

Invitation to Alexandrov geometry:  
CAT[0] spaces

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# Preface

These notes arise as a coproduct of the book on Alexandrov geometry we have been writing for a number of years. They were shaped in a number of lectures given by Anton Petrunin to undergraduate students at different occasions at the MASS program at Penn State University and the Summer School “Algebra and Geometry” in Yaroslavl.

The idea is to demonstrate the beauty and power of Alexandrov geometry by reaching interesting applications with a minimum of preparation.

Here is a list of some sources providing a good introduction to Alexandrov spaces with curvature bounded above, which we recommend for further information:

- ◊ The book by Martin Bridson and André Haefliger [15];
- ◊ Lecture notes of Werner Ballmann [10];
- ◊ Chapter 9 in the book [17] by Dmitri Burago, Yurii Burago and Sergei Ivanov.

## Early history of Alexandrov geometry

Alexandrov geometry can use “back to Euclid” as a slogan. Alexandrov spaces are defined via axioms similar to those given by Euclid, but certain equalities are changed to inequalities. Depending on the sign of the inequalities we get Alexandrov spaces with *curvature bounded above* or *curvature bounded below*; for more on this see the manifesto below. Although the definitions of the two classes of spaces are similar, their properties and known applications are quite different.

The first synthetic description of curvature is due to Abraham Wald; it was given in a single publication on “coordinate-free differential geometry” published in 1936, see [55]. In 1941, similar definitions were rediscovered independently by Aleksandr Danilovich Aleksandrov, see [5]. In Alexandrov’s work the first fruitful applications of this approach were given. Mainly:

- ◇ Alexandrov's embedding theorem — *metrics of non-negative curvature on the sphere, and only they, are isometric to closed convex surfaces in Euclidean 3-space.*
- ◇ Gluing theorem, which tells when the sphere obtained by gluing of two discs along their boundaries has non-negative curvature in the sense of Alexandrov.

These two results together gave a very intuitive geometric tool for studying embeddings and bending of surfaces in Euclidean space, and changed this subject dramatically. They formed the foundation of the branch of geometry now called *Alexandrov geometry*.

The study of spaces with curvature bounded above started later. The first paper on the subject was written by Alexandrov; it appeared in 1951, see [6]. An analogous weaker definition was considered earlier by Herbert Busemann in [20].

Fundamental results in this direction were obtained by Yuri Grigorievich Reshetnyak. They include his majorization theorem and gluing theorem. The gluing theorem states that if two non-positively curved spaces have isometric convex sets, then the space obtained by gluing these sets along an isometry is also non-positively curved.

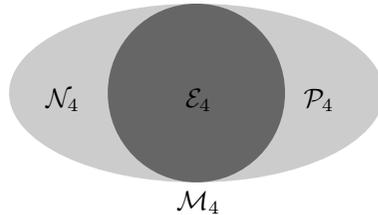
## Manifesto of Alexandrov geometry

Consider the space  $\mathcal{M}_4$  of all isometry classes of 4-point metric spaces. Each element in  $\mathcal{M}_4$  can be described by 6 numbers — the distances between all 6 pairs of its points, say  $l_{i,j}$  for  $1 \leq i < j \leq 4$  modulo permutations of  $(1, 2, 3, 4)$ . These 6 numbers are subject to 12 triangle inequalities; that is,

$$l_{i,j} + l_{j,k} \geq l_{i,k}$$

holds for all  $i, j$  and  $k$ , where we assume that  $l_{j,i} = l_{i,j}$  and  $l_{i,i} = 0$ .

Consider the subset  $\mathcal{E}_4 \subset \mathcal{M}_4$  of all isometry classes of 4-point metric spaces which admit isometric embeddings into Euclidean space. The complement  $\mathcal{M}_4 \setminus \mathcal{E}_4$  has two connected components.



**0.0.1. Exercise.** *Prove the latter statement.*

One of the components will be denoted by  $\mathcal{P}_4$  and the other by  $\mathcal{N}_4$ . Here  $\mathcal{P}$  and  $\mathcal{N}$  stand for *positive* and *negative curvature* because

spheres have no quadruples of type  $\mathcal{N}_4$  and hyperbolic space has no quadruples of type  $\mathcal{P}_4$ .

A metric space, with length metric, which has no quadruples of points of type  $\mathcal{P}_4$  or  $\mathcal{N}_4$  is called an Alexandrov space with non-positive or non-negative curvature respectively.

Here is an exercise, solving which would force the reader to rebuild a considerable part of Alexandrov geometry.

**0.0.2. Advanced exercise.** *Assume  $\mathcal{X}$  is a complete metric space with length metric, containing only quadruples of type  $\mathcal{E}_4$ . Show that  $\mathcal{X}$  is isometric to a convex set in a Hilbert space.*

In fact, it might be helpful to spend some time thinking about this exercise before proceeding.

In the definition above, instead of Euclidean space one can take hyperbolic space of curvature  $-1$ . In this case, one obtains the definition of spaces with curvature bounded above or below by  $-1$ .

To define spaces with curvature bounded above or below by  $1$ , one has to take the unit 3-sphere and specify that only the quadruples of points such that each of the four triangles has perimeter at most  $2\cdot\pi$  are checked. The latter condition could be thought as a part of the *spherical triangle inequality*.

## Acknowledgment

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We want to thank the mathematical institutions where we worked on this material, including BIRS, MFO, Henri Poincaré Institute, University of Colone, Max Planck Institute for Mathematics.



# Chapter 1

## Preliminaries

In this chapter we fix some conventions and recall the main definitions. The chapter may be used as a quick reference when reading the book.

To learn background in metric geometry, the reader may consult the book of Dmitri Burago, Yuri Burago and Sergei Ivanov [17].

### 1.1 Metric spaces

The distance between two points  $x$  and  $y$  in a metric space  $\mathcal{X}$  will be denoted by  $|x - y|$  or  $|x - y|_{\mathcal{X}}$ . The latter notation is used if we need to emphasize that the distance is taken in the space  $\mathcal{X}$ .

The function

$$\text{dist}_x: y \mapsto |x - y|$$

is called the *distance function* from  $x$ .

◇ The *diameter* of metric space  $\mathcal{X}$  is defined as

$$\text{diam } \mathcal{X} = \sup\{|x - y|_{\mathcal{X}} \mid x, y \in \mathcal{X}\}.$$

◇ Given  $R \in [0, \infty]$  and  $x \in \mathcal{X}$ , the sets

$$\begin{aligned} \mathbb{B}(x, R) &= \{y \in \mathcal{X} \mid |x - y| < R\}, \\ \overline{\mathbb{B}}[x, R] &= \{y \in \mathcal{X} \mid |x - y| \leq R\} \end{aligned}$$

are called respectively the *open* and the *closed balls* of radius  $R$  with center  $x$ . Again, if we need to emphasize that these balls are taken in the metric space  $\mathcal{X}$ , we write

$$\mathbb{B}(x, R)_{\mathcal{X}} \quad \text{and} \quad \overline{\mathbb{B}}[x, R]_{\mathcal{X}}.$$

A metric space  $\mathcal{X}$  is called *proper* if all closed bounded sets in  $\mathcal{X}$  are compact. This condition is equivalent to each of the following statements:

1. For some (and therefore any) point  $p \in \mathcal{X}$  and any  $R < \infty$ , the closed ball  $\bar{B}[p, R] \subset \mathcal{X}$  is compact.
2. The function  $\text{dist}_p: \mathcal{X} \rightarrow \mathbb{R}$  is proper for some (and therefore any) point  $p \in \mathcal{X}$ ; that is, for any compact set  $K \subset \mathbb{R}$ , its inverse image  $\{x \in \mathcal{X} \mid |p - x|_{\mathcal{X}} \in K\}$  is compact.

**1.1.1. Exercise.** Let  $K$  be a compact metric space and

$$f: K \rightarrow K$$

be a non-contracting map. Prove that  $f$  is an isometry.

## 1.2 Constructions

**Product space.** Given two metric spaces  $\mathcal{U}$  and  $\mathcal{V}$ , the *product space*  $\mathcal{U} \times \mathcal{V}$  is defined as the set of all pairs  $(u, v)$  in the set  $\mathcal{U} \times \mathcal{V}$  with the metric defined by formula

$$|(u^1, v^1) - (u^2, v^2)|_{\mathcal{U} \times \mathcal{V}} = \sqrt{|u^1 - u^2|_{\mathcal{U}}^2 + |v^1 - v^2|_{\mathcal{V}}^2}.$$

**Cone.** The *cone*  $\mathcal{V} = \text{Cone}\mathcal{U}$  over a metric space  $\mathcal{U}$  is defined as the metric space whose underlying set consists of equivalence classes in  $[0, \infty) \times \mathcal{U}$  with the equivalence relation “ $\sim$ ” given by  $(0, p) \sim (0, q)$  for any points  $p, q \in \mathcal{U}$ , and whose metric is given by the cosine rule

$$|(p, s) - (q, t)|_{\mathcal{V}} = \sqrt{s^2 + t^2 - 2 \cdot s \cdot t \cdot \cos \alpha},$$

where  $\alpha = \min\{\pi, |p - q|_{\mathcal{U}}\}$ .

The point in the cone  $\mathcal{V}$  formed by the equivalence class of  $0 \times \mathcal{U}$  is called the *tip of the cone* and is denoted by  $0$  or  $0_{\mathcal{V}}$ . The distance  $|0 - v|_{\mathcal{V}}$  is called the norm of  $v$  and is denoted by  $|v|$  or  $|v|_{\mathcal{V}}$ .

**Suspension.** The *suspension*  $\mathcal{V} = \text{Susp}\mathcal{U}$  over a metric space  $\mathcal{U}$  is defined as the metric space whose underlying set consists of equivalence classes in  $[0, \pi] \times \mathcal{U}$  with the equivalence relation “ $\sim$ ” given by  $(0, p) \sim (0, q)$  and  $(\pi, p) \sim (\pi, q)$  for any points  $p, q \in \mathcal{U}$ , and whose metric is given by the spherical cosine rule

$$\cos |(p, s) - (q, t)|_{\text{Susp}\mathcal{U}} = \cos s \cdot \cot t - \sin s \cdot \sin t \cdot \cos \alpha,$$

where  $\alpha = \min\{\pi, |p - q|_{\mathcal{U}}\}$ .

The points in  $\mathcal{V}$  formed by the equivalence class of  $0 \times \mathcal{U}$  and  $\pi \times \mathcal{U}$  are called the *north* and the *south poles* of the suspension.

**1.2.1. Exercise.** *Let  $\mathcal{U}$  be a metric space. Show that the spaces*

$$\mathbb{R} \times \text{Cone}\mathcal{U} \quad \text{and} \quad \text{Cone}[\text{Susp}\mathcal{U}]$$

*are isometric.*

## 1.3 Geodesics, triangles and hinges

**Geodesics.** Let  $\mathcal{X}$  be a metric space and  $\mathbb{I}$  be a real interval. A globally isometric map  $\gamma: \mathbb{I} \rightarrow \mathcal{X}$  is called a *geodesic*<sup>1</sup>; in other words,  $\gamma: \mathbb{I} \rightarrow \mathcal{X}$  is a geodesic if

$$|\gamma(s) - \gamma(t)|_{\mathcal{X}} = |s - t|$$

for any pair  $s, t \in \mathbb{I}$ .

We say that  $\gamma: \mathbb{I} \rightarrow \mathcal{X}$  is a geodesic from point  $p$  to point  $q$  if  $\mathbb{I} = [a, b]$  and  $p = \gamma(a)$ ,  $q = \gamma(b)$ . In this case the image of  $\gamma$  is denoted by  $[pq]$  and with an abuse of notations we also call it a *geodesic*. Given a geodesic  $[pq]$  we can parametrize it by distance to  $p$ ; this parametrization will be denoted as  $\text{geod}_{[pq]}(t)$ .

We may write  $[pq]_{\mathcal{X}}$  to emphasize that the geodesic  $[pq]$  is in the space  $\mathcal{X}$ . Also we use the following short-cut notation:

$$]pq[ = [pq] \setminus \{p, q\}, \quad ]pq] = [pq] \setminus \{p\}, \quad [pq[ = [pq] \setminus \{q\}.$$

In general, a geodesic between  $p$  and  $q$  need not exist and if it exists, it need not be unique. However, once we write  $[pq]$  we mean that we made a choice of a geodesic.

A metric space is called *geodesic* if any pair of its points can be joined by a geodesic.

A *geodesic path* is a geodesic with constant-speed parameterization by  $[0, 1]$ . Given a geodesic  $[pq]$ , we denote by  $\text{path}_{[pq]}$  the corresponding geodesic path; that is,

$$\text{path}_{[pq]}(t) \equiv \text{geod}_{[pq]}(t \cdot |p - q|).$$

A curve  $\gamma: \mathbb{I} \rightarrow \mathcal{X}$  is called a *local geodesic* if for any  $t \in \mathbb{I}$  there is a neighborhood  $U$  of  $t$  in  $\mathbb{I}$  such that the restriction  $\gamma|_U$  is a geodesic. A constant speed parametrization of local geodesic by the unit interval  $[0, 1]$  is called a *local geodesic path*.

<sup>1</sup>Various authors call it differently: *shortest path*, *minimizing geodesic*.

**Triangles.** For a triple of points  $p, q, r \in \mathcal{X}$ , a choice of a triple of geodesics  $([qr], [rp], [pq])$  will be called a *triangle*; we will use the short notation  $[pqr] = ([qr], [rp], [pq])$ . Again, given a triple  $p, q, r \in \mathcal{X}$  there may be no triangle  $[pqr]$  simply because one of the pairs of these points can not be joined by a geodesic, and also many different triangles with these vertices may exist, any of which can be denoted by  $[pqr]$ . Once we write  $[pqr]$ , it means that we made a choice of such a triangle, that is, a choice of each  $[qr]$ ,  $[rp]$  and  $[pq]$ . The value  $|p - q| + |q - r| + |r - p|$  will be called the *perimeter of the triangle*  $[pqr]$ .

**Hinges.** Let  $p, x, y \in \mathcal{X}$  be a triple of points such that  $p$  is distinct from  $x$  and  $y$ . A pair of geodesics  $([px], [py])$  will be called a *hinge* and it will be denoted by  $[p_x] = ([px], [py])$ .

**Convex sets.** A set  $A$  in a metric space  $\mathcal{X}$  is called *convex* if for every two points  $p, q \in A$ , every geodesic  $[pq]$  in  $\mathcal{X}$  lies in  $A$ .

A set  $A \subset \mathcal{X}$  is called *locally convex* if every point  $a \in A$  admits an open neighborhood  $\Omega \ni a$  such that any geodesic lying in  $\Omega$  and with ends in  $A$  lies completely in  $A$ .

Note that any open set is locally convex by definition.

## 1.4 Length spaces

Recall that a path or a curve is a continuous map from a real interval to a space.

**1.4.1. Definition.** Let  $\mathcal{X}$  be a metric space and  $\alpha: \mathbb{I} \rightarrow \mathcal{X}$  be a curve. We define the length of  $\alpha$  as

$$\text{length } \alpha \stackrel{\text{def}}{=} \sup_{t_0 \leq t_1 \leq \dots \leq t_n} \sum_i |\alpha(t_i) - \alpha_{i+1}|$$

Directly from the definition, it follows that if a path  $\alpha: [0, 1] \rightarrow \mathcal{X}$  connects two points  $x$  and  $y$  (that is, such that  $\alpha(0) = x$  and  $\alpha(1) = y$ ) then

$$\text{length } \alpha \geq |x - y|.$$

If for any  $\varepsilon > 0$  and any pair of points  $x$  and  $y$  in a metric space  $\mathcal{X}$  there is a path  $\alpha$  connecting  $x$  to  $y$  such that

$$\text{length } \alpha < |x - y| + \varepsilon,$$

then  $\mathcal{X}$  is called a *length space*.

Note that any geodesic space is a length space; as can be seen from the following example, the converse does not hold.

