi|^2 \psi \\
& \Box A = (1 - \nabla \Delta \nabla) \Im \overline{\psi} \nabla_A \psi,
\end{aligned}
$$

where $w(x) = \frac{1}{4\pi \|x\|}$. The first equation in this system is the magnetic Hartree equation. Another physical situation where the magnetic Hartree equation appears is the mean-field limit of bosons in the presence of magnetic potentials, which will be addressed in details in this letter.

There has been a lot of development in the study of the mean-field dynamics of many-body bosonic systems, in the *absence* of magnetic potentials. Early results using coherent states were given in [19] for bounded pair interaction, and extended in [15, 16] to more singular potentials. This approach is considerably extended in [21] to the case where the initial condition corresponds to a factorized state. The analysis of [21] has been simplified and generalized to the case of rotating bosons in [2]. Another approach based on the reduced density matrix was developed in [22] and was extended to more general potentials and to the derivation of the Gross-Pitaevskii equation (and quintic NLS) in [11], [6], [7], [12], [20], [10]. Furthermore, a new approach was developed in [13] that treats the quantum many-body dynamics as a deformation of the classical Hamiltonian dynamics, see also [14, 4, 5].

The main ingredients of our analysis is the coherent state approach that was introduced in [19] and used in [16, 21, 2]. We note that the analysis below is simpler than the one in [21], in particular, some terms in the difference between the true and the Hartree evolution are estimated using the stationary phase method rather than direct series expansions.

1.2. **The model and statement of the main result.** We consider a quantum system of N bosons ($N \in \mathbb{N}$) in the presence of a (generally time-dependent and unbounded) magnetic vector potential A. The Hilbert space of the system is $L^2_S(\mathbb{R}^{3N})$, the symmetrized (bosonic) space of L^2 functions. The dynamics of the system is generated by the (N-body) quantum Hamiltonian

$$H_N(t) = \sum_{j=1}^{N} -\Delta_{A_j(t)} + \frac{1}{N} \sum_{1\le i<j\le N} w(x_i - x_j), \tag{1}$$

where $x \in \mathbb{R}^3$ is a point is configuration space, and w is the (rescaled) pair interaction. The rescaling is such that the kinetic energy term and the pair interaction scale similarly in the large N limit. We assume that there exists a positive constant K such that pair interaction satisfies the operator inequality

$$|w(x)| \le K\sqrt{1-\Delta}, \tag{2}$$

where $\Delta = \nabla^2$. [1] We also assume that

$$A(t,x) = A_0(x) + A_1(t,x), \tag{3}$$

with $A_0(x) = \frac{1}{2}B_0 \wedge x$, the vector potential corresponding to a constant magnetic field B_0, and $A_1 \in C^1(\mathbb{R}, W^{1,\infty}(\mathbb{R}^3))$. Note that $A \in L^2_{loc}(\mathbb{R}^3)$, which, together with the above assumption w, implies the self-adjointness of H_N, with *fixed* dense domain[2]

$$\mathcal{D}_N = \{\psi \in H^2(\mathbb{R}^{3N}), \quad (1+\|x\|)^2\psi \in L^2(\mathbb{R}^{3N})\}.$$

In the Schrödinger picture, the dynamics of the N-body system is given by

$$i\partial_t \Psi_{N,t} = H_N(t)\Psi_{N,t}. \tag{4}$$

Given an initial condition $\Psi_{N,0}$ at time $t_0 = 0$, the solution of (4) at time t is

$$\Psi_{N,t} = S_N(t,0)\Psi_{N,0},$$

where the unitary propagator S_N satisfies

$$\begin{aligned} i\partial_t S_N(t,s) &= H_N(t) S_N(t,s) \\ S_N(s,s) &= 1. \end{aligned} \tag{5}$$

In what follows, we normalize the initial condition, $\|\Psi_{N,0}\|_{L^2} = 1$. The density matrix associated with $\Psi_{N,t}$ is

$$\Gamma_{N,t} = |\Psi_{N,t}\rangle\langle\Psi_{N,t}|,$$

[1] It follows from Kato's inequality that the Coulomb interaction $w(x) = \pm\frac{1}{\|x\|}$ satisfies this assumption in $\mathbb{R}^3$.