**1.4.2. Example.** Let  $\mathcal{X}$  be obtained by gluing a countable collection of disjoint intervals  $\mathbb{I}_i$  of length  $1 + 1/i$ , where for each  $\mathbb{I}_i$  one end is glued to  $p = \{0\}$  and the other to  $q = \{1\}$ . Then  $\mathcal{X}$  carries a natural complete length metric with respect to which  $|p - q| = 1$  but there is no geodesic connecting  $p$  to  $q$ .

**1.4.3. Exercise.** Give an example of a complete length space for which no pair of distinct points can be joined by a geodesic.

Let  $\mathcal{X}$  be a metric space and  $x, y \in \mathcal{X}$ .

(i) A point  $z \in \mathcal{X}$  is called a *midpoint* between  $x$  and  $y$  if

$$|x - z| = |y - z| = \frac{1}{2} \cdot |x - y|.$$

(ii) Assume  $\varepsilon \geq 0$ . A point  $z \in \mathcal{X}$  is called an  $\varepsilon$ -*midpoint* between  $x$  and  $y$  if

$$|x - z|, |y - z| \leq \frac{1}{2} \cdot |x - y| + \varepsilon.$$

Note that a 0-midpoint is the same as a midpoint.

**1.4.4. Lemma.** Let  $\mathcal{X}$  be a complete metric space.

- a) Assume that for any pair of points  $x, y \in \mathcal{X}$  and any  $\varepsilon > 0$  there is a  $\varepsilon$ -midpoint  $z$ . Then  $\mathcal{X}$  is a length space.
- b) Assume that for any pair of points  $x, y \in \mathcal{X}$  there is a midpoint  $z$ . Then  $\mathcal{X}$  is a geodesic space.

*Proof.* We first prove (a). Let  $x, y \in \mathcal{X}$  be a pair of points.

Set  $\varepsilon_n = \frac{\varepsilon}{2^{2-n}}$ .

Set  $\alpha(0) = x$  and  $\alpha(1) = y$ .

Let  $\alpha(\frac{1}{2})$  be an  $\varepsilon_1$ -midpoint of  $\alpha(0)$  and  $\alpha(1)$ . Further, let  $\alpha(\frac{1}{4})$  and  $\alpha(\frac{3}{4})$  be  $\varepsilon_2$ -midpoints for the pairs  $(\alpha(0), \alpha(\frac{1}{2}))$  and  $(\alpha(\frac{1}{2}), \alpha(1))$  respectively. Applying the above procedure recursively, on the  $n$ -th step we define  $\alpha(\frac{k}{2^n})$ , for every odd integer  $k$  such that  $0 < \frac{k}{2^n} < 1$ , as an  $\varepsilon_n$ -midpoint of the already defined  $\alpha(\frac{k-1}{2^n})$  and  $\alpha(\frac{k+1}{2^n})$ .

In this way we define  $\alpha(t)$  for  $t \in W$ , where  $W$  denotes the set of dyadic rationals in  $[0, 1]$ . The map  $\alpha$  can be extended continuously to  $[0, 1]$ . Moreover,

$$\begin{aligned} \text{length } \alpha &\leq |x - y| + \sum_{n=1}^{\infty} 2^{n-1} \cdot \varepsilon_n \leq \\ &\leq |x - y| + \frac{\varepsilon}{2}. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, we get (a).

To prove (b), one should repeat the same argument taking mid-points instead of  $\varepsilon_n$ -midpoints. In this case ❶ holds for  $\varepsilon_n = \varepsilon = 0$ .  $\square$

Since in a compact space a sequence of  $1/n$ -midpoints  $z_n$  contains a convergent subsequence, Lemma 1.4.4 immediately implies

**1.4.5. Proposition.** *A proper length space is geodesic.*

**1.4.6. Hopf–Rinow theorem.** *Any complete, locally compact length space is proper.*

*Proof.* Let  $\mathcal{X}$  be a locally compact length space. Given  $x \in \mathcal{X}$ , denote by  $\rho(x)$  the supremum of all  $R > 0$  such that the closed ball  $\overline{B}[x, R]$  is compact. Since  $\mathcal{X}$  is locally compact

❷  $\rho(x) > 0$  for any  $x \in \mathcal{X}$ .

It is sufficient to show that  $\rho(x) = \infty$  for some (and therefore any) point  $x \in \mathcal{X}$ .

Assume the contrary; that is,  $\rho(x) < \infty$ .

❸  $B = \overline{B}[x, \rho(x)]$  is compact for any  $x$ .

Indeed,  $\mathcal{X}$  is a length space; therefore for any  $\varepsilon > 0$ , the set  $\overline{B}[x, \rho(x) - \varepsilon]$  forms a compact  $\varepsilon$ -net in  $B$ . Since  $B$  is closed and hence complete, it has to be compact.  $\triangle$

❹  $|\rho(x) - \rho(y)| \leq |x - y|_{\mathcal{X}}$ , in particular  $\rho: \mathcal{X} \rightarrow \mathbb{R}$  is a continuous function.

Indeed, assume the contrary; that is,  $\rho(x) + |x - y| < \rho(y)$  for some  $x, y \in \mathcal{X}$ . Then  $\overline{B}[x, \rho(x) + \varepsilon]$  is a closed subset of  $\overline{B}[y, \rho(y)]$  for some  $\varepsilon > 0$ . Then compactness of  $\overline{B}[y, \rho(y)]$  implies compactness of  $\overline{B}[x, \rho(x) + \varepsilon]$ , a contradiction.  $\triangle$

Set  $\varepsilon = \min_{y \in B} \{\rho(y)\}$ ; the minimum is defined since  $B$  is compact. From ❷, we have  $\varepsilon > 0$ .

Choose a finite  $\frac{\varepsilon}{10}$ -net  $\{a_1, a_2, \dots, a_n\}$  in  $B$ . The union  $W$  of the closed balls  $\overline{B}[a_i, \varepsilon]$  is compact. Clearly  $\overline{B}[x, \rho(x) + \frac{\varepsilon}{10}] \subset W$ . Therefore  $\overline{B}[x, \rho(x) + \frac{\varepsilon}{10}]$  is compact, a contradiction.  $\square$

**1.4.7. Exercise.** *Construct a geodesic space that is locally compact, but whose completion is neither geodesic nor locally compact.*

## 1.5 Model angles and triangles.

Let  $\mathcal{X}$  be a metric space,  $p, q, r \in \mathcal{X}$ . Let us define its *model triangle*  $[\tilde{p}\tilde{q}\tilde{r}]$  (briefly,  $[\tilde{p}\tilde{q}\tilde{r}] = \tilde{\Delta}(pqr)_{\mathbb{E}^2}$ ) to be a triangle in the plane  $\mathbb{E}^2$  with the same sides; that is,

$$|\tilde{p} - \tilde{q}| = |p - q|, \quad |\tilde{q} - \tilde{r}| = |q - r|, \quad |\tilde{r} - \tilde{p}| = |r - p|.$$

In the same way we can define the *hyperbolic* and the *spherical model triangles*  $\tilde{\Delta}(pqr)_{\mathbb{H}^2}$ ,  $\tilde{\Delta}(pqr)_{\mathbb{S}^2}$  in the hyperbolic plane  $\mathbb{H}^2$  and the unit sphere  $\mathbb{S}^2$ . In the latter case the model triangle is said to be defined if in addition

$$|p - q| + |q - r| + |r - p| < 2 \cdot \pi.$$

In this case the model triangle again exists and is unique up to an motion of  $\mathbb{S}^2$ .

If  $[\tilde{p}\tilde{q}\tilde{r}] = \tilde{\Delta}(pqr)_{\mathbb{E}^2}$  and  $|p - q|, |p - r| > 0$ , the angle measure of  $[\tilde{p}\tilde{q}\tilde{r}]$  at  $\tilde{p}$  will be called the *model angle* of triple  $p, q, r$  and it will be denoted by  $\tilde{\angle}(p^q_r)_{\mathbb{E}^2}$ . In the same way we define  $\tilde{\angle}(p^q_r)_{\mathbb{H}^2}$  and  $\tilde{\angle}(p^q_r)_{\mathbb{S}^2}$ ; in the latter case we assume in addition that the model triangle  $\tilde{\Delta}(pqr)_{\mathbb{S}^2}$  is defined. We may use the notation  $\tilde{\angle}(p^q_r)$  if it is evident which of the model spaces  $\mathbb{H}^2$ ,  $\mathbb{E}^2$  or  $\mathbb{S}^2$  is meant.

**1.5.1. Alexandrov's lemma.** *Let  $p, x, y, z$  be distinct points in a metric space such that  $z \in ]xy[$ . Then the following expressions for the Euclidean model angles have the same sign:*

- a)  $\tilde{\angle}(x^p_x) - \tilde{\angle}(x^p_y)$ ,
- b)  $\tilde{\angle}(z^p_x) + \tilde{\angle}(z^p_y) - \pi$ .

Moreover,

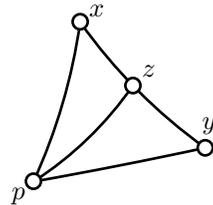
$$\tilde{\angle}(p^x_y) \geq \tilde{\angle}(p^x_z) + \tilde{\angle}(p^z_y),$$

with equality if and only if the expressions in (a) and (b) vanish.

The same holds for the hyperbolic and spherical model angles, but in the latter case one has to assume in addition that

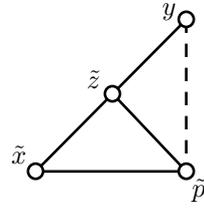
$$|p - z| + |p - y| + |x - y| < 2 \cdot \pi.$$

*Proof.* Consider the model triangle  $[\tilde{x}\tilde{p}\tilde{z}] = \tilde{\Delta}(xpz)$ . Take a point  $\tilde{y}$  on the extension of  $[\tilde{x}\tilde{z}]$  beyond  $\tilde{z}$  so that  $|\tilde{x} - \tilde{y}| = |x - y|$  (and therefore  $|\tilde{x} - \tilde{z}| = |x - z|$ ).



Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:

- (i)  $\angle[\tilde{x} \frac{\tilde{p}}{\tilde{y}}] - \angle(x \frac{p}{y})$ ;
- (ii)  $|\tilde{p} - \tilde{y}| - |p - y|$ ;
- (iii)  $\angle[\tilde{z} \frac{\tilde{p}}{\tilde{y}}] - \angle(z \frac{p}{y})$ .



Since

$$\angle[\tilde{x} \frac{\tilde{p}}{\tilde{y}}] = \angle[\tilde{x} \frac{\tilde{p}}{\tilde{z}}] = \angle(x \frac{p}{z})$$

and

$$\angle[\tilde{z} \frac{\tilde{p}}{\tilde{y}}] = \pi - \angle[\tilde{z} \frac{\tilde{x}}{\tilde{p}}] = \pi - \angle(z \frac{x}{p}),$$

the first statement follows.

For the second statement, construct a model triangle  $[\tilde{p}\tilde{z}\tilde{y}'] = = \tilde{\Delta}(pzy)_{\mathbb{E}^2}$  on the opposite side of  $[\tilde{p}\tilde{z}]$  from  $[\tilde{x}\tilde{p}\tilde{z}]$ . Note that

$$\begin{aligned} |\tilde{x} - \tilde{y}'| &\leq |\tilde{x} - \tilde{z}| + |\tilde{z} - \tilde{y}'| = \\ &= |x - z| + |z - y| = \\ &= |x - y|. \end{aligned}$$

Therefore

$$\begin{aligned} \angle(p \frac{x}{y}) + \angle(p \frac{z}{y}) &= \angle[\tilde{p} \frac{\tilde{x}}{\tilde{z}}] + \angle[\tilde{p} \frac{\tilde{z}}{\tilde{y}'}] = \\ &= \angle[\tilde{p} \frac{\tilde{x}}{\tilde{y}'}] \leq \\ &\leq \angle(p \frac{x}{y}). \end{aligned}$$

Equality holds if and only if  $|\tilde{x} - \tilde{y}'| = |x - y|$ , as required.  $\square$

## 1.6 Angles and the first variation.

Given a hinge  $[p \frac{x}{y}]$ , we define its *angle* as the limit

$$\bullet \quad \angle[p \frac{x}{y}] \stackrel{\text{def}}{=} \lim_{\bar{x}, \bar{y} \rightarrow p} \angle(p \frac{\bar{x}}{\bar{y}})_{\mathbb{E}^2},$$

where  $\bar{x} \in ]px]$  and  $\bar{y} \in ]py]$ , provided the limit exists. The angle under the limit can be calculated from the cosine law:

$$\cos \angle(p \frac{x}{y})_{\mathbb{E}^2} = \frac{|p - x|^2 + |p - y|^2 - |x - y|^2}{2 \cdot |p - x| \cdot |p - y|}.$$

The following lemma implies that in the angle definition  $\bullet$ , one can use  $\angle(p \frac{\bar{x}}{\bar{y}})_{\mathbb{S}^2}$  or  $\angle(p \frac{\bar{x}}{\bar{y}})_{\mathbb{H}^2}$  instead of  $\angle(p \frac{\bar{x}}{\bar{y}})_{\mathbb{E}^2}$ .

**1.6.1. Lemma.** *For any three points  $p, x, y$  in a metric space the following inequalities*

$$\begin{aligned} \textcircled{2} \quad & |\check{Z}(p_y^x)_{\mathbb{S}^2} - \check{Z}(p_y^x)_{\mathbb{E}^2}| \leq |p - x| \cdot |p - y|, \\ & |\check{Z}(p_y^x)_{\mathbb{H}^2} - \check{Z}(p_y^x)_{\mathbb{E}^2}| \leq |p - x| \cdot |p - y|, \end{aligned}$$

hold whenever the left-hand side is defined.

*Proof.* Note that

$$\check{Z}(p_y^x)_{\mathbb{H}^2} \leq \check{Z}(p_y^x)_{\mathbb{E}^2} \leq \check{Z}(p_y^x)_{\mathbb{S}^2}.$$

Therefore

$$\begin{aligned} 0 & \leq \check{Z}(p_y^x)_{\mathbb{S}^2} - \check{Z}(p_y^x)_{\mathbb{H}^2} \leq \\ & \leq \check{Z}(p_y^x)_{\mathbb{S}^2} + \check{Z}(x_y^p)_{\mathbb{S}^2} + \check{Z}(y_x^p)_{\mathbb{S}^2} - \check{Z}(p_y^x)_{\mathbb{H}^2} - \check{Z}(x_y^p)_{\mathbb{H}^2} - \check{Z}(y_x^p)_{\mathbb{H}^2} = \\ & = \text{area } \tilde{\Delta}(pxy)_{\mathbb{S}^2} + \text{area } \tilde{\Delta}(pxy)_{\mathbb{H}^2}. \end{aligned}$$

The inequality  $\textcircled{2}$  follows since

$$\begin{aligned} 0 & \leq \text{area } \tilde{\Delta}(pxy)_{\mathbb{H}^2} \leq \\ & \leq \text{area } \tilde{\Delta}(pxy)_{\mathbb{S}^2} \leq \\ & \leq |p - x| \cdot |p - y|. \end{aligned} \quad \square$$

**1.6.2. Triangle inequality for angles.** *Let  $[px^1]$ ,  $[px^2]$  and  $[px^3]$  be three geodesics in a metric space. If all the angles  $\alpha^{ij} = \angle [p_{x^i}^{x^j}]$  are defined, then they satisfy the triangle inequality:*

$$\alpha^{13} \leq \alpha^{12} + \alpha^{23}.$$

*Proof.* Since  $\alpha^{13} \leq \pi$ , we can assume that  $\alpha^{12} + \alpha^{23} < \pi$ . Set  $\gamma^i = \text{geod}_{[px^i]}$ . Given any  $\varepsilon > 0$ , for all sufficiently small  $t, \tau, s \in \mathbb{R}_+$  we have

$$\begin{aligned} |\gamma^1(t) - \gamma^3(\tau)| & \leq |\gamma^1(t) - \gamma^2(s)| + |\gamma^2(s) - \gamma^3(\tau)| < \\ & < \sqrt{t^2 + s^2 - 2 \cdot t \cdot s \cdot \cos(\alpha^{12} + \varepsilon)} + \\ & \quad + \sqrt{s^2 + \tau^2 - 2 \cdot s \cdot \tau \cdot \cos(\alpha^{23} + \varepsilon)} \leq \end{aligned}$$

(Below we define  $s(t, \tau)$  so that for  $s = s(t, \tau)$ , this chain of inequalities can be continued as follows:)

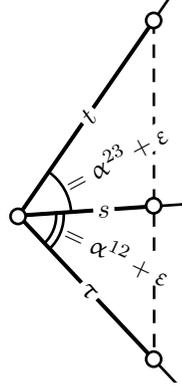
$$\leq \sqrt{t^2 + \tau^2 - 2 \cdot t \cdot \tau \cdot \cos(\alpha^{12} + \alpha^{23} + 2 \cdot \varepsilon)}.$$

Thus for any  $\varepsilon > 0$ ,

$$\alpha^{13} \leq \alpha^{12} + \alpha^{23} + 2 \cdot \varepsilon.$$

Hence the result.