[2] See [3].

which is an orthogonal projection (in L^2) onto the state $\Psi_{N,t}$. It is a positive trace-class operator on $L^2(\mathbb{R}^{3N})$ with trace 1 and kernel

$$\Gamma_{N,t}(\mathbf{x}_N;\mathbf{x}'_N) = \Psi_{N,t}(\mathbf{x}_N)\overline{\Psi_{N,t}(\mathbf{x}'_N)}.$$

Here, $\mathbf{x}_m = (x_1, \cdots, x_m) \in \mathbb{R}^{3m}$, and $\bar{\cdot}$ denotes complex conjugation. The k^{th} *marginal density*, $\Gamma^{(k)}_{N,t}$, is defined by taking the partial trace of $\Gamma_{N,t}$ with respect to the last $N-k$ entries. Its kernel is given by

$$\Gamma^{(k)}_{N,t}(\mathbf{x}_k;\mathbf{x}'_k) := \int d\mathbf{x}_{N-k}\Gamma_{N,t}(\mathbf{x}_k,\mathbf{x}_{N-k};\mathbf{x}'_k,\mathbf{x}_{N-k}), \quad k = 1, \cdots, N.$$

Note that $\Gamma^{(k)}_{N,t}$ is a positive trace-class operator in $L^2(\mathbb{R}^{3k})$ with trace 1. In the mean-field limit, we expect that for an initial condition

$$\Psi_{N,0}(\mathbf{x}_N) = \prod_{j=1}^{N} \phi(x_j), \quad \|\phi\|_{L^2} = 1,$$

the marginal density matrix

$$\Gamma^{(k)}_{N,t} \to |\phi_t\rangle\langle\phi_t|^{\otimes k}$$

as $N \to \infty$, in trace-norm topology, where ϕ_t satisfies the magnetic Hartree equation

$$i\partial_t\phi_t = -\Delta_{A(t)}\phi_t + w \star |\phi_t|^2\phi_t, \quad \phi_{t=0} = \phi. \tag{6}$$

We introduce the Banach space

$$\Sigma = \{u \in H^1(\mathbb{R}^3),\ (1 + \|x\|)u \in L^2(\mathbb{R}^3)\}$$

with norm

$$\|u\|_\Sigma = \|u\|_{H^1} + \|u\|_{L^2}.$$

We show in Subsect. 2.1 that (6) is locally well-posed in Σ. Proving global well-posedness is a little bit more delicate, since the usual energy conservation argument fails if A is time-dependent. We discuss different situations where global well-posedness holds in Subsect. 2.1.

The following is the main result of this letter.

Theorem 1. *Consider $\Psi_{N,t}$ the solution of the N-body Schrödinger equation (4) with initial condition*

$$\Psi_{N,0}(\mathbf{x}_N) = \prod_{j=1}^{N} \phi(x_j), \quad \phi \in \Sigma, \quad \|\phi\|_{L^2} = 1.$$

Suppose that the pair interaction satisfies (2) and that the magnetic potential satisfies (3). Then there exists a maximal time $T \le \infty$, and positive constants C and α such that

$$Tr\left|\Gamma^{(1)}_{N,t} - |\phi_t\rangle\langle\phi_t|\right| \le C\frac{e^{\alpha t}}{\sqrt{N}}, \quad t \in [0,T),$$

where the constants C and α depend only on K and $\sup_{s\in[0,t]}\|\phi_t\|_\Sigma$. Here, ϕ_t satisfies the magnetic Hartree equation (6) with initial condition $\phi_{t=0} = \phi$, and T is the maximal time for which (6) is well-posed in Σ.

Remark 1. *The dependence of the constants C and α on $\|\phi_t\|_\Sigma$ can be replaced by $\|\phi\|_\Sigma$ if $T=\infty$, i.e., if (6) is globally well-posed.*

Remark 2. *An analysis similar to the proof of Theorem 1 can be used to prove that*

$$Tr(\left|\Gamma_{N,t}^{(m)} - |\phi_t\rangle\langle\phi_t|^{\otimes m}\right|) \leq C_m \frac{e^{\alpha_m t}}{\sqrt{N}}, \quad m<\infty.$$

This implies that for a bounded operator $A^{(m)}$ on $L^2(\mathbb{R}^{3m})$,

$$\langle\Psi_{N,t}, (A^{(m)}\otimes 1^{N-m})\Psi_{N,t}\rangle \to \langle\phi_t^{\otimes m}, A^{(m)}\phi_t^{\otimes m}\rangle$$

as $N\to\infty$.

Remark 3. *The main result can be extended to the case where there is a random external potential or random pair interaction, see* [1].

Remark 4. *The analysis below applies "in toto" to the mean-field dynamics of bosons on a smooth Riemannian manifold $(\mathcal{M},g)$ whose Laplace-Beltrami operator Δ_{LB} is bounded from below. Using the GNS construction (see* [8]*), the Fock space representation holds in $\mathcal{M}$ (as opposed to $\mathbb{R}^3$), and the result of Theorem 1 holds with Δ_A replaced by Δ_{LB} (with suitable boundary conditions, if applicable). This result is relevant to the mean-field dynamics of bosons in strongly anisotropic confining traps, where the effective dynamics of the N-body system is restricted to a submanifold of $\mathbb{R}^3$. Another physical setting where the effective dynamics is on manifold rather than $\mathbb{R}^3$ is dynamics in the presence of a periodic background potential, say that of a crystal. In this case, $\mathcal{M}=\mathbb{T}^n$, the n-dimensional torus.*

The organization of this letter is as follows. In Sect. 2, we discuss the well-posedness of the magnetic Hartree equation, and recall basic results about the Fock space representation and coherent states. In Sect. 3, we prove Theorem 1.

2. Mathematical preliminary

2.1. **Well-posedness of the magnetic Hartree equation.** To set the stage, we discuss the well-posedness of (6). We have the following lemma.

Lemma 1. *Consider the magnetic Hartree equation (6). Suppose that (2) and (3) hold. Then there exists $T>0$ such that $\phi_t\in C([0,T);\Sigma)\cap C^1([o,T);\Sigma^*)$, where Σ^* is the dual of Σ (in the L^2 sense). Moreover,*

$$\|\phi_t\|_{L^2} = \|\phi\|_{L^2},$$

and $T=\infty$, or $T<\infty$ with $\lim_{t\nearrow T}\|\phi_t\|_\Sigma=\infty$.

Proof. Note first that the Hartree nonlinearity is locally Lipschitz in Σ,

$$\|w\star|u|^2u - w\star|v|^2v\|_\Sigma \lesssim (\|u\|_\Sigma^2+\|v\|_\Sigma^2)\|u-v\|_\Sigma. \tag{7}$$

Furthermore, the one particle Hamiltonian $-\Delta_{A(t)}$ gives rise to a self-adjoint operator on L^2 with fixed dense domain. Its extension to Σ generates a C^0-group of isometries $\{S(t)\}_{t\geq 0}$ on Σ. The claim of the lemma follows from a standard argument for evolution equations with locally Lipschitz nonlinearities (contraction argument for short time), see [9]. ∎

Proving global well-posedness is a little bit more delicate, since the energy functional

$$\mathcal{E}(t,\phi) := \frac{1}{2}\langle \nabla_{A(t)}\phi, \nabla_{A(t)}\phi\rangle + \frac{1}{4}\langle \phi, w\star|\phi|^2\phi\rangle \tag{8}$$

is generally *not* conserved. In fact, using (6) (and a standard regularization procedure), we have

$$\mathcal{E}(t,\phi_t) = \mathcal{E}(t_0,\phi_0) + \Im\int_{t_0}^{t} ds' \langle \partial_{s'}A(s')\phi_{s'}, \nabla_{A(s')}\phi_{s'}\rangle, \quad t\in[0,T). \tag{9}$$

Global well-posedness in the *autonomous* case is straight forward. Assumption (2) implies

$$\sup_{x\in\mathbb{R}^3}\int dy\, w(x-y)|\phi(y)|^2 \le \epsilon\|\nabla\phi\|_{L^2}^2 + \epsilon^{-1}\|\phi\|_{L^2}^2, \quad \forall\epsilon, \tag{10}$$

which implies in the autonomous case that

$$\begin{aligned} \mathcal{E}(t,\phi_t) &\lesssim \|\phi_t\|_{\Sigma}^2(1+\|\phi\|_{L^2}^2) \\ \|\phi_t\|_{\Sigma}^2 &\lesssim \mathcal{E}(t,\phi_t) + \|\phi\|_{L^2}^2 + \|\phi\|_{L^2}^4. \end{aligned} \tag{11}$$

Together with the blow-up alternative, this implies that $T=\infty$, irrespective of the sign of the pair interaction w.