To define  $s(t, \tau)$  so that it satisfies the above inequality, consider three rays  $\tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3$  on a Euclidean plane starting at one point, such that  $\angle(\tilde{\gamma}^1, \tilde{\gamma}^2) = \alpha^{12} + \varepsilon$ ,  $\angle(\tilde{\gamma}^2, \tilde{\gamma}^3) = \alpha^{23} + \varepsilon$  and  $\angle(\tilde{\gamma}^1, \tilde{\gamma}^3) = \alpha^{12} + \alpha^{23} + 2 \cdot \varepsilon$ . We parametrize each ray by the distance from the starting point. Given two positive numbers  $t, \tau \in \mathbb{R}_+$ , let  $s = s(t, \tau)$  be the number such that  $\tilde{\gamma}^2(s) \in [\tilde{\gamma}^1(t), \tilde{\gamma}^3(\tau)]$ . Clearly  $s \leq \max\{t, \tau\}$ , so  $t, \tau, s$  may be taken sufficiently small.  $\square$



**1.6.3. Exercise.** Prove that the sum of adjacent angles is at least  $\pi$ .

More precisely: let  $\mathcal{X}$  be a complete length space and  $p, x, y, z \in \mathcal{X}$ . If  $p \in ]xy[$  then

$$\angle[p_x^x] + \angle[p_z^y] \geq \pi$$

whenever each angle on the left-hand side is defined.

**1.6.4. First variation inequality.** Assume that for a hinge  $[q_x^p]$  the angle  $\alpha = \angle[q_x^p]$  is defined. Then

$$|p - \text{geod}_{[qx]}(t)| \leq |q - p| - t \cdot \cos \alpha + o(t).$$

*Proof.* Take a sufficiently small  $\varepsilon > 0$ . For all sufficiently small  $t > 0$ , we have

$$\begin{aligned} |\text{geod}_{[qp]}(t/\varepsilon) - \text{geod}_{[qx]}(t)| &\leq \frac{t}{\varepsilon} \cdot \sqrt{1 + \varepsilon^2 - 2 \cdot \varepsilon \cdot \cos \alpha} + o(t) \leq \\ &\leq \frac{t}{\varepsilon} - t \cdot \cos \alpha + t \cdot \varepsilon. \end{aligned}$$

Applying the triangle inequality, we get

$$\begin{aligned} |p - \text{geod}_{[qx]}(t)| &\leq |p - \text{geod}_{[qp]}(t/\varepsilon)| + |\text{geod}_{[qp]}(t/\varepsilon) - \text{geod}_{[qx]}(t)| \leq \\ &\leq |p - q| - t \cdot \cos \alpha + t \cdot \varepsilon \end{aligned}$$

for any fixed  $\varepsilon > 0$  and all sufficiently small  $t$ . Hence the result.  $\square$

## 1.7 Space of directions and tangent space

Let  $\mathcal{X}$  be a metric space with defined angles for all hinges. Fix a point  $p \in \mathcal{X}$ .

Consider the set  $\mathfrak{S}_p$  of all nontrivial geodesics which start at  $p$ . By 1.6.2, the triangle inequality holds for  $\angle$  on  $\mathfrak{S}_p$ , so  $(\mathfrak{S}_p, \angle)$  forms a

*pseudometric space*; that is,  $\angle$  satisfies all the conditions of a metric on  $\mathfrak{S}_p$ , except that the angle between distinct geodesics might vanish.

The metric space corresponding to  $(\mathfrak{S}_p, \angle)$  is called the *space of geodesic directions* at  $p$ , denoted by  $\Sigma'_p$  or  $\Sigma'_p \mathcal{X}$ . Elements of  $\Sigma'_p$  are called *geodesic directions* at  $p$ . Each geodesic direction is formed by an equivalence class of geodesics in  $\mathfrak{S}_p$  for the equivalence relation

$$[px] \sim [py] \iff \angle[p^x_y] = 0.$$

The completion of  $\Sigma'_p$  is called the *space of directions* at  $p$  and is denoted by  $\Sigma_p$  or  $\Sigma_p \mathcal{X}$ . Elements of  $\Sigma_p$  are called *directions* at  $p$ .

The Euclidean cone  $\text{Cone } \Sigma_p$  over the space of directions  $\Sigma_p$  is called the *tangent space* at  $p$  and is denoted by  $T_p$  or  $T_p \mathcal{X}$ .

The tangent space  $T_p$  could also be defined directly, without introducing the space of directions. To do so, consider the set  $\mathfrak{T}_p$  of all geodesics with constant-speed parameterizations starting at  $p$ . Given  $\alpha, \beta \in \mathfrak{T}_p$ , set

$$\bullet \quad |\alpha - \beta|_{\mathfrak{T}_p} = \lim_{\varepsilon \rightarrow 0} \frac{|\alpha(\varepsilon) - \beta(\varepsilon)|_{\mathcal{X}}}{\varepsilon}$$

Since the angles in  $\mathcal{X}$  are defined,  $\bullet$  defines a pseudometric on  $\mathfrak{T}_p$ .

The corresponding metric space admits a natural isometric identification with the cone  $T'_p = \text{Cone } \Sigma'_p$ . The elements of  $T'_p$  are equivalence classes for the relation

$$\alpha \sim \beta \iff |\alpha(t) - \beta(t)|_{\mathcal{X}} = o(t).$$

The completion of  $T'_p$  is therefore naturally isometric to  $T_p$ .

Elements of  $T_p$  will be called *tangent vectors* at  $p$ , even though  $T_p$  is only a metric cone and need not be a vector space. Elements of  $T'_p$  will be called *geodesic tangent vectors* at  $p$ .

## 1.8 Hausdorff convergence

It seems that Hausdorff convergence was first introduced by Felix Hausdorff in [36], and a couple of years later an equivalent definition was given by Wilhelm Blaschke in [13]. A refinement of this definition was introduced by Zdeněk Frolík in [30], and then rediscovered by Robert Wijsman in [56]. However, this refinement was a reversion to the *closed convergence* introduced by Hausdorff in the original book. For this reason we call it Hausdorff convergence instead of *Hausdorff–Blaschke–Frolík–Wijsman convergence*.

Let  $\mathcal{X}$  be a metric space and  $A \subset \mathcal{X}$ . We will denote by  $\text{dist}_A(x)$  the distance from  $A$  to a point  $x$  in  $\mathcal{X}$ ; that is,

$$\text{dist}_A(x) \stackrel{\text{def}}{=} \inf \{ |a - x|_{\mathcal{X}} \mid a \in A \}.$$

**1.8.1. Definition of Hausdorff convergence.** *Given a sequence of closed sets  $(A_n)_{n=1}^{\infty}$  in a metric space  $\mathcal{X}$ , a closed set  $A_{\infty} \subset \mathcal{X}$  is called the Hausdorff limit of  $(A_n)_{n=1}^{\infty}$ , briefly  $A_n \rightarrow A_{\infty}$ , if*

$$\text{dist}_{A_n}(x) \rightarrow \text{dist}_{A_{\infty}}(x) \quad \text{as } n \rightarrow \infty$$

for any fixed  $x \in \mathcal{X}$ .

Then the sequence of closed sets  $(A_n)_{n=1}^{\infty}$  is said to converge in the sense of Hausdorff.

**Example.** Let  $D_n$  be the disc in the coordinate plane with center  $(0, n)$  and radius  $n$ . Then  $D_n$  converges to the upper half-plane as  $n \rightarrow \infty$ .

**1.8.2. Exercise.** *In a metric space  $\mathcal{X}$ , show that if  $A_n \rightarrow A_{\infty}$  as in Definition 1.8.1, then  $A_{\infty}$  is the set of all points  $p$  such that  $p_n \rightarrow p$  for some sequence of points  $p_n \in A_n$ .*

*Show that the converse fails.*

**1.8.3. Selection theorem.** *Let  $\mathcal{X}$  be a proper metric space and  $(A_n)_{n=1}^{\infty}$  be a sequence of closed sets in  $\mathcal{X}$ . Assume that for some (and therefore any) point  $x \in \mathcal{X}$ , the sequence  $\text{dist}_{A_n}(x)$  is bounded. Then the sequence  $(A_n)_{n=1}^{\infty}$  has a convergent subsequence in the sense of Hausdorff.*

*Proof.* Since  $\mathcal{X}$  is proper, we can choose a countable dense set  $\{x_1, x_2, \dots\}$  in  $\mathcal{X}$ . Note that the sequence  $a_n = \text{dist}_{A_n}(x_k)$  is bounded for each  $k$ . Therefore, passing to a subsequence of  $(A_n)_{n=1}^{\infty}$ , we can assume that  $\text{dist}_{A_n}(x_k)$  converges as  $n \rightarrow \infty$  for any fixed  $k$ .

Note that for each  $n$ , the function  $\text{dist}_{A_n} : \mathcal{X} \rightarrow \mathbb{R}$  is 1-Lipschitz and nonnegative. Therefore the sequence  $\text{dist}_{A_n}$  converges pointwise to a 1-Lipschitz nonnegative function  $f : \mathcal{X} \rightarrow \mathbb{R}$ .

Set  $A_{\infty} = f^{-1}(0)$ . Since  $f$  is 1-Lipschitz,

$$\text{dist}_{A_{\infty}}(y) \geq f(y)$$

for any  $y \in \mathcal{X}$ . It remains to show that

$$\text{dist}_{A_{\infty}}(y) \leq f(y)$$

for any  $y$ .

Assume the contrary; that is,

$$f(z) < R < \text{dist}_{A_\infty}(z)$$

for some  $z \in \mathcal{X}$  and  $R > 0$ . Then for any large enough  $n$ , there is a point  $z_n \in A_n$  such that  $|x - z_n| \leq R$ . Since  $\mathcal{X}$  is proper, we can pass to a partial limit  $z_\infty$  of  $z_n$  as  $n \rightarrow \infty$ .

It is clear that  $f(z_\infty) = 0$ , that is,  $z_\infty \in A_\infty$ . On the other hand,

$$\text{dist}_{A_\infty}(y) \leq |z_\infty - y| \leq R < \text{dist}_{A_\infty}(y),$$

a contradiction.  $\square$

## 1.9 Gromov–Hausdorff convergence

**1.9.1. Definition.** Let  $\{\mathcal{X}_\alpha \mid \alpha \in \mathcal{A}\}$  be a set of metric spaces. A metric  $\rho$  on the disjoint union

$$\mathbf{X} = \bigsqcup_{\alpha \in \mathcal{A}} \mathcal{X}_\alpha$$

is called a compatible metric if the restriction of  $\rho$  to every  $\mathcal{X}_\alpha$  is the original metric on  $\mathcal{X}_\alpha$ .

**1.9.2. Definition.** Let  $\mathcal{X}_1, \mathcal{X}_2, \dots$  and  $\mathcal{X}_\infty$  be proper metric spaces and  $\rho$  be a compatible metric on their disjoint union  $\mathbf{X}$ . Assume that  $\mathcal{X}_n$  is an open set in  $(\mathbf{X}, \rho)$  for each  $n \neq \infty$ , and  $\mathcal{X}_n \rightarrow \mathcal{X}_\infty$  in  $(\mathbf{X}, \rho)$  as  $n \rightarrow \infty$  in the sense of Hausdorff (see Definition 1.8.1).

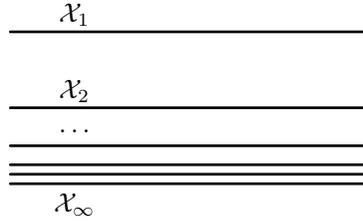
Then we say  $\rho$  defines a convergence<sup>2</sup> in the sense of Gromov–Hausdorff, and write  $\mathcal{X}_n \rightarrow \mathcal{X}_\infty$  or  $\mathcal{X}_n \xrightarrow{\rho} \mathcal{X}_\infty$ . The space  $\mathcal{X}_\infty$  is called the limit space of the sequence  $(\mathcal{X}_n)$  along  $\rho$ .

Usually Gromov–Hausdorff convergence is defined differently. We prefer this definition since it induces convergence for a sequence of points  $x_n \in \mathcal{X}_n$  (Exercise 1.8.2), as well as weak convergence of measures  $\mu_n$  on  $\mathcal{X}_n$ , and so on, corresponding to convergence in the ambient space  $(\mathbf{X}, \rho)$ .

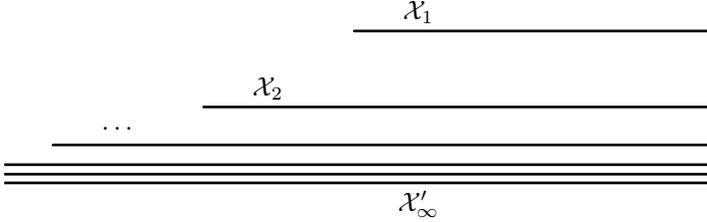
Once we write  $\mathcal{X}_n \rightarrow \mathcal{X}_\infty$ , we mean that we made a choice of convergence. Note that for a fixed sequence of metric spaces  $\mathcal{X}_1, \mathcal{X}_2, \dots$ , one may construct different Gromov–Hausdorff convergences, say  $\mathcal{X}_n \xrightarrow{\rho} \mathcal{X}_\infty$  and  $\mathcal{X}_n \xrightarrow{\rho'} \mathcal{X}'_\infty$ , whose limit spaces  $\mathcal{X}_\infty$  and  $\mathcal{X}'_\infty$  need not be isometric to each other.

For example, for the constant sequence  $\mathcal{X}_n \stackrel{\text{iso}}{=} \mathbb{R}_{\geq 0}$ , there is a convergence with limit  $\mathcal{X}_\infty \stackrel{\text{iso}}{=} \mathbb{R}_{\geq 0}$ ; guess the metric  $\rho$  from the diagram.

<sup>2</sup>Formally speaking, convergence is the topology induced by  $\rho$  on  $\mathbf{X}$ .



For another metric  $\rho'$  — also guess it from the diagram — the limit space  $\mathcal{X}'_\infty$  is isometric to the real line.



**1.9.3. Selection theorem.** *Let  $\mathcal{X}_n$  be a sequence of proper metric spaces with marked points  $x_n \in \mathcal{X}_n$ . Assume that for any fixed  $R, \varepsilon > 0$ , there is  $N = N(R, \varepsilon) \in \mathbb{N}$  such that for each  $n$  the ball  $\overline{B}[x_n, R]_{\mathcal{X}_n}$  admits a finite  $\varepsilon$ -net with at most  $N$  points. Then there is a subsequence of  $\mathcal{X}_n$  admitting a Gromov–Hausdorff convergence such that the sequence of marked points  $x_n \in \mathcal{X}_n$  converges.*

*Proof.* From the main assumption in the theorem, in each space  $\mathcal{X}_n$  there is a sequence of points  $z_{i,n} \in \mathcal{X}_n$  such that the following condition holds for a fixed sequence of integers  $M_1 < M_2 < \dots$

- ◇  $|z_{i,n} - x_n|_{\mathcal{X}_n} \leq k + 1$  if  $i \leq M_k$ ;
- ◇ the points  $z_{1,n}, \dots, z_{M_k,n}$  form an  $\frac{1}{k}$ -net in  $\overline{B}[x_n, k]_{\mathcal{X}_n}$ .

Passing to a subsequence, we can assume that the sequence

$$\ell_n = |z_{i,n} - z_{j,n}|_{\mathcal{X}_n}$$

converges for any  $i$  and  $j$ .

Consider a countable set of points  $\mathcal{W} = \{w_1, w_2, \dots\}$  equipped with the pseudometric defined as

$$|w_i - w_j|_{\mathcal{W}} = \lim_{n \rightarrow \infty} |z_{i,n} - z_{j,n}|_{\mathcal{X}_n}.$$

Let  $\hat{\mathcal{W}}$  be the metric space corresponding to  $\mathcal{W}$ ; that is, points in  $\hat{\mathcal{W}}$  are equivalence classes in  $\mathcal{W}$  for the relation  $\sim$ , where  $w_i \sim w_j$  if and only if  $|w_i - w_j|_{\mathcal{W}} = 0$ , and where

$$|[w_i] - [w_j]|_{\hat{\mathcal{W}}} \stackrel{\text{def}}{=} |w_i - w_j|_{\mathcal{W}}.$$

Denote by  $\mathcal{X}_\infty$  the completion of  $\hat{\mathcal{W}}$ .

It remains to show that there is a Gromov–Hausdorff convergence  $\mathcal{X}_n \rightarrow \mathcal{X}_\infty$  such that the sequence  $x_n \in \mathcal{X}_n$  converges. To prove it, we need to construct a metric  $\rho$  on the disjoint union of

$$\mathbf{X} = \mathcal{X}_\infty \sqcup \mathcal{X}_1 \sqcup \mathcal{X}_2 \sqcup \dots$$

satisfying definitions 1.9.1 and 1.9.2. The metric  $\rho$  can be constructed as the maximal compatible metric such that

$$\rho(z_{i,n}, w_i) \leq \frac{1}{m}$$

for any  $n \geq N_m$  and  $i < I_m$  for a suitable choice of two sequences  $I_m, N_m$  with  $I_1 = N_1 = 1$ .  $\square$

**1.9.4. Exercise.** Let  $\mathcal{X}_n$  be a sequence of metric spaces which admits two convergences  $\mathcal{X}_n \xrightarrow{\rho} \mathcal{X}_\infty$  and  $\mathcal{X}_n \xrightarrow{\rho'} \mathcal{X}'_\infty$ .

- a) If  $\mathcal{X}_\infty$  is compact, then  $\mathcal{X}_\infty \stackrel{iso}{=} \mathcal{X}'_\infty$ .
- b) If  $\mathcal{X}_\infty$  is proper and there is a sequence of points  $x_n \in \mathcal{X}_n$  which converges in both convergences, then  $\mathcal{X}_\infty \stackrel{iso}{=} \mathcal{X}'_\infty$ .



# Chapter 2

## Gluings theorem and billiards

In this chapter we define  $\text{CAT}[\kappa]$  spaces and give the first application, to billiards.

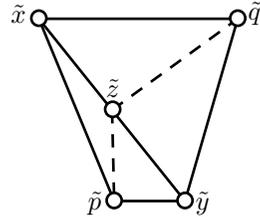
### 2.1 4-point condition

Given a quadruple of points  $p, q, x, y$  in a metric space  $\mathcal{X}$ , consider two model triangles in the plane  $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\Delta}(pxy)_{\mathbb{E}^2}$  and  $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\Delta}(qxy)_{\mathbb{E}^2}$  with common side  $[\tilde{x}\tilde{y}]$ .

If the inequality

$$|p - q|_{\mathcal{X}} \leq |\tilde{p} - \tilde{z}|_{\mathbb{E}^2} + |\tilde{z} - \tilde{q}|_{\mathbb{E}^2}$$

holds for any point  $\tilde{z} \in [\tilde{x}\tilde{y}]$  then we say that the quadruple  $p, q, x, y$  satisfies  $\text{CAT}[0]$  comparison.



If we do the same for spherical model triangles  $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\Delta}(pxy)_{\mathbb{S}^2}$  and  $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\Delta}(qxy)_{\mathbb{S}^2}$ , then we arrive at the definition of  $\text{CAT}[1]$  comparison. If one of the spherical model triangles is undefined<sup>1</sup> then it is assumed that  $\text{CAT}[1]$  comparison automatically holds for this quadruple.

We can do the same for the model plane with curvature  $\kappa$ ; that is a sphere if  $\kappa > 0$ , Euclidean plane if  $\kappa = 0$  and Lobachevsky plane if

<sup>1</sup> that is, if

$$|p - x| + |p - y| + |x - y| \geq 2 \cdot \pi \quad \text{or} \quad |q - x| + |q - y| + |x - y| \geq 2 \cdot \pi.$$

$\kappa < 0$ . In this case we arrive at the definition of  $\text{CAT}[\kappa]$  comparison. However in these notes we will consider mostly  $\text{CAT}[0]$  comparison and occasionally  $\text{CAT}[1]$  comparison; so, if you see  $\text{CAT}[\kappa]$ , you can assume that  $\kappa$  is 0 or 1.

If all quadruples in a metric space  $\mathcal{X}$  satisfy  $\text{CAT}[\kappa]$  comparison then we say that the space  $\mathcal{X}$  is  $\text{CAT}[\kappa]$ . (Note that  $\text{CAT}[\kappa]$  is an adjective.)

Note that in order to check  $\text{CAT}[\kappa]$  comparison, it is sufficient to know 6 distances between all the pairs of points in the quadruple. The latter observation implies the following.

**2.1.1. Proposition.** *The Gromov–Hausdorff limit of a sequence of  $\text{CAT}[\kappa]$  spaces is  $\text{CAT}[\kappa]$ .*

In the proposition above, it does not matter which definition of convergence for metric spaces you use, as long as any quadruple of points in the limit space can be arbitrarily well approximated by quadruples in the sequence of metric spaces.

**2.1.2. Exercise.** *Let  $\mathcal{V}$  be a metric space and  $\mathcal{U} = \text{Cone } \mathcal{V}$ . Show that  $\mathcal{U}$  is  $\text{CAT}[0]$  if and only if  $\mathcal{V}$  is  $\text{CAT}[1]$ .*

*Analogously, if  $\mathcal{U} = \text{Susp } \mathcal{V}$  then  $\mathcal{U}$  is  $\text{CAT}[1]$  if and only if  $\mathcal{V}$  is  $\text{CAT}[1]$ .*

The cone and suspension constructions are defined in Section 1.2.

The following exercise is bit simpler, but can be proved in essentially the same way.

**2.1.3. Exercise.** *Assume  $\mathcal{U}$  and  $\mathcal{V}$  are  $\text{CAT}[0]$  spaces. Show that the product space  $\mathcal{U} \times \mathcal{V}$  is  $\text{CAT}[0]$ .*

**2.1.4. Exercise.** *Show that any complete length  $\text{CAT}[0]$  space is geodesic.*

## 2.2 Thin triangles

The inheritance lemma 2.2.9 proved below plays a central role in the theory. It will lead to two fundamental constructions: patchwork globalization (3.3.2) and Reshetnyak gluing (2.3.1), which in turn are used to prove the globalization theorem (3.3.1).

Recall that a *triangle*  $[x^1 x^2 x^3]$  in a space  $\mathcal{X}$  is a triple of minimizing geodesics  $[x^1 x^2]$ ,  $[x^2 x^3]$  and  $[x^3 x^1]$ . Consider the model triangle  $[\tilde{x}^1 \tilde{x}^2 \tilde{x}^3] = \tilde{\Delta}(x^1 x^2 x^3)_{\mathbb{E}^2}$  in the Euclidean plane. The *natural map*  $[\tilde{x}^1 \tilde{x}^2 \tilde{x}^3] \rightarrow [x^1 x^2 x^3]$  sends a point  $\tilde{z} \in [\tilde{x}^i \tilde{x}^j]$  to the corresponding

point  $z \in [x^i x^j]$ ; that is,  $z$  is the point such that  $|\tilde{x}^i - \tilde{z}| = |x^i - z|$  and therefore  $|\tilde{x}^j - \tilde{z}| = |x^j - z|$ .

In the same way, a natural map can be defined by the spherical model triangle  $\tilde{\Delta}(x^1 x^2 x^3)_{\mathbb{S}^2}$ .

**2.2.1. Definition of thin triangles.** A triangle  $[x^1 x^2 x^3]$  in the metric space  $\mathcal{X}$  is called thin if the natural map  $\tilde{\Delta}(x^1 x^2 x^3)_{\mathbb{E}^2} \rightarrow [x^1 x^2 x^3]$  is short (that is, a distance nonincreasing map).

Analogously, triangle  $[x^1 x^2 x^3]$  is called spherically thin if the natural map from the spherical model triangle  $\tilde{\Delta}(x^1 x^2 x^3)_{\mathbb{S}^2}$  to  $[x^1 x^2 x^3]$  is short.

**2.2.2. Proposition.** A geodesic space is CAT[0] (respectively CAT[1]) if and only if all its triangles are thin (respectively, all its triangles of perimeter  $< 2 \cdot \pi$  are spherically thin).

*Proof; “If” part.* Apply the triangle inequality and thinness of triangles  $[pxy]$  and  $[qxy]$ , where  $p, q, x$  and  $y$  are as in the definition of CAT[ $\kappa$ ] comparison (page 21).

*“Only-if” part.* Applying CAT[0] comparison to a quadruple  $p, q, x, y$  with  $q \in [xy]$  shows that any triangle satisfies *point-side comparison*, that is, the distance from a vertex to a point on the opposite side is no greater than the corresponding distance in the Euclidean model triangle.

Now consider triangle  $[x^1 x^2 x^3]$ , with  $y \in [x^1 x^2]$  and  $z \in [x^1 x^3]$ . Let  $\tilde{y}, \tilde{z}$  be the corresponding points on the sides of the model triangle  $\tilde{\Delta}(x^1 x^2 x^3)_{\mathbb{E}^2}$ . Applying point-side comparison first to triangle  $[x^1 x^2 x^3]$  with  $y \in [x^1 x^2]$ , and then to triangle  $[x^1 y x^3]$  with  $z \in [x^1 x^3]$ , implies that model angles satisfy  $\angle(x^1 x^2 x^3)_{\mathbb{E}^2} \geq \angle(x^1 y x^3)_{\mathbb{E}^2} \geq \angle(x^1 y z)_{\mathbb{E}^2}$ . Therefore  $|\tilde{y} - \tilde{z}|_{\mathbb{E}^2} \geq |y - z|$ .

The CAT[1] argument is the same. □

**2.2.3. Uniqueness of geodesics.** In a CAT[0] proper length space, pairs of points are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.

Analogously, in a CAT[1] proper length space, pairs of points at distance less than  $\pi$  are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.

*Proof.* Given 4 points  $p^1, p^2, q^1, q^2$  in a CAT[0] proper length space  $\mathcal{U}$ , consider two triangles  $[p^1 q^1 p^2]$  and  $[p^2 q^2 q^1]$ . Since both these triangles are thin, we get

$$|\text{path}_{[p^1 q^1]}(t) - \text{path}_{[p^2 q^1]}(t)|_{\mathcal{U}} \leq (1-t) \cdot |p^1 - p^2|_{\mathcal{U}},$$

$$|\text{path}_{[p^2 q^1]}(t) - \text{path}_{[p^2 q^2]}(t)|_{\mathcal{U}} \leq t \cdot |q^1 - q^2|_{\mathcal{U}}.$$

It follows that

$$|\text{path}_{[p^1 q^1]}(t) - \text{path}_{[p^2 q^2]}(t)|_{\mathcal{U}} \leq \max\{|p^1 - p^2|_{\mathcal{U}}, |q^1 - q^2|_{\mathcal{U}}\}.$$

Hence continuity and uniqueness in the CAT[0] case. The CAT[1] case is done essentially the same way.  $\square$

Adding the first two inequalities of the preceding proof gives:

**2.2.4. Proposition.** *Suppose  $p^1, p^2, q^1, q^2$  are points in a CAT[0] proper length space  $\mathcal{U}$ . Then*

$$|\text{path}_{[p^1 q^1]}(t) - \text{path}_{[p^2 q^2]}(t)|_{\mathcal{U}}$$

*is a convex function.*

**2.2.5. Corollary.** *Let  $K$  be a closed convex subset in a CAT[0] proper length space  $\mathcal{U}$ . Then  $\text{dist}_K$  is convex on  $\mathcal{U}$ .*

*In particular,  $\text{dist}_p$  is convex for any point  $p$  in  $\mathcal{U}$ .*

**2.2.6. Corollary.** *Any CAT[0] proper length space is contractible.*

*Analogously, any CAT[1] proper length space with diameter  $< \pi$  is contractible.*

*Proof.* Let  $\mathcal{U}$  be a CAT[0] proper length space. Fix a point  $p \in \mathcal{U}$ .

For each point  $x$  consider the geodesic path  $\gamma_x: [0, 1] \rightarrow \mathcal{U}$  from  $p$  to  $x$ . Consider the map  $h_t: x \mapsto \gamma_x(t)$  for each  $t$ . By uniqueness of geodesics (2.2.3) the map  $(t, x) \mapsto h_t(x)$  is continuous.

It remains to note that  $h_1(x) = x$  and  $h_0(x) = p$  for any  $x$ .

The proof of the CAT[1] case is identical.  $\square$

**2.2.7. Proposition.** *Suppose  $\mathcal{U}$  is a CAT[0] proper length space. Then any local geodesic in  $\mathcal{U}$  is a geodesic.*

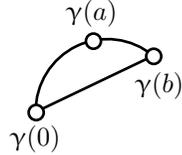
*Analogously, if  $\mathcal{U}$  is a CAT[1] proper length space then any local geodesic with length  $< \pi$  in  $\mathcal{U}$  is a geodesic.*

*Proof.* Suppose  $\gamma: [0, \ell] \rightarrow \mathcal{U}$  is a local geodesic that is not a geodesic. Choose  $a$  to be the maximal value such that  $\gamma$  is a geodesic on  $[0, a]$ . Further choose  $b > a$  so that  $\gamma$  is a geodesic on  $[a, b]$ .

Since the triangle  $[\gamma(0) \gamma(a) \gamma(b)]$  is thin, we have

$$|\gamma(a - \varepsilon) - \gamma(a + \varepsilon)| < 2 \cdot \varepsilon$$

for all small  $\varepsilon > 0$ . That is,  $\gamma$  is not length-minimizing on the interval  $[a - \varepsilon, a + \varepsilon]$  for any  $\varepsilon > 0$ , a contradiction.



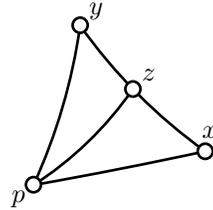
The spherical case is done in the same way.  $\square$

**2.2.8. Exercise.** Assume  $\mathcal{U}$  is a  $\text{CAT}[\kappa]$  proper length space with extendable geodesics; that is, any geodesic in  $\mathcal{U}$  is an arc in an infinite local geodesic defined on  $\mathbb{R}$ .

Show that the space of geodesic directions at any point in  $\mathcal{U}$  is complete.

Now let us formulate the main result of this section.