When A is *time-dependent*, one can also show global well-posedness if the nonlinearity is *defocusing*, i.e., w is negative. It follows from (9) and (11) that in the defocusing case

$$\begin{aligned} \|\nabla_{A(t)}\phi_t\|_{L^2}^2 &\le C_1 + C_2\Big|\int_{t_0}^{t} ds\ \langle \partial_s A(s)\phi_s, \nabla_{A(s)}\phi_s\rangle\Big| \\ &\le C_1 + C_3\int_{t_0}^{t} ds\ \|\nabla_{A(s)}\phi_s\|_{L^2}^2. \end{aligned}$$

By the Gronwall lemma, $\|\nabla_{A(t)}\phi_t\|_{L^2}$ grows at most exponentially fast in time, which, together with the blow-up alternative, implies global well-posedness.

2.2. **Bosonic Fock space and second quantization.** We recall in this subsection standard results on the Fock space representation of bosonic systems, see [8] for a detailed discussion and for proofs. The bosonic Fock space over $L^2(\mathbb{R}^3)$ is defined as the Hilbert space

$$\mathcal{F} = \bigoplus_{n\ge 0} L^2(\mathbb{R}^3)^{\otimes_s n} = \mathbb{C}\oplus\bigoplus_{n\ge1} L_s^2(\mathbb{R}^{3n}).$$

For $\psi\in\mathcal{F}$, $\psi = \{\psi^{(n)}\}_{n\ge0}$, which is a sequence of n-particle wave functions $\psi^{(n)}\in L_s^2(\mathbb{R}^{3n})$. The scalar product on $\mathcal{F}$ is given by

$$\begin{aligned} \langle\psi_1,\psi_2\rangle &= \sum_{n\ge0}\langle\psi_1^{(n)},\psi_2^{(n)}\rangle_{L^2(\mathbb{R}^{3n})} \\ &= \overline{\psi_1^{(0)}}\psi_2^{(0)} + \sum_{n\ge1}\int dx_1\dots dx_n\,\overline{\psi_1^{(n)}}(x_1,\dots,x_n)\psi_2^{(n)}(x_1,\dots,x_n). \end{aligned}$$

The vacuum vector in $\mathcal{F}$ is given by $\Omega = \{1, 0, 0, \dots\}$, while the N-particle wavefunction ψ_N is given by $\{\psi^{(n)}\}_{n\geq 0}$ where $\psi^{(N)} = \psi_N$ and $\psi^{(n)} = 0$ for all $n \neq N$.

On $\mathcal{F}$, we introduce the creation operator $a^*(f)$ and the annihilation operator $a(f)$, which are defined through their action

$$\left(a^*(f)\psi\right)^{(n)}(x_1, \dots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \psi^{(n-1)}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$$

$$\left(a(f)\psi\right)^{(n)}(x_1, \dots, x_n) = \sqrt{n+1} \int dx\, \overline{f}(x)\, \psi^{(n+1)}(x, x_1, \dots, x_n)\,.$$

Note that $a^*(f)$ and $a(f)$ are unbounded, densely defined, closed operators, and they satisfy the canonical commutation relations (CCR)

$$[a(f), a^*(g)] = \langle f, g\rangle_{L^2(\mathbb{R}^3)}, \qquad [a^{\#}(f), a^{\#}(g)] = 0, \tag{12}$$

where $a^{\#}$ is either a or a^*. Furthermore,

$$\begin{aligned} &\|a(f)\psi\| \leq \|f\|_{L^2}\, \|\widehat{N}^{1/2}\psi\| \\ &\|a^*(f)\psi\| \leq \|f\|_{L^2}\, \| \left(\widehat{N}+1\right)^{1/2} \psi\|. \end{aligned} \tag{13}$$

The operator valued distributions a^*_x and a_x $(x \in \mathbb{R}^3)$ are defined by

$$a^*(f) = \int dx\;\, f(x)\, a^*_x, \quad a(f) = \int dx\, \overline{f}(x)\, a_x,$$

for every $f \in L^2(\mathbb{R}^3)$. The CCR in this case are

$$[a_x, a^*_y] = \delta(x-y), \qquad [a^{\#}_x, a^{\#}_y] = 0.$$

We define the particle number operator $\widehat{N}$ by

$$(\widehat{N}\psi)^{(n)} = n\psi^{(n)}.$$

Expressed through the distributions a_x and a^*_x, it is given by

$$\widehat{N} = \int dx\, a^*_x a_x.$$

For $\psi \in \mathcal{F}$, the one-particle density $\widehat{\Gamma}^{(1)}_{\psi}$ associated with ψ is the positive trace-class operator on $L^2(\mathbb{R}^3)$ with kernel

$$\widehat{\Gamma}^{(1)}_{\psi}(x;y) = \frac{1}{\langle \psi, \widehat{N}\psi\rangle}\, \langle \psi, a^*_y a_x \psi\rangle \tag{14}$$

and trace 1.