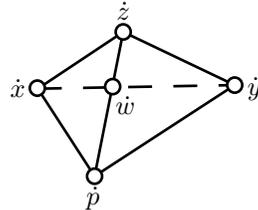
**2.2.9. Inheritance lemma.** Assume that a triangle  $[pxy]$  in a metric space is decomposed into two triangles  $[pxz]$  and  $[pyz]$ ; that is,  $[pxz]$  and  $[pyz]$  have a common side  $[pz]$ , and the sides  $[xz]$  and  $[zy]$  together form the side  $[xy]$  of  $[pxy]$ .



If both triangles  $[pxz]$  and  $[pyz]$  are thin, then the triangle  $[pxy]$  is also thin.

Analogously, if  $[pxy]$  has perimeter  $< 2 \cdot \pi$  and both triangles  $[pxz]$  and  $[pyz]$  are spherically thin, then triangle  $[pxy]$  is spherically thin.

*Proof.* Construct model triangles  $[\dot{p}\dot{x}\dot{z}] = \tilde{\Delta}(pxz)_{\mathbb{E}^2}$  and  $[\dot{p}\dot{y}\dot{z}] = \tilde{\Delta}(pyz)_{\mathbb{E}^2}$  so that  $\dot{x}$  and  $\dot{y}$  lie on opposite sides of  $[\dot{p}\dot{z}]$ .



Let us show that

❶  $\chi(z^p_x) + \chi(z^p_y) \geq \pi.$

Suppose the contrary, that is

$$\chi(z^p_x) + \chi(z^p_y) < \pi.$$

Then for some point  $\dot{w} \in [\dot{p}\dot{z}]$ , we have

$$|\dot{x} - \dot{w}| + |\dot{w} - \dot{y}| < |\dot{x} - \dot{z}| + |\dot{z} - \dot{y}| = |\dot{x} - \dot{y}|.$$

Let  $w \in [pz]$  correspond to  $\dot{w}$ ; that is  $|z - w| = |\dot{z} - \dot{w}|$ . Since  $[pxz]$  and  $[pyz]$  are thin, we have

$$|x - w| + |w - y| < |x - y|,$$

contradicting the triangle inequality.

Let us denote by  $\dot{D}$  the union of two solid triangles  $[\dot{p}\dot{x}\dot{z}]$  and  $[\dot{p}\dot{y}\dot{z}]$ . By ❶, there is a short map  $F$  that sends  $[\tilde{p}\tilde{x}\tilde{y}]$  to  $\dot{D}$  in such a way that

$$\tilde{p} \mapsto \dot{p}, \quad \tilde{x} \mapsto \dot{x}, \quad \tilde{z} \mapsto \dot{z}, \quad \tilde{y} \mapsto \dot{y}.$$

**2.2.10. Exercise.** Use Alexandrov's lemma (1.5.1) to prove the last statement.

By assumption, the natural maps  $[\dot{p}\dot{x}\dot{z}] \rightarrow [pxz]$  and  $[\dot{p}\dot{y}\dot{z}] \rightarrow [pyz]$  are short. By composition, the natural map from  $[\tilde{p}\tilde{x}\tilde{y}]$  to  $[pyz]$  is short, as claimed.

The spherical case is done along the same lines.  $\square$

**2.2.11. Exercise.** Show that any ball in a CAT[0] proper length space is a convex set.

Analogously, show that any ball of radius  $R < \frac{\pi}{2}$  in a CAT[1] proper length space is a convex set.

Recall that a set  $A$  in a metric space  $\mathcal{U}$  is called locally convex if for any point  $p \in A$  there is a open neighborhood  $\mathcal{U} \ni p$  such that any geodesic in  $\mathcal{U}$  with ends in  $A$  lies in  $A$ .

**2.2.12. Exercise.** Show that in any CAT[0] proper length space, any closed, connected, locally convex set is convex.

**2.2.13. Exercise.** Let  $\mathcal{U}$  be a CAT[0] proper length space and  $K \subset \mathcal{U}$  be a closed convex set. Show that

- a) For each point  $p \in \mathcal{U}$  there is unique point  $p^* \in K$  that minimizes the distance  $|p - p^*|$ .
- b) The closest-point projection  $p \mapsto p^*$  defined by (a) is short.

## 2.3 Reshetnyak's gluing theorem

Suppose  $\mathcal{U}^1$  and  $\mathcal{U}^2$  are proper length spaces with isometric closed convex sets  $A^i \subset \mathcal{U}^i$  and  $\iota: A^1 \rightarrow A^2$  be an isometry. Consider the space  $\mathcal{W}$  of all equivalence classes on  $\mathcal{U}^1 \sqcup \mathcal{U}^2$  with the equivalence relation given by  $a \sim \iota(a)$  for any  $a \in A^1$ .

It is straightforward to see that  $\mathcal{W}$  is a proper length space if equipped with the following metric

$$\begin{aligned} |x - y|_{\mathcal{W}} &= |x - y|_{\mathcal{U}^i} \\ &\quad \text{if } x, y \in \mathcal{U}^i, \text{ and} \\ |x - y|_{\mathcal{W}} &= \min \{ |x - a|_{\mathcal{U}^1} + |y - \iota(a)|_{\mathcal{U}^2} \mid a \in A^1 \} \\ &\quad \text{if } x \in \mathcal{U}^1 \text{ and } y \in \mathcal{U}^2. \end{aligned}$$

Abusing notation, we denote by  $x$  and  $y$  the points in  $\mathcal{U}^1 \sqcup \mathcal{U}^2$  and their equivalence classes in  $\mathcal{U}^1 \sqcup \mathcal{U}^2 / \sim$ .

The space  $\mathcal{W}$  is called the *gluing* of  $\mathcal{U}^1$  and  $\mathcal{U}^2$  along  $\iota$ . If one applies this construction to two copies of one space  $\mathcal{U}$  with a set  $A \subset \mathcal{U}$  and identity map  $\iota: A \rightarrow A$ , then the obtained space is called the *doubling* of  $\mathcal{U}$  along  $A$ .

We can (and will) identify  $\mathcal{U}^i$  with its image in  $\mathcal{W}$ ; this way both subsets  $A^i \subset \mathcal{U}^i$  will be identified and denoted further by  $A$ . Note that  $A = \mathcal{U}^1 \cap \mathcal{U}^2 \subset \mathcal{W}$ , therefore  $A$  is also a convex set in  $\mathcal{W}$ .

The following theorem was proved by Yuri Reshetnyak in [47].

**2.3.1. Reshetnyak gluing.** *Suppose  $\mathcal{U}^1$  and  $\mathcal{U}^2$  are CAT[0] proper length spaces with isometric closed convex sets  $A^i \subset \mathcal{U}^i$ , and  $\iota: A^1 \rightarrow A^2$  is an isometry. Then the gluing of  $\mathcal{U}^1$  and  $\mathcal{U}^2$  along  $\iota$  is a CAT[0] proper length space.*

*Proof.* By construction of the gluing space, the statement can be reformulated in the following way.

**2.3.2. Reformulation of 2.3.1.** *Let  $\mathcal{W}$  be a proper length space which has two closed convex sets  $\mathcal{U}^1, \mathcal{U}^2 \subset \mathcal{W}$  such that  $\mathcal{U}^1 \cup \mathcal{U}^2 = \mathcal{W}$  and  $\mathcal{U}^1, \mathcal{U}^2$  are CAT[0]. Then  $\mathcal{W}$  is a CAT[0] space.*

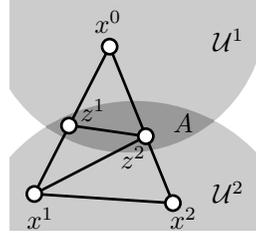
It suffices to show that any triangle  $[x^0 x^1 x^2]$  in  $\mathcal{W}$  is thin. This is obviously true if all three points  $x^0, x^1, x^2$  lie in one of  $\mathcal{U}^i$ . Thus, without loss of generality, we may assume that  $x^0 \in \mathcal{U}^1$  and  $x^1, x^2 \in \mathcal{U}^2$ .

Choose points  $z^1, z^2 \in A = \mathcal{U}^1 \cap \mathcal{U}^2$  that lie respectively on the sides  $[x^0 x^1], [x^0 x^2]$ . Note that

- ◊ triangle  $[x^0 z^1 z^2]$  lies in  $\mathcal{U}^1$ ,
- ◊ both triangles  $[x^1 z^1 z^2]$  and  $[x^1 z^2 x^2]$  lie in  $\mathcal{U}^2$ .

In particular each triangle  $[x^0 z^1 z^2]$ ,  $[x^1 z^1 z^2]$  and  $[x^1 z^2 x^2]$  is thin.

Applying the inheritance lemma for thin triangles (2.2.9) twice we get that  $[x^0 x^1 z^2]$  and consequently  $[x^0 x^1 x^2]$  is thin.  $\square$

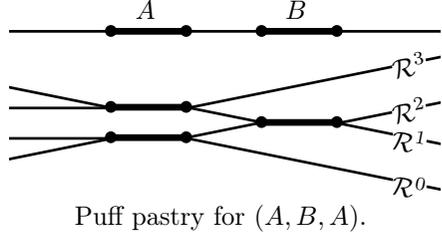


## 2.4 Reshetnyak's puff pastry

In this section we introduce Reshetnyak's puff pastry, a construction which will be used in the next section to prove the collision theorem (2.5.1).

Let  $\mathbf{A} = (A^1, \dots, A^N)$  be an array of convex closed sets in the Euclidean space  $\mathbb{E}^m$ . Consider an array of  $N+1$  copies of  $\mathbb{E}^m$ . Assume that the space  $\mathcal{R}$  is obtained by gluing successive pairs of spaces along  $A^1, \dots, A^N$  respectively.

The resulting space  $\mathcal{R}$  will be called *Reshetnyak's puff pastry* for the array  $A$ . The copies of  $\mathbb{E}^m$  in the puff pastry  $\mathcal{R}$  will be called *levels*; they will be denoted by  $\mathcal{R}^0, \dots, \mathcal{R}^N$ . The point in the  $k$ -th level  $\mathcal{R}^k$  which corresponds to  $x \in \mathbb{E}^m$  will be denoted by  $x^k$ .



Given  $x \in \mathbb{E}^m$ , any point  $x^k \in \mathcal{R}$  is called a *lifting* of  $x$ . The map  $x \mapsto x^k$  defines an isometry  $\mathbb{E}^m \rightarrow \mathcal{R}^k$  in particular we can talk about liftings of subsets in  $\mathbb{E}^m$ .

Note that

- ◊ The intersection  $A^1 \cap \dots \cap A^N$  admits a unique lifting in  $\mathcal{R}$ .
- ◊ Moreover,  $x^i = x^j$  for some  $i < j$  if and only if

$$x \in A^{i+1} \cap \dots \cap A^j.$$

- ◊ The restriction  $\mathcal{R}^k \rightarrow \mathbb{E}^m$  of the natural projection  $x^k \mapsto x$  is an isometry.

**2.4.1. Observation.** *Any Reshetnyak's puff pastry is a CAT[0] proper length space.*

*Proof.* Apply Reshetnyak's gluing theorem (2.3.1) recursively for the convex sets in the array. □

**2.4.2. Proposition.** *Assume  $(A^1, \dots, A^N)$  and  $(\check{A}^1, \dots, \check{A}^N)$  are two arrays of convex closed sets in  $\mathbb{E}^m$  such that  $A^k \subset \check{A}^k$  for each  $k$ . Let  $\mathcal{R}$  and  $\check{\mathcal{R}}$  be the corresponding Reshetnyak's puff pastries. Then the map  $\mathcal{R} \rightarrow \check{\mathcal{R}}$  defined as  $x^k \mapsto \check{x}^k$  is short.*

Moreover, if

$$|x^i - y^j|_{\mathcal{R}} = |\check{x}^i - \check{y}^j|_{\check{\mathcal{R}}}$$

for some  $x, y \in \mathbb{E}^m$  and  $i, j \in \{0, \dots, n\}$ , then the unique geodesic  $[\check{x}^i \check{y}^j]_{\check{\mathcal{R}}}$  is the image of the unique geodesic  $[x^i y^j]_{\mathcal{R}}$  under the map  $x^i \mapsto \check{x}^i$ .

*Proof.* The first statement in the proposition follows from the construction of Reshetnyak's puff pastries.

By Observation 2.4.1,  $\mathcal{R}$  and  $\check{\mathcal{R}}$  are CAT[0] proper length spaces, hence  $[x^i y^j]_{\mathcal{R}}$  and  $[\check{x}^i \check{y}^j]_{\check{\mathcal{R}}}$  are unique. Since the map  $\mathcal{R} \rightarrow \check{\mathcal{R}}$  is short, the image of  $[x^i y^j]_{\mathcal{R}}$  is a geodesic of  $\check{\mathcal{R}}$  joining  $\check{x}^i$  to  $\check{y}^j$ . Hence the second statement follows. □

**2.4.3. Definition.** Consider a Reshetnyak's puff pastry  $\mathcal{R}$  with the levels  $\mathcal{R}^0, \dots, \mathcal{R}^N$ . We say that  $\mathcal{R}$  is end-to-end convex if  $\mathcal{R}^0 \cup \mathcal{R}^N$ , the union of its lower and upper levels, forms a convex set in  $\mathbb{R}$ .

If  $\mathcal{R}$  is the Reshetnyak's puff pastry for an array of convex sets  $\mathbf{A} = (A^1, \dots, A^N)$  then  $\mathcal{R}$  is end-to-end convex if and only if the union of the lower and the upper levels  $\mathcal{R}^0 \cup \mathcal{R}^N$  is isometric to the doubling of  $\mathbb{E}^m$  along the nonempty intersection  $A^1 \cap \dots \cap A^N$ .

**2.4.4. Observation.** Let  $\check{\mathbf{A}}$  and  $\mathbf{A}$  be arrays of convex bodies in  $\mathbb{E}^m$ . Assume the array  $\mathbf{A}$  is obtained by inserting in  $\check{\mathbf{A}}$  several copies of the bodies which were already listed in  $\check{\mathbf{A}}$ .

For example, if  $\check{\mathbf{A}} = (A, C, B, C, A)$ , by placing  $B$  in the second place and  $A$  in the fourth place, we obtain  $\mathbf{A} = (A, B, C, A, B, C, A)$ .

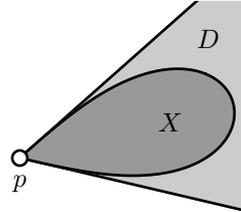
Denote by  $\check{\mathcal{R}}$  and  $\mathcal{R}$  the Reshetnyak's puff pastries for  $\check{\mathbf{A}}$  and  $\mathbf{A}$  respectively.

If  $\check{\mathcal{R}}$  is end-to-end convex then so is  $\mathcal{R}$ .

*Proof.* Without loss of generality we can assume that  $\mathbf{A}$  is obtained by inserting one element in  $\check{\mathbf{A}}$ , say at the place number  $k$ .

Note that  $\check{\mathcal{R}}$  is isometric to the puff pastry for  $\mathbf{A}$  with  $A^k$  replaced by  $\mathbb{E}^m$ . It remains to apply Proposition 2.4.2. □

Let  $X$  be a convex set in a Euclidean space. By a *dihedral angle* we understand an intersection of two half-spaces; the intersection of corresponding hyperplanes is called the *edge* of the angle. We say that a dihedral angle  $D$  is supporting  $X$  at a point  $p \in X$  if  $D$  contains  $X$  and the edge of  $D$  contains  $p$ .



**2.4.5. Lemma.** Let  $A$  and  $B$  be two convex sets in  $\mathbb{E}^m$ . Assume that any dihedral angle supporting  $A \cap B$  at some point has angle measure at least  $\alpha$ . Then the Reshetnyak's puff pastry for the array

$$\underbrace{(A, B, A, \dots)}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \text{ times}}$$

is end-to-end convex.

The proof of the lemma is based on a partial case which we formulate as a sublemma.

**2.4.6. Sublemma.** Let  $\check{A}$  and  $\check{B}$  be two half-planes in  $\mathbb{E}^2$ , where  $\check{A} \cap \check{B}$  is an angle with measure  $\alpha$ . Then the Reshetnyak's puff pastry for the array

$$\underbrace{(\check{A}, \check{B}, \check{A}, \dots)}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \text{ times}}$$

is end-to-end convex.

*Proof.* Note that the puff pastry  $\check{\mathcal{R}}$  is isometric to the cone over the space glued from the unit circles as shown on the diagram.

All the short arcs on the diagram have length  $\alpha$ ; the long arcs have length  $\pi - \alpha$ , so making a circuit along any path will take  $2 \cdot \pi$ .

Observe that end-to-end convexity of  $\check{\mathcal{R}}$  is equivalent to the fact that any geodesic shorter than  $\pi$  with the ends on the inner and the outer circles lies completely in the union of these two circles.

The latter holds if the zigzag line has length at least  $\pi$ . This line is formed by  $\lceil \frac{\pi}{\alpha} \rceil$  arcs with length  $\alpha$  each. Hence the sublemma follows.  $\square$

*Proof of Lemma 2.4.5.* Fix arbitrary  $x, y \in \mathbb{E}^m$ . Choose a point  $z \in A \cap B$  for which the sum

$$|x - z| + |y - z|$$

is minimal. To show the end-to-end convexity of  $\mathcal{R}$ , it is sufficient to prove the following:

- ❶ The geodesic  $[x^0 y^N]_{\mathcal{R}}$  passes through  $z^0 = z^N \in \mathcal{R}$ .

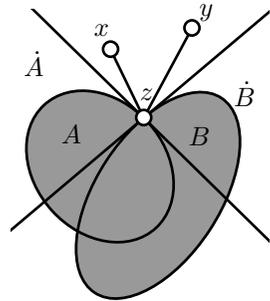
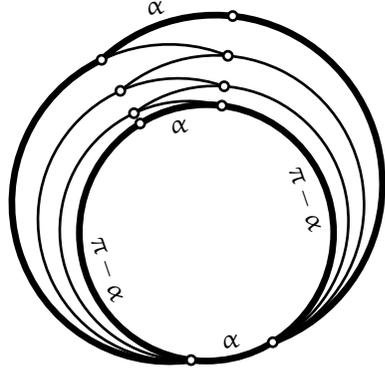
Without loss of generality we can assume that  $z \in \partial A \cap \partial B$ . Indeed, since the puff pastry for 1-array ( $B$ ) is end-to-end convex, Proposition 2.4.2 together with Observation 2.4.4 imply ❶ in case  $z$  lies in the interior of  $A$ . In the same way we can treat the case when  $z$  lies in the interior of  $B$ .

Further we will use the following exercise in convex geometry

**2.4.7. Exercise.** Let  $A$  and  $B$  be two closed convex sets in  $\mathbb{E}^m$  and  $A \cap B \neq \emptyset$ . Given two points  $x, y \in \mathbb{E}^m$  let  $f(z) = |x - z| + |y - z|$ .

Let  $z_0 \in A \cap B$  be a point of minimum of  $f|_{A \cap B}$ .

Show that there are half-spaces  $\dot{A}$  and  $\dot{B}$  such that  $\dot{A} \supset A$  and  $\dot{B} \supset B$  and  $z_0$  is also a point of minimum of  $f|_{\dot{A} \cap \dot{B}}$ .



Note that the angle measure of  $\dot{A} \cap \dot{B}$  is at least  $\alpha$ .

Note that  $\mathbb{E}^m$  admits isometric splitting  $\mathbb{E}^{m-2} \times \mathbb{E}^2$  such that

$$\begin{aligned}\dot{A} &= \mathbb{E}^{m-2} \times \dot{\ddot{A}} \\ \dot{B} &= \mathbb{E}^{m-2} \times \dot{\ddot{B}}\end{aligned}$$

where  $\ddot{A}$  and  $\ddot{B}$  are half-planes in  $\mathbb{E}^2$ .

Using Exercise 2.4.7 let us replace each  $A$  by  $\dot{A}$  and each  $B$  by  $\dot{B}$  in the array, to get the array

$$\underbrace{(\dot{A}, \dot{B}, \dot{A}, \dots)}_{\lceil \frac{\alpha}{\alpha} \rceil + 1 \text{ times}}$$

The corresponding puff pastry  $\dot{\mathcal{R}}$  splits as a product of  $\mathbb{E}^{m-2}$  and a puff pastry, call it  $\ddot{\mathcal{R}}$ , glued from the copies of the plane  $\mathbb{E}^2$  for the array

$$\underbrace{(\ddot{A}, \ddot{B}, \ddot{A}, \dots)}_{\lceil \frac{\alpha}{\alpha} \rceil + 1 \text{ times}}$$

According to Sublemma 2.4.6 and Observation 2.4.4,  $\ddot{\mathcal{R}}$  is end-to-end convex.

Since  $\mathcal{R} \stackrel{iso}{=} \mathbb{E}^{m-2} \times \ddot{\mathcal{R}}$ , the puff pastry  $\dot{\mathcal{R}}$  is also end-to-end convex.

It follows that the geodesic  $[x^0 y^N]_{\dot{\mathcal{R}}}$  passes through  $\dot{z}^0 = \dot{z}^N \in \dot{\mathcal{R}}$ . By Proposition 2.4.2, the image of  $[x^0 y^N]_{\dot{\mathcal{R}}}$  under the map  $\dot{x}^k \mapsto x^k$  is the geodesic  $[x^0 y^N]_{\mathcal{R}}$ . Hence Claim **1** and the Lemma follow.  $\square$

The proof of the following proposition is based on Lemma 2.4.5, which is essentially the case  $n = 2$ .

We say that a closed convex set  $A \subset \mathbb{E}^m$  has  $\varepsilon$ -wide corners for given  $\varepsilon > 0$  if together with each point  $p$  the set  $A$  contains a small right circular cone with tip at  $p$  and aperture  $\varepsilon$ ; that is,  $\varepsilon$  is the maximum angle between two generating lines of the cone.

For example, a plane polygon has  $\varepsilon$ -wide corners if all its interior angles are at least  $\varepsilon$ .

We will consider finite collections of closed convex sets  $A^1, \dots, A^n \subset \mathbb{E}^m$  such that for any subset  $F \subset \{1, \dots, n\}$  the intersection  $\bigcap_{i \in F} A^i$  has  $\varepsilon$ -wide corners. In this case we may say briefly *all intersections of  $A^i$  have  $\varepsilon$ -wide corners*.

**2.4.8. Exercise.** *Assume  $A^1, \dots, A^n \subset \mathbb{E}^m$  are compact, convex sets with a common interior point. Show that all intersections of  $A^i$  have  $\varepsilon$ -wide corners for some positive  $\varepsilon$ .*

**2.4.9. Exercise.** Assume  $A^1, \dots, A^n \subset \mathbb{E}^m$  are convex sets with nonempty interior which have a common center of symmetry. Show that all intersections of an arbitrary number of sets  $A^i$  have  $\varepsilon$ -wide corners for some positive  $\varepsilon$ .

**2.4.10. Proposition.** Given  $\varepsilon > 0$  and a positive integer  $n$  there is an array of integers  $\mathbf{j}_\varepsilon(n) = (j_1, \dots, j_N)$  such that

- a) For each  $k$  we have  $1 \leq j_k \leq n$  and each number  $1, \dots, n$  appears in  $\mathbf{j}_\varepsilon$  at least once.
- b) If  $A^1, \dots, A^n$  is a collection of closed convex sets in  $\mathbb{E}^m$  with a common point and all their intersections have  $\varepsilon$ -wide corners then the puff pastry for the array  $(A^{j_1}, \dots, A^{j_N})$  is end-to-end convex.

Moreover we can assume that  $N \leq (\lceil \frac{n}{\varepsilon} \rceil + 1)^n$ .

*Proof.* The array  $\mathbf{j}_\varepsilon(n) = (j_1, \dots, j_N)$  is constructed recursively. For  $n = 1$ , the array is (1).

Assume that  $\mathbf{j}_\varepsilon(n)$  is constructed. Let us exchange each occurrence of  $n$  in  $\mathbf{j}_\varepsilon(n)$  by the alternating string

$$\underbrace{n, n+1, n, \dots}_{\lceil \frac{n}{\varepsilon} \rceil + 1 \text{ times}}$$

Denote the obtained array by  $\mathbf{j}_\varepsilon(n+1)$ .

By Lemma 2.4.5, end-to-end convexity of the puff pastry for  $\mathbf{j}_\varepsilon(n+1)$  follows from end-to-end convexity of the puff pastry for the array where each string

$$\underbrace{A^n, A^{n+1}, A^n, \dots}_{\lceil \frac{n}{\varepsilon} \rceil + 1 \text{ times}}$$

is exchanged to  $Q = A^n \cap A^{n+1}$ . The latter follows by assumption on  $\mathbf{j}_\varepsilon(n)$ , since all the intersections of  $A^1, \dots, A^{n-1}, Q$  have  $\varepsilon$ -wide corners.

The upper bound on  $N$  follows directly from the construction.  $\square$

## 2.5 Billiards

Let  $A^1, A^2, \dots, A^n$  be a finite collection of closed convex sets in  $\mathbb{E}^m$ . Assume that for each  $i$  the boundary  $\partial A^i$  is a smooth hypersurface.

Consider the billiard table formed by the closure of the complement

$$T = \overline{\mathbb{E}^m \setminus \bigcup_i A^i}.$$

The sets  $A^i$  will be called *walls* of the table  $T$  and the billiards described above will be called *billiards with convex walls*.

A *billiard trajectory* on the table  $T$  is a unit-speed broken line  $\gamma$  which follows the standard law of billiards at the break points on  $\partial A^i$  — in particular, the angle of reflection is equal to the angle of incidence. The break points of the trajectory will be called *collisions*. We assume the trajectory meets only one wall at a time.

Recall that the definition of sets with  $\varepsilon$ -wide corners is given on page 31.

**2.5.1. Collision theorem.** *Assume  $T \subset \mathbb{E}^m$  is a billiard table with convex walls. Assume that the walls of  $T$  have common interior point and all their intersections have  $\varepsilon$ -wide corners. Then the number of collisions of any trajectory in  $T$  is bounded by a number  $N$  which depends only on the number  $n$  of walls and  $\varepsilon$ .*

As we will see from the proof, the value  $N$  can be found explicitly, say one can take

$$N = (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^{n^2}.$$

The collision theorem was proved by Dmitri Burago, Serge Ferleger and Alexey Kononenko in [18]; we present their proof with minor improvements.

Let us formulate and prove a corollary of the collision theorem:

**2.5.2. Corollary.** *Consider  $n$  homogeneous hard balls moving freely and colliding elastically in  $\mathbb{R}^3$ . Every ball moves along a straight line with constant speed until two balls collide, and then the new velocities of the two balls are determined by the laws of classical mechanics. We assume that only two balls can collide at the same time.*

*Then the total number of collisions cannot exceed some number  $N$  which depends on the radii and masses of the balls. If the balls are identical then  $N$  depends only on  $n$ .*

The proof below admits a straightforward generalization to all dimensions.

*Proof.* Denote by  $a_i = (x_i, y_i, z_i) \in \mathbb{R}^3$  the center of the  $i$ -th ball. Consider the corresponding point in  $\mathbb{R}^{3 \cdot n}$

$$\begin{aligned} \mathbf{a} &= (a_1, a_2, \dots, a_n) = \\ &= (x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_n, y_n, z_n). \end{aligned}$$

The  $i$ -th and  $j$ -th ball intersect if

$$|a_i - a_j| \leq R_i + R_j,$$

where  $R_i$  denoted the radius of the  $i$ -th ball. These inequalities define  $\frac{n \cdot (n-1)}{2}$  cylinders

$$C_{i,j} = \{ (a_1, a_2, \dots, a_n) \in \mathbb{R}^{3 \cdot n} \mid |a_i - a_j| \leq R_i + R_j \}.$$

The closure of the complement

$$T = \overline{\mathbb{R}^{3 \cdot n} \setminus \bigcup_{i < j} C_{i,j}}$$

is the configuration space of our system. Its points correspond to valid positions of the system of balls.

The evolution of the system of balls is described by the motion of the point  $\mathbf{a} \in \mathbb{R}^{3 \cdot n}$ . It moves along a straight line at a constant speed until it hits one of the cylinders  $C_{i,j}$ ; this event corresponds to a collision in the system of balls.

Consider the norm of  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^{3 \cdot n}$  defined by

$$\|\mathbf{a}\| = \sqrt{M_1 \cdot |a_1|^2 + \dots + M_n \cdot |a_n|^2},$$

where  $|a_i| = \sqrt{x_i^2 + y_i^2 + z_i^2}$  and  $M_i$  denotes the mass of the  $i$ -th ball. In the metric defined by  $\|\cdot\|$ , the collisions follow the standard law of billiards: the angle of reflection is equal to the angle of incidence.

By construction, the number of collisions of hard balls that we need to estimate is the same as the number of collisions of the corresponding billiard trajectory on the table  $T$  with  $C_{i,j}$  as the walls.

Note that each cylinder  $C_{i,j}$  is a convex set; it has smooth boundary and it is centrally symmetric around the origin. By Exercise 2.4.9, all the intersections of the walls have  $\varepsilon$ -wide corners for some  $\varepsilon > 0$  that depends on the radiuses  $R_i$  and the masses  $M_i$ . It remains to apply the collision theorem (2.5.1).  $\square$

Now we present the proof of the collision theorem (2.5.1) based on the results developed in the previous section.

*Proof of Theorem 2.5.1.* Let us apply induction on  $n$ .

*Base:*  $n = 1$ . The number of collisions cannot exceed 1. Indeed, by the convexity of  $A^1$ , if the trajectory is reflected once in  $\partial A^1$ , then it cannot return to  $A^1$ .

*Step.* Assume  $\gamma$  is a trajectory which meets the walls in the order  $A^{i_1}, \dots, A^{i_N}$  for a large integer  $N$ .

Consider the array

$$\mathbf{A}_\gamma = (A^{i_1}, \dots, A^{i_N}).$$

The induction hypothesis implies:

❶ *There is a positive integer  $M$  such that any  $M$  consecutive elements of  $\mathbf{A}_\gamma$  contain each  $A^i$  at least once.*

Let  $\mathcal{R}_\gamma$  be the Reshetnyak's puff pastry for  $\mathbf{A}_\gamma$ .

Consider the lift of  $\gamma$  to  $\mathcal{R}_\gamma$  defined as  $\bar{\gamma}(t) = \gamma^k(t) \in \mathcal{R}_\gamma$  for any moment of time  $t$  between  $k$  and  $(k+1)$ -th collisions. Since  $\gamma$  follows the standard law of billiards at break points — in particular, the angle of reflection is equal to the angle of incidence — the lift  $\bar{\gamma}$  is locally a geodesic in  $\mathcal{R}_\gamma$ . Since  $\mathcal{R}_\gamma$  is a CAT[0] proper length space (see Observation 2.4.1),  $\bar{\gamma}$  is a geodesic.

Since  $\gamma$  does not pass through  $A^1 \cap \dots \cap A^n$ , the lift  $\bar{\gamma}$  does not lie in  $\mathcal{R}_\gamma^0 \cup \mathcal{R}_\gamma^N$ . In particular,  $\mathcal{R}_\gamma$  is not end-to-end convex.

Let

$$\mathbf{B} = (A^{j_1}, \dots, A^{j_K})$$

be the array provided by Proposition 2.4.10; so  $\mathbf{B}$  contains each  $A^i$  at least once and the puff pastry  $\mathcal{R}_\mathbf{B}$  for  $\mathbf{B}$  is end-to-end convex. If  $N$  is sufficiently large, namely  $N \geq K \cdot M$ , then ❶ implies that  $\mathbf{A}_\gamma$  can be obtained by inserting a finite number of  $A^i$ 's in  $\mathbf{B}$ .