We introduce the *second quantized* Hamiltonian $\widehat{H}_N$ acting on $\mathcal{F}$, which is given by

$$(\widehat{H}_N\psi)^{(n)} = \widehat{H}^{(n)}_N \psi^{(n)},$$

$$\widehat{H}^{(n)}_N(t) = -\sum_{j=1}^{n} \Delta_{A_j(t)} + \frac{1}{N}\sum_{i<j}^{n} w(x_i - x_j).$$

Equivalently,

$$\widehat{H}_N(t) = \int dx\; \nabla_{A(t)} a_x^* \nabla_{A(t)} a_x + \frac{1}{2N}\int dxdy\, w(x-y) a_x^* a_y^* a_y a_x.$$

Note that when restricted to the N-particle sector, $\widehat{H}_N(t)$ coincides with the Hamiltonian $H_N(t)$ defined in (1). Furthermore, $[\widehat{H}_N(t), \widehat{N}] = 0$. In what follows, we denote the unitary propagator generated by $\widehat{H}_N(t)$ by $\widehat{S}_N$, which satisfies

$$i\partial_t \widehat{S}_N(t,s) = \widehat{H}_N(t)\widehat{S}_N(t,s), \quad \widehat{S}_N(s,s) = 1.$$

We now introduce the notion of *coherent states*, which will be useful in the subsequent analysis. For $f \in L^2(\mathbb{R}^3)$, we define the Weyl-operator

$$W(f) := \exp\left(a^*(f) - a(f)\right).$$

The coherent state $\psi(f) \in \mathcal{F}$ with one-particle wave function f is given by

$$\psi(f) := W(f)\Omega.$$

We refer the reader to [8] for further discussion about the Fock space representation and coherent states.

3. Proof of the main result

We first represent the initially factorized state in terms of coherent states. This representation already appeared in [21]. Since

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{2\pi} e^{iN\theta} W(e^{-i\theta}\sqrt{N}\phi)\Omega &= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{iN\theta} \sum_{n\geq 0} e^{-N\|\phi\|_{L^2}^2/2} \frac{e^{-in\theta} N^{n/2} (a^*(\phi))^n}{n!}\Omega \\
&= \sum_{n\geq 0} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-N/2} e^{iN\theta} \frac{e^{-in\theta} N^{n/2}}{\sqrt{n!}} \phi^{\otimes n} \\
&= \frac{N^{N/2} e^{-N/2}}{\sqrt{N!}} \phi^{\otimes N},
\end{aligned}$$

we have

$$\phi^{\otimes N} = a_N \int_0^{2\pi} \frac{d\theta}{2\pi} e^{iN\theta} W(e^{-i\theta}\sqrt{N}\phi)\Omega \tag{15}$$

with $a_N = \frac{e^{N/2}\sqrt{N!}}{N^{N/2}}$. Using Stirling's formula, $a_N \sim N^{1/4}$ for $N \gg 1$.

We now use (15) in (14). We have

$$\begin{aligned}
\widehat{\Gamma}^{(1)}_{N,t}(x;y) &= \frac{1}{\langle \psi, \widehat{N}\psi\rangle} \langle \widehat{\Psi}_{N,t}, a_y^* a_x \widehat{\Psi}_{N,t}\rangle \\
&= \frac{1}{N} \langle \phi^{\otimes N}, a_y^*(t) a_x(t) \phi^{\otimes N}\rangle \\
&= \frac{a_N^2}{N} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} e^{iN(\theta_2-\theta_1)} \langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega, a_y^*(t) a_x(t) W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle,
\end{aligned}$$

where we have used the fact that the time evolution commutes with $\widehat{N}$ in the denominator. Here, $a^{\#}(t) = \widehat{S}_N^*(t,0) a^{\#} \widehat{S}_N(t,0)$, the time evolved operator in the Heisenberg picture. Adding and subtracting $\sqrt{N}\phi_t$ to the creation and annihilation operators (modulo a phase factor), we have