By Observation 2.4.4,  $\mathcal{R}_\gamma$  is end-to-end convex, a contradiction.  $\square$

## 2.6 Comments

The gluing theorem (2.3.1) extends to the class of CAT[0] geodesic spaces, which by Exercise 2.1.4 includes all CAT[0] complete length spaces. It also admits a natural generalization to CAT[ $\kappa$ ] length spaces; see the book of Martin Bridson and André Haefliger [15] and our book [4] for details.

The idea described above has a number of applications to billiards. In particular, it can be used to bound topological entropy of the billiard flow; see another paper of Dmitri Burago, Serge Ferleger and Alexey Kononenko [19]. The lecture [16] of Dmitri Burago gives a short survey on the subject.

Note that the walls in the puff pastry play a key role in the proof despite the fact that trajectories never go inside the walls. In a similar fashion, puff pastry was used by Stephanie Alexander and Richard Bishop in [3] to find the upper curvature bound for warped products.

In [35], Joel Hass constructed an example of a Riemannian metric on the 3-ball with negative curvature and concave boundary. This example might decrease your appetite for generalizing the collision theorem — while locally such a 3-ball looks as good as the billiards table in the theorem, the number of collisions is obviously infinite.



## Chapter 3

# Globalization and asphericity

In this chapter we introduce locally  $\text{CAT}[0]$  spaces and show that the universal cover of a locally  $\text{CAT}[0]$  proper length space is a  $\text{CAT}[0]$  proper length space. It follows that any locally  $\text{CAT}[0]$  proper length space is aspherical; that is, its universal cover is contractible.

This globalization theorem leads to a *construction toy set*, described by Flag condition 3.5.5. Playing with this toy set, we produce examples of exotic aspherical spaces.

### 3.1 Locally $\text{CAT}$ spaces

We say that a space  $\mathcal{U}$  is *locally*  $\text{CAT}[0]$  (or *locally*  $\text{CAT}[1]$ ) if a small closed ball centered at any point  $p$  in  $\mathcal{U}$  forms a  $\text{CAT}[0]$  space (or  $\text{CAT}[1]$  space, respectively).

For example,  $\mathbb{S}^1$  is locally isometric to  $\mathbb{R}$ , and so  $\mathbb{S}^1$  is locally  $\text{CAT}[0]$ . On the other hand,  $\mathbb{S}^1$  is not  $\text{CAT}[0]$ , since closed local geodesics in  $\mathbb{S}^1$  are not geodesics, so  $\mathbb{S}^1$  does not satisfy Proposition 2.2.7.

If  $\mathcal{U}$  is a proper length space then it is locally  $\text{CAT}[0]$  (or locally  $\text{CAT}[1]$ ) if and only if each point  $p \in \mathcal{U}$  admits an open neighborhood  $\Omega$  which is geodesic and any triangle in  $\Omega$  is thin (or spherically thin, respectively). The proof goes along the same lines as in Exercise 2.2.11.

### 3.2 Space of local geodesic paths

In this section we will study behavior of local geodesics in locally  $\text{CAT}[\kappa]$  spaces. The results will be used in the proof of the globaliza-

tion theorem (3.3.1).

Recall that *path* denotes a curve parametrized by  $[0, 1]$ . The space of paths in  $\mathcal{U}$  comes with the natural metric

$$\bullet \quad |\alpha - \beta| = \sup\{|\alpha(t) - \beta(t)|_{\mathcal{U}} \mid t \in [0, 1]\}.$$

**3.2.1. Proposition.** *Let  $\mathcal{U}$  be a proper locally CAT $[\kappa]$  length space.*

*Assume  $\gamma_n: [0, 1] \rightarrow \mathcal{U}$  is a sequence of local geodesic paths converging to a path  $\gamma_\infty: [0, 1] \rightarrow \mathcal{U}$ . Then  $\gamma_\infty$  is a local geodesic path. Moreover*

$$\text{length } \gamma_n \rightarrow \text{length } \gamma_\infty$$

*as  $n \rightarrow \infty$ .*

*Proof; CAT[0] case.* Fix  $t \in [0, 1]$ . Let  $R > 0$  be the value such that  $\overline{\mathcal{B}}[\gamma_\infty(t), R]$  forms a CAT[0] proper length space.

Assume a local geodesic  $\sigma$  is shorter than  $R/2$  and intersects the ball  $\mathcal{B}(\gamma_\infty(t), R/2)$ . Then  $\sigma$  cannot leave the ball  $\overline{\mathcal{B}}[\gamma_\infty(t), R]$  and hence is a geodesic by Proposition 2.2.7. In particular, for all sufficiently large  $n$ , any arc of  $\gamma_n$  of length  $R/2$  or less is a geodesic.

Since  $\mathcal{B} = \overline{\mathcal{B}}[\gamma_\infty(t), R]$  is a CAT[0] proper length space, by Theorem 2.2.3, geodesic segments in  $\mathcal{B}$  depend uniquely and continuously on their endpoint pairs. Thus there is a subinterval  $\mathbb{I}$  of  $[0, 1]$ , which contains a neighborhood of  $t$  in  $[0, 1]$  and such that the arc  $\gamma_n|_{\mathbb{I}}$  is minimizing for all large  $n$ . It follows that  $\gamma_\infty|_{\mathbb{I}}$  is a geodesic, and therefore  $\gamma_\infty$  is a local geodesic.

The CAT[1] case is done the same way, but one has to assume in addition that  $R < \pi$ .  $\square$

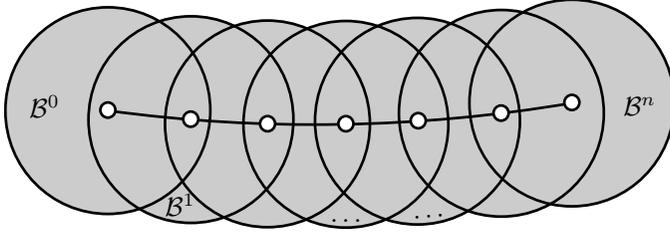
The following lemma and its proof were suggested to us by Alexander Lytchak. This lemma allows a local geodesic path to be moved continuously so that its endpoints follow given trajectories; such a lemma was originally obtained by Stephanie Alexander and Richard Bishop by a different method in [2].

**3.2.2. Patchwork along a curve.** *Let  $\mathcal{U}$  be a locally CAT[0] proper length space, and  $\gamma: [0, 1] \rightarrow \mathcal{U}$  be a path.*

*Then there is a CAT[0] proper length space  $\mathcal{N}$ , an open set  $\hat{\Omega} \subset \mathcal{N}$ , and a path  $\hat{\gamma}: [0, 1] \rightarrow \hat{\Omega}$ , such that there is an open locally isometric immersion  $\Phi: \hat{\Omega} \rightarrow \mathcal{U}$  satisfying  $\Phi \circ \hat{\gamma} = \gamma$ .*

*If  $\text{length } \gamma < \pi$ , then the same holds in CAT[1] case. Namely we assume that  $\mathcal{U}$  is a CAT[1] proper length space and construct a CAT[1] proper length space  $\mathcal{N}$  with the same property as above.*

*Proof.* Fix  $r > 0$  so that for each  $t \in [0, 1]$ , the closed ball  $\overline{\mathcal{B}}[\gamma(t), r]$  forms a CAT $[\kappa]$  proper length space.



Choose a partition  $0 = t^0 < t^1 < \dots < t^n = 1$  so that

$$B(\gamma(t^i), r) \supset \gamma([t^{i-1}, t^i])$$

for all  $n > i > 0$ . Set  $\mathcal{B}^i = \overline{B}[\gamma(t^i), r]$ .

Consider the disjoint union  $\bigsqcup_i \mathcal{B}^i = \{(i, x) \mid x \in \mathcal{B}^i\}$  with the minimal equivalence relation  $\sim$  such that  $(i, x) \sim (i-1, x)$  for all  $i$ . Let  $\mathcal{N}$  be the space obtained by gluing the  $\mathcal{B}^i$  along  $\sim$ .

Note that  $A^i = \mathcal{B}^i \cap \mathcal{B}^{i-1}$  is convex in  $\mathcal{B}^i$  and in  $\mathcal{B}^{i-1}$ . Applying the Reshetnyak gluing theorem (2.3.1)  $n$  times, we conclude that  $\mathcal{N}$  is a CAT[0] proper length space.

For  $t \in [t^{i-1}, t^i]$ , define  $\hat{\gamma}(t)$  as the equivalence class of  $(i, \gamma(t))$  in  $\mathcal{N}$ . Let  $\hat{\Omega}$  be the  $\varepsilon$ -neighborhood of  $\hat{\gamma}$  in  $\mathcal{N}$ , where  $\varepsilon > 0$  is chosen so that  $B(\gamma(t), \varepsilon) \subset \mathcal{B}^i$  for all  $t \in [t^{i-1}, t^i]$ .

Define  $\Phi: \hat{\Omega} \rightarrow \mathcal{U}$  by sending the equivalence class of  $(i, x)$  to  $x$ . It is straightforward to check that  $\Phi$ ,  $\hat{\gamma}$  and  $\hat{\Omega} \subset \mathcal{N}$  satisfy the conclusion of the lemma.

The CAT[1] case is proved the same way. □

The following two corollaries follow from: (1) the patchwork (3.2.2); (2) Proposition 2.2.7, which states that local geodesics are geodesics in any CAT space; and (3) Theorem 2.2.3 on uniqueness of geodesics.

**3.2.3. Corollary.** *If  $\mathcal{U}$  is a locally CAT[0] proper length space, then for any pair of points  $p, q \in \mathcal{U}$ , the space of all local geodesic paths from  $p$  to  $q$  is discrete; that is, for any local geodesic path  $\gamma$  connecting  $p$  to  $q$ , there is  $\varepsilon > 0$  such that for any other local geodesic path  $\delta$  from  $p$  to  $q$  we have  $|\gamma(t) - \delta(t)|_{\mathcal{U}} > \varepsilon$  for some  $t \in [0, 1]$ .*

*Analogously, if  $\mathcal{U}$  is a locally CAT[1] proper length space, then for any pair of points  $p, q \in \mathcal{U}$ , the space of all local geodesic paths from  $p$  to  $q$  with length less than  $\pi$  is discrete.*

**3.2.4. Corollary.** *If  $\mathcal{U}$  is a locally CAT[0] proper length space, then for any path  $\alpha$  there is a choice of a local geodesic path  $\gamma_\alpha$  connecting the ends of  $\alpha$  such that the map  $\alpha \mapsto \gamma_\alpha$  is continuous, and if  $\alpha$  is a local geodesic path then  $\gamma_\alpha = \alpha$ .*

Analogously, if  $\mathcal{U}$  is a locally CAT[1] proper length space, then for any path  $\alpha$  shorter than  $\pi$ , there is a choice of a local geodesic path  $\gamma_\alpha$  shorter than  $\pi$  connecting the ends of  $\alpha$  such that the map  $\alpha \mapsto \gamma_\alpha$  is continuous, and if  $\alpha$  is a local geodesic path then  $\gamma_\alpha = \alpha$ .

*Proof of Corollary 3.2.4.* We do the CAT[0] case; the CAT[1] case is analogous.

Consider the maximal interval  $\mathbb{I} \subset [0, 1]$  containing 0, such that there is a continuous one-parameter family of local geodesic paths  $\gamma_t$  for  $t \in \mathbb{I}$  connecting  $\alpha(0)$  to  $\alpha(t)$  with  $\gamma_t(0) = \gamma_0(t) = \alpha(0)$  for any  $t$ .

By Proposition 3.2.1,  $\mathbb{I}$  is closed, so we can assume  $\mathbb{I} = [0, s]$  for some  $s \in [0, 1]$ .

Applying the patchwork (3.2.2) to  $\gamma_s$ , we get that  $\mathbb{I}$  is also open in  $[0, 1]$ . Hence  $\mathbb{I} = [0, 1]$ . Set  $\gamma_\alpha = \gamma_1$ .

By construction, if  $\alpha$  is a local geodesic path, then  $\gamma_\alpha = \alpha$ .

Moreover, from Corollary 3.2.3, the construction  $\alpha \mapsto \gamma_\alpha$  produces close results for sufficiently close paths in the metric defined by  $\bullet$ ; that is, the map  $\alpha \mapsto \gamma_\alpha$  is continuous.  $\square$

Given a path  $\alpha: [0, 1] \rightarrow \mathcal{U}$ , we denote by  $\bar{\alpha}$  the same path traveled in the opposite direction; that is,

$$\bar{\alpha}(t) = \alpha(1 - t).$$

The *product* of two paths will be denoted with “\*”; if two paths  $\alpha$  and  $\beta$  connect the same pair of points then the product  $\bar{\alpha} * \beta$  forms a closed curve.

**3.2.5. Exercise.** Assume  $\mathcal{U}$  is a locally CAT[1] proper length space. Consider the construction  $\alpha \mapsto \gamma_\alpha$  provided by Corollary 3.2.4.

Assume  $\alpha$  and  $\beta$  are two paths connecting the same pair of points in  $\mathcal{U}$ , where each has length less than  $\pi$  and the product  $\bar{\alpha} * \beta$  is null-homotopic in the class of closed curves with length smaller than  $2\cdot\pi$ . Then  $\gamma_\alpha = \gamma_\beta$ .

### 3.3 Globalization

The original formulation of the *globalization theorem*, or *Hadamard–Cartan theorem*, states that if  $M$  is a complete Riemannian manifold with sectional curvature  $\leq 0$ , then the exponential map at any point  $p \in M$  is a covering; in particular it implies that the universal cover of  $M$  is diffeomorphic to the Euclidean space of the same dimension.

In this generality, this theorem appeared in the lectures of Elie Cartan, see [22]. This theorem was proved for surfaces in Euclidean 3-space by Hans von Mangoldt [41], and a few years later independently for 2-dimensional Riemannian manifolds by Jacques Hadamard [33].

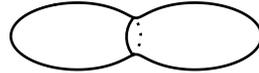
Formulations for metric spaces of different generality were proved by Herbert Busemann in [20], Willi Rinow in [49], Mikhael Gromov in [31, p.119]. A detailed proof of Gromov's statement was given by Werner Ballmann in [9] when  $\mathcal{U}$  is proper, and by Stephanie Alexander and Richard Bishop in [2] in more generality; also see references in our book [4].

For CAT[1] proper spaces, the globalization theorem was proved by Brian Bowditch in [14].

**3.3.1. Globalization theorem.** *Any simply connected, locally CAT[0] proper length space is CAT[0].*

*Analogously, assume  $\mathcal{U}$  is a locally CAT[1] proper length space such that any closed curve  $\gamma: \mathbb{S}^1 \rightarrow \mathcal{U}$  with length smaller than  $2\cdot\pi$  is null-homotopic in the class of closed curves with length smaller than  $2\cdot\pi$ . Then  $\mathcal{U}$  is a CAT[1] space.*

The surface of revolution illustrated in the diagram is an example of a simply connected space that is locally CAT[1] but not CAT[1]. To contract the marked curve one has to increase its length to  $2\cdot\pi$  or more; in particular the surface does not satisfy the assumption of the globalization theorem.



The proof of the globalization theorem relies on the following theorem, which essentially is [7, Satz 9].