(16)
$$\begin{aligned}\widehat{\Gamma}^{(1)}_{N,t}(x;y) =& \frac{a_N^2}{N}\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}e^{iN(\theta_2-\theta_1)}\langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega, (a^*_y(t)-e^{i\theta_1}\sqrt{N}\overline{\phi}_t(x))\times\\ &\times (a_x(t)-e^{-i\theta_2}\sqrt{N}\phi_t(y))W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle\\ &+\overline{\phi}_t(y)\phi_t(x)+\frac{\overline{\phi}_t(y)f_N(x)}{\sqrt{N}}+\frac{\overline{f}_N(y)\phi_t(x)}{\sqrt{N}},\end{aligned}$$

where

(17)
$$\begin{aligned}f_N(x) =& a_N^2\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}e^{iN\theta_2}e^{-i(N-1)\theta_1}\langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega, (a_x(t)-e^{-i\theta_2}\sqrt{N}\phi_t(x))\times\\ &\times W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle.\end{aligned}$$

We introduce the unitary propagator

$$\widehat{U}_N(t,s) := W^*(\sqrt{N}\phi_t)\widehat{S}_N(t,s)W(\sqrt{N}\phi_s),$$

which first appeared (in a different setting) in [19]. It describes the dynamics of the fluctuations away from the Hartree evolution. Using

$$\begin{aligned}W^*(f)a_xW(f) &= a_x+f(x),\\ W^*(f)a^*_xW(f) &= a^*_x+\overline{f}(x),\end{aligned}$$

one can verify that $\widehat{U}_N$ satisfies the initial value problem

$$\begin{aligned}&i\partial_t\widehat{U}_N(t,s) = \widehat{L}_N(t)\widehat{U}_N(t,s),\\ &\widehat{U}_N(s,s) = 1,\end{aligned} \tag{18}$$

with

(19)
$$\begin{aligned}\widehat{L}_N(t) =& \int dx\ \nabla_{A(t)}a^*_x\nabla_{A(t)}a_x+\int dx\ w\star|\phi_t|^2(x)a^*_xa_x+\int dx\ dy\ w(x-y)\overline{\phi}_t(y)\phi_t(x)a^*_xa_y\\ &+\frac{1}{2}\int dx\ dy\ w(x-y)\phi_t(y)\phi_t(x)a^*_ya^*_x+\frac{1}{2}\int dx\ dy\ w(x-y)\overline{\phi}_t(y)\overline{\phi}_t(x)a_ya_x\\ &+\frac{1}{\sqrt{N}}\int dx\ dy\ w(x-y)a^*_x\phi_t(y)a^*_ya_x+\frac{1}{\sqrt{N}}\int dx\ dy\ w(x-y)a^*_x\overline{\phi}_t(y)a_ya_x\\ &+\frac{1}{2N}\int dx\ dy\ w(x-y)a^*_xa^*_ya_ya_x.\end{aligned}$$

Here, ϕ_t satisfies (6) with initial condition ϕ. Note that

$$W^*(\sqrt{N}\phi_t)\widehat{S}_N(0,t)(a_x-\sqrt{N}\phi_t)\widehat{S}_N(t,0)W(\sqrt{N}\phi) = \widehat{U}_N(0,t)a_x\widehat{U}_N(t,0). \tag{20}$$

We will need the following estimate.

Lemma 2. *There exist positive constants C and α that depend on $\sup_{s\in[0,t]}\|\phi_s\|_\Sigma$ and K only, such that*

$$\|\widehat{N}^{1/2}\widehat{U}_N(t,0)\Omega\|^2 \le Ce^{\alpha t},$$

uniformly in $N \geq 1$.

Proof. Step 1. Auxiliary dynamics.

We introduce the parity preserving auxiliary dynamics

$$i\partial_t \tilde{U}_N(t,s) = \tilde{L}_N(t)\tilde{U}_N(t,s), \quad \tilde{U}_N(s,s) = 1, \tag{21}$$

with

$$\begin{aligned}\tilde{L}_N(t) = &\int dx\ \nabla_{A(t)} a_x^* \nabla_{A(t)} a_x + \int dx\ w \star |\phi_t|^2(x) a_x^* a_x + \int dx\ dy\ w(x-y)\overline{\phi}_t(y)\phi_t(x) a_x^* a_y \\ &+ \frac{1}{2}\int dx\ dy\ w(x-y)\phi_t(y)\phi_t(x) a_y^* a_x^* + \frac{1}{2}\int dx\ dy\ w(x-y)\overline{\phi}_t(y)\overline{\phi}_t(x) a_y a_x \\ &+ \frac{1}{2N}\int dx\ dy\ w(x-y) a_x^* a_y^* a_y a_x.\end{aligned}$$

We claim that the following estimates hold,

$$\langle \Omega, \tilde{U}_N(0,t)\widehat{N}\tilde{U}_N(t,0)\Omega\rangle \lesssim e^{\alpha t} \tag{22}$$

$$\langle \Omega, \tilde{U}_N(0,t)\widehat{N}^3\tilde{U}_N(t,0)\Omega\rangle \lesssim e^{\alpha t} \tag{23}$$

uniformly in $N \geq 1$. To see this,

$$I_N^{(m)} := \langle \Omega, \tilde{U}_N(0,t)\widehat{N}^m\tilde{U}_N(t,0)\Omega\rangle, \quad m = 1,3.$$

It follows from (21) that

$$\begin{aligned}\frac{dI_N^{(1)}}{dt} &= \langle \Omega, \tilde{U}_N(0,t) i[\tilde{L}_N(t), \widehat{N}]\tilde{U}_N(t,0)\Omega\rangle \\ &= i\Re \int dx\ dy\ w(x-y)\phi_t(x)\phi_t(y)\langle \Omega, \tilde{U}_N(0,t)[a_x^* a_y^*, \widehat{N}]\tilde{U}_N(t,0)\Omega\rangle.\end{aligned} \tag{24}$$

Using the push-through relations

$$a_x\widehat{N} = (\widehat{N}+1)a_x, \quad a_x^*\widehat{N} = (\widehat{N}-1)a_x \tag{25}$$

we have

$$\begin{aligned}&|\int dx\ dy\ w(x-y)\phi_t(x)\phi_t(y)\langle \Omega, \tilde{U}_N(0,t)[a_x^* a_y^*, \widehat{N}]\tilde{U}_N(t,0)\Omega\rangle| \\ &\leq 2\sup_x \|w(x-\cdot)\phi_t\|_{L^2}\|\phi_t\|_{L^2}\|(\widehat{N}+1)^{1/2}\tilde{U}_N(t)\Omega\|^2 \\ &\leq 2K\|\phi_t\|_\Sigma \|\phi_t\|_{L^2} I_N^{(1)}\end{aligned} \tag{26}$$

for some constant C'.Similarly, using (25), we have

$$\frac{dI_N^{(3)}}{dt} \leq CK\|\phi_t\|_\Sigma I_N^{(3)}. \tag{27}$$

Estimates (22) and (23) follow from Gronwall lemma and (26) and (27), respectively.

Step 2. Comparison between auxiliary and true dynamics.

We need to estimate $\|(\tilde{U}_N(t,0) - \widehat{U}_N(t,0))\Omega\|$. We have

$$
\begin{aligned}
&|\langle \Omega, (1 - \tilde{U}_N(0,t)\widehat{U}_N(t,0))\Omega\rangle| = |\int_0^t ds\ \langle \Omega, \partial_s(\tilde{U}_N^*(s,0)\widehat{U}_N(s,0))\Omega\rangle| \\
&= |\int_0^t ds\ \langle \Omega, \tilde{U}_N^*(s,0)(\widehat{L}_N(s) - \tilde{L}_N(s))\widehat{U}_N(s,0))\Omega\rangle| \\
&= \frac{1}{\sqrt{N}}|\Im \int_0^t ds\ \int dx\ dy\ \langle \Omega, \tilde{U}_N^*(s,0)(w(x-y)a_x a_y^* a_x \phi_s(y))\widehat{U}_N(s,0))\Omega\rangle| \\
&\leq \frac{C}{\sqrt{N}} \sup_{s\in[o,t]} \|\phi_s\|_\Sigma \int_0^t \|(\widehat{N}+3)^{3/2}\tilde{U}_N(s)\Omega\|
\end{aligned}
$$

Together with (23), this implies that

$$
|\langle \Omega, (1 - \tilde{U}_N(0,t)\widehat{U}_N(t,0))\Omega\rangle| \leq \frac{C}{\sqrt{N}} e^{\alpha t}. \tag{28}
$$

It follows from (28) and the unitarity of $\tilde{U}_N$ that

$$
\|(\tilde{U}_N(t,0) - \widehat{U}_N(t,0))\Omega\| \leq \frac{C}{\sqrt{N}} e^{\alpha t} \tag{29}
$$

where C and α are positive constants that depend only on $\sup_{s\in[0,t]} \|\phi\|_\Sigma$ and K. The claim of the lemma follows directly from (22) and (29). $\square$

Proof of Theorem 1. We want to estimate the Hilbert-Schmidt norm of the difference $\widehat{\Gamma}^{(1)}_{N,t}(x;y) - \overline{\phi}_t(y)\phi_t(x)$. It follows from (20) that

$$
\begin{aligned}
&|\frac{a_N^2}{N}\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi} e^{iN(\theta_2-\theta_1)}\langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega, (a_y^*(t) - e^{i\theta_1}\sqrt{N}\overline{\phi}_t(y))\times \\
&\times (a_x(t) - e^{-i\theta_2}\sqrt{N}\phi_t(x))W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle| \\
&\leq \frac{a_N^2}{N}\|a_y^*\widehat{U}_N^{\theta_1}(t,0)\Omega\|\|a_x\widehat{U}_N^{\theta_2}(t,0)\Omega\|
\end{aligned}
$$

where $\widehat{U}_N^\theta$ corresponds to $\widehat{U}_N$ with ϕ replaced by $e^{-i\theta}\phi$. Together with Lemma 2, this implies that

$$
\begin{aligned}
&\frac{a_N^4}{N^2}\int dxdy\ |\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi} e^{iN(\theta_2-\theta_1)}\langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega, (a_y^*(t) - e^{i\theta_1}\sqrt{N}\overline{\phi}_t(y))\times \\
&\times (a_x(t) - e^{-i\theta_2}\sqrt{N}\phi_t(x))W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle|^2 \\
&\leq \frac{Ca_N^4}{N^2}\|\widehat{N}^{1/2}\widehat{U}_N^{(\theta_1)}(t)\Omega\|^2\|\widehat{N}^{1/2}\widehat{U}_N^{(\theta_2)}(t)\Omega\|^2 \\
&\leq \frac{C}{N}e^{\alpha t}.
\end{aligned} \tag{30}
$$

Estimating f_N, which appears in the third and fourth terms of (16), is somewhat more delicate, but the approach presented here is much simpler than the one in [21]. We make use of the fast oscillating phase of the integrand in (17). Integrating by parts in θ_1, we have

$$
\begin{aligned}
f_N(x) =& a_N^2 \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} e^{iN\theta_2} [\frac{1}{i(N-1)} \frac{d}{d\theta_1} e^{-i\theta_1 (N-1)}] \times \\
& \times \langle W(e^{-i\theta_1}\sqrt{N}\phi)\Omega, (a_x(t) - e^{-i\theta_2}\sqrt{N}\phi_t(x)) W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle \\
& = \frac{a_N^2}{N-1} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} e^{iN\theta_2} e^{-i(N-1)\theta_1} \langle (a^*(e^{-i\theta_1}\sqrt{N}\phi) - a(e^{-i\theta_1}\sqrt{N}\phi) + \sqrt{N}) \times \\
& \times W(e^{-i\theta_1}\sqrt{N}\phi)\Omega, (a_x(t) - e^{-i\theta_2}\sqrt{N}\phi_t(x)) W(e^{-i\theta_2}\sqrt{N}\phi)\Omega\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|f_N\|_{L^2} &\leq \frac{4a_N^2}{N-1} \|(\widehat{N}^{1/2} + \sqrt{N}) W(e^{-i\theta_1}\sqrt{N}\phi)\Omega\| \|\widehat{N}^{1/2} \widehat{U}_N^{\theta_2}(t)\Omega\| \\
&\leq \frac{Ca_N^2}{\sqrt{N}} \|\widehat{N}^{1/2} \widehat{U}_N^{\theta_2}(t)\Omega\|,
\end{aligned}
$$

which, together with Lemma 2, yields the estimate

$$
\|f_N\|_{L^2} \leq Ce^{\alpha t}. \tag{31}
$$

Eq. (16) together with estimates (30) and (31) imply that the Hilbert-Schmidt norm of the difference

$$
\|\widehat{\Gamma}^{(1)}_{N,t} - |\phi_t\rangle\langle\phi_t|\|_{HS} \leq \frac{C}{\sqrt{N}} e^{\alpha t},
$$

which gives the claim of the theorem. □

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