**3.3.2. Patchwork globalization theorem.** *A locally CAT[0] proper length space  $\mathcal{U}$  is CAT[0] if and only if all pairs of points in  $\mathcal{U}$  are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

*Analogously, a locally CAT[1] proper length space  $\mathcal{U}$  is CAT[1] if and only if all pairs of points in  $\mathcal{U}$  at distance less than  $\pi$  are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

The proof uses a thin-triangle decomposition with the inheritance lemma (2.2.9) and the following construction:

**3.3.3. Definition of line-of-sight map.** *Let  $p$  be a point and  $\alpha$  be a curve of finite length in a length space  $\mathcal{X}$ . Let  $\bar{\alpha}: [0, 1] \rightarrow \mathcal{U}$  be the constant-speed parameterization of  $\alpha$ . If  $\gamma_t: [0, 1] \rightarrow \mathcal{U}$  is a geodesic path from  $p$  to  $\bar{\alpha}(t)$ , we say*

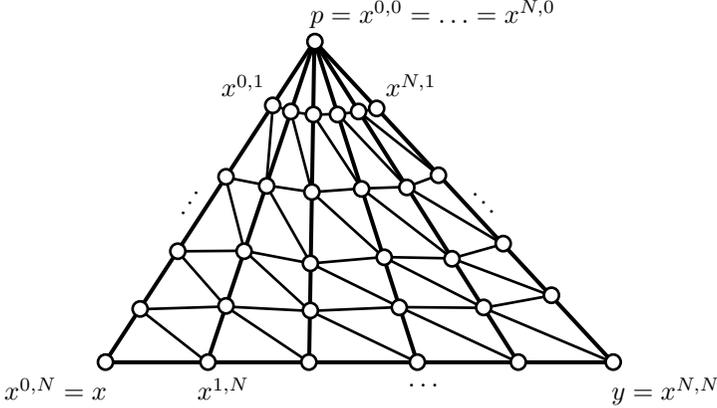
$$[0, 1] \times [0, 1] \rightarrow \mathcal{U}: (t, s) \mapsto \gamma_t(s)$$

*is a line-of-sight map from  $p$  to  $\alpha$ .*

*Proof of the patchwork globalization theorem (3.3.2).* Note that the implication “only if” is already proved in Theorem 2.2.3; it only remains to prove “if” part.

Fix a triangle  $[pxy]$  in  $\mathcal{U}$ . We need to show that  $[pxy]$  is thin.

By the assumptions, the line-of-sight map  $(t, s) \mapsto \gamma_t(s)$  from  $p$  to  $[xy]$  is uniquely defined and continuous.



Fix a partition

$$0 = t^0 < t^1 < \dots < t^N = 1,$$

and set  $x^{i,j} = \gamma_{t^i}(t^j)$ . Since the line-of-sight map is continuous and  $\mathcal{U}$  is locally CAT[0], we may assume that each triangle

$$[x^{i,j} x^{i,j+1} x^{i+1,j+1}] \quad \text{and} \quad [x^{i,j} x^{i+1,j} x^{i+1,j+1}]$$

is thin.

Now we show that the thin property propagates to  $[pxy]$  by repeated application of the inheritance lemma (2.2.9):

- ◊ For fixed  $i$ , sequentially applying the lemma shows that the triangles  $[x x^{i,1} x^{i+1,2}]$ ,  $[x x^{i,2} x^{i+1,2}]$ ,  $[x x^{i,2} x^{i+1,3}]$ , and so on are thin.

In particular, for each  $i$ , the long triangle  $[x x^{i,N} x^{i+1,N}]$  is thin.

- ◊ Applying the same lemma again shows that the triangles  $[x x^{0,N} x^{2,N}]$ ,  $[x x^{0,N} x^{3,N}]$ , and so on are thin.

In particular,  $[pxy] = [p x^{0,N} x^{N,N}]$  is thin.  $\square$

*Proof of the globalization theorem; CAT[0] case.* Let  $\mathcal{U}$  be a simply connected, locally CAT[0] proper length space. Given a path  $\alpha$  in  $\mathcal{U}$ , denote by  $\gamma_\alpha$  the local geodesic path provided by Corollary 3.2.4. Since the map  $\alpha \mapsto \gamma_\alpha$  is continuous, by Corollary 3.2.3 we have  $\gamma_\alpha = \gamma_\beta$  for any pair of paths  $\alpha$  and  $\beta$  homotopic relative to the ends.

Since  $\mathcal{U}$  is simply connected, any pair of paths with common ends are homotopic. In particular, if  $\alpha$  and  $\beta$  are local geodesics from  $p$  to  $q$ , then  $\alpha = \gamma_\alpha = \gamma_\beta = \beta$  by Corollary 3.2.4. It follows that any two points  $p, q \in \mathcal{U}$  are joined by a unique local geodesic that depends continuously on  $(p, q)$ . Since  $\mathcal{U}$  is geodesic, it remains to apply the patchwork globalization theorem (3.3.2).

CAT[1] case. The proof goes along the same lines, but one needs to use Exercise 3.2.5.  $\square$

**3.3.4. Corollary.** *Any compact locally CAT[0] space that contains no closed local geodesics is CAT[0].*

*Analogously, any compact locally CAT[1] space that contains no closed local geodesics of length smaller than  $2\cdot\pi$  is CAT[1].*

*Proof.* By the globalization theorem (3.3.1), we need to show that the space is simply connected. Assume the contrary. Fix a nontrivial homotopy class of closed curves.

Denote by  $\ell$  the exact lower bound for the lengths of curves in the class. Note that  $\ell > 0$ ; otherwise there would be a closed non-contractible curve in a CAT[0] spherical neighborhood of some point, contradicting Corollary 2.2.6.

Since the space is compact, the class contains a length-minimizing curve, which must be a closed local geodesic.

The CAT[1] case is analogous, one only has to consider a homotopy class of closed curves shorter than  $2\cdot\pi$ .  $\square$

**3.3.5. Exercise.** *Prove that any compact locally CAT[0] space that is not CAT[0] contains a geodesic circle, that is, a simple closed curve  $\gamma$  such that for any two points  $p, q \in \gamma$ , one of the arcs of  $\gamma$  with endpoints  $p$  and  $q$  is a geodesic. Prove the analogous statement for CAT[1].*

## 3.4 Polyhedral spaces

**3.4.1. Definition.** *A length space  $\mathcal{P}$  is called a (spherical) polyhedral space if it admits a finite triangulation  $\tau$  such that every simplex in  $\tau$  is isometric to a simplex in a Euclidean space (or correspondingly a unit sphere) of appropriate dimension.*

*By a triangulation of a polyhedral space we will always understand the triangulation as above.*

Note that according to the above definition, all polyhedral spaces are compact. However, most of the statements below admit straightforward generalizations to *locally polyhedral spaces*; that is, complete

length spaces, any point of which admits a closed neighborhood isometric to a polyhedral space. The latter class of spaces includes in particular infinite covers of polyhedral spaces.

The dimension of a polyhedral space  $\mathcal{P}$  is defined as the maximal dimension of the simplices in one (and therefore any) triangulation of  $\mathcal{P}$ .

**Links.** Let  $\mathcal{P}$  be a polyhedral space and  $\sigma$  be a simplex in a triangulation  $\tau$  of  $\mathcal{P}$ .

The simplices which contain  $\sigma$  form an abstract simplicial complex called the *link* of  $\sigma$ , denoted by  $\text{Link}_\sigma$ . If  $m$  is the dimension of  $\sigma$  then the set of vertices of  $\text{Link}_\sigma$  is formed by the  $(m + 1)$ -simplices which contain  $\sigma$ ; the set of its edges are formed by the  $(m + 2)$ -simplices which contain  $\sigma$ ; and so on.

The link  $\text{Link}_\sigma$  can be identified with the subcomplex of  $\tau$  formed by all the simplices  $\sigma'$  such that  $\sigma \cap \sigma' = \emptyset$  but both  $\sigma$  and  $\sigma'$  are faces of a simplex of  $\tau$ .

The points in  $\text{Link}_\sigma$  can be identified with the normal directions to  $\sigma$  at a point in its interior. The angle metric between directions makes  $\text{Link}_\sigma$  into a spherical polyhedral space. We will always consider the link with this metric.

**Tangent space and space of directions.** Let  $\mathcal{P}$  be a polyhedral space (Euclidean or spherical) and  $\tau$  be its triangulation. If a point  $p \in \mathcal{P}$  lies in the interior of a  $k$ -simplex  $\sigma$  of  $\tau$  then the tangent space  $T_p = T_p\mathcal{P}$  is naturally isometric to

$$\mathbb{E}^k \times (\text{Cone Link}_\sigma).$$

Equivalently, the space of directions  $\Sigma_p = \Sigma_p\mathcal{P}$  can be isometrically identified with the  $k$ -times-iterated suspension over  $\text{Link}_\sigma$ ; that is,

$$\Sigma_p \stackrel{\text{iso}}{=} \text{Susp}^k(\text{Link}_\sigma).$$

If  $\mathcal{P}$  is an  $m$ -dimensional polyhedral space, then for any  $p \in \mathcal{P}$  the space of directions  $\Sigma_p$  is a spherical polyhedral space of dimension at most  $m - 1$ .

In particular, for any point  $p$  in  $\sigma$ , the isometry class of  $\text{Link}_\sigma$  together with  $k = \dim \sigma$  determines the isometry class of  $\Sigma_p$ , and the other way around.

A small neighborhood of  $p$  is isometric to a neighborhood of the tip of the cone over  $\Sigma_p$  (or, for the spherical polyhedral space, the north pole in the spherical suspension over  $\Sigma_p$ ). In fact, if this property holds at any point of a compact length space  $\mathcal{P}$  then  $\mathcal{P}$  is a polyhedral space, see [38] by Nina Lebedeva and Anton Petrunin.

The following theorem provides a combinatorial description of polyhedral spaces with curvature bounded above.

**3.4.2. Theorem.** *Let  $\mathcal{P}$  be a polyhedral space and  $\tau$  be its triangulation. Then  $\mathcal{P}$  is locally CAT[0] if and only if the link of each simplex in  $\tau$  has no closed local geodesic shorter than  $2\cdot\pi$ .*

*Analogously, let  $\mathcal{P}$  be a spherical polyhedral space and  $\tau$  be its triangulation. Then  $\mathcal{P}$  is CAT[1] if and only if neither  $\mathcal{P}$  nor the link of any simplex in  $\tau$  has a closed local geodesic shorter than  $2\cdot\pi$ .*

*Proof of Theorem 3.4.2.* The “only if” part follows from Proposition 2.2.7 and Exercise 2.1.2.

To prove “if” part, we apply induction on  $\dim \mathcal{P}$ . The base case  $\dim \mathcal{P} = 0$  is evident. Let us start with the CAT[1] case.

*Step.* Assume that the theorem is proved in the case  $\dim \mathcal{P} < m$ . Suppose  $\dim \mathcal{P} = m$ .

Fix a point  $p \in \mathcal{P}$ . A neighborhood of  $p$  is isometric to a neighborhood of the north pole in the suspension over the space of directions  $\Sigma_p$ .

Note that  $\Sigma_p$  is a spherical polyhedral space and its links are isometric to links of  $\mathcal{P}$ . By the induction hypothesis,  $\Sigma_p$  is CAT[1].

Thus, by the second part of Exercise 2.1.2,  $\mathcal{P}$  is locally CAT[1].

Applying the second part of Corollary 3.3.4, we get the statement.

The CAT[0] case is exactly the same except we need to use the first part of Exercise 2.1.2 and the first part of Corollary 3.3.4.  $\square$

**3.4.3. Exercise.** *Let  $\mathcal{P}$  be a polyhedral space such that any two points can be connected by a unique geodesic. Show that  $\mathcal{P}$  is a CAT[0] space.*

**3.4.4. Advanced exercise.** *Construct a polyhedral metric on  $\mathbb{S}^3$  such that the total angle around each edge is at least  $2\cdot\pi$ .*

## 3.5 Flag complexes

**3.5.1. Definition.** *A simplicial complex  $\mathcal{S}$  is called flag if whenever  $\{v^0, \dots, v^k\}$  is a set of distinct vertices of  $\mathcal{S}$  which are pairwise joined by edges, then the vertices  $v^0, \dots, v^k$  span a  $k$ -simplex in  $\mathcal{S}$ .*

*If the above condition is satisfied for  $k = 2$ , then we say  $\mathcal{S}$  satisfies the no-triangle condition.*

Note that every flag complex is determined by its 1-skeleton.

**3.5.2. Exercise.** *Show that the barycentric subdivision of any simplicial complex is a flag complex.*

Use Flag condition (see 3.5.5 below) to conclude that any finite simplicial complex is homeomorphic to a CAT[1] proper length space.

**3.5.3. Proposition.** *A simplicial complex  $\mathcal{S}$  is flag if and only if  $\mathcal{S}$  as well as all the links of all its simplices satisfy the no-triangle condition.*

From the definition of flag complex we get the following.

**3.5.4. Lemma.** *Any link of any simplex in a flag complex is flag.*

*Proof of Proposition 3.5.3.* By Lemma 3.5.4, the no-triangle condition holds for any flag complex and the links of all its simplices.

Now assume a complex  $\mathcal{S}$  and all its links satisfy the no-triangle condition. It follows that  $\mathcal{S}$  includes a 2-simplex for each triangle. Applying the same observation for each edge we get that  $\mathcal{S}$  includes a 3-simplex for any complete graph with 4 vertices. Repeating this observation for triangles, 4-simplices, 5-simplices and so on we get that  $\mathcal{S}$  is flag.  $\square$

**All-right triangulation.** A triangulation of a spherical polyhedral space is called an *all-right triangulation* if each simplex of the triangulation is isometric to a spherical simplex all of whose angles are right. Similarly, we say that a simplicial complex is equipped with an *all-right spherical metric* if it is a length metric and each simplex is isometric to a spherical simplex all of whose angles are right.

Spherical polyhedral CAT[1] spaces glued from of right-angled simplices admit the following characterization discovered by Mikhael Gromov [31, p. 122].

**3.5.5. Flag condition.** *Assume that a spherical polyhedral space  $\mathcal{P}$  admits an all-right triangulation  $\tau$ . Then  $\mathcal{P}$  is a CAT[1] space if and only if  $\tau$  is flag.*

*Proof; “Only-if” part.* Assume there are three vertices  $v^1, v^2$  and  $v^3$  of  $\tau$  that are pairwise joined by edges but do not span a simplex. Note that in this case

$$\angle[v^1 v^2]_{v^3} = \angle[v^2 v^3]_{v^1} = \angle[v^3 v^1]_{v^2} = \pi.$$

Equivalently,

❶ *The product of the geodesics  $[v^1 v^2]$ ,  $[v^2 v^3]$  and  $[v^3 v^1]$  forms a locally geodesic loop in  $\mathcal{P}$ .*

Now assume that  $\mathcal{P}$  is a CAT[1] space. Then by Theorem 3.4.2,  $\text{Link}_\sigma \mathcal{P}$  is a CAT[1] space for every simplex  $\sigma$  in  $\tau$ .

Each of these links is an all-right spherical complex and by Theorem 3.4.2, none of these links can contain a geodesic circle of length less than  $2\cdot\pi$ .

Therefore Proposition 3.5.3 and **1** imply the “only-if” part.

“If” part. By Lemma 3.5.4 and Theorem 3.4.2, it is sufficient to show that any closed local geodesic  $\gamma$  in a flag complex  $\mathcal{S}$  with all-right metric has length at least  $2\cdot\pi$ .

Recall that the *closed star* of a vertex  $v$  (briefly  $\overline{\text{Star}}_v$ ) is formed by all the simplices containing  $v$ . Similarly,  $\text{Star}_v$ , the open star of  $v$ , is the union of all simplices containing  $v$  with faces opposite  $v$  removed.

Choose a simplex  $\sigma$  which contains a point  $\gamma(t_0)$ . Let  $v$  be a vertex of  $\sigma$ . Consider the maximal arc  $\gamma_v$  of  $\gamma$  which contains the point  $\gamma(t_0)$  and runs in  $\text{Star}_v$ . Note that the distance  $|v - \gamma_v(t)|_{\mathcal{P}}$  behaves exactly the same way as the distance from north pole to a geodesic in the north hemisphere; that is, there is a geodesic  $\tilde{\gamma}_v$  in the north hemisphere of  $\mathbb{S}^2$  such that for any  $t$  we have

$$|v - \gamma_v(t)|_{\mathcal{P}} = |n - \tilde{\gamma}_v(t)|_{\mathbb{S}^2},$$

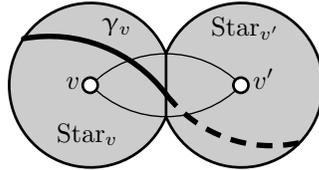
where  $n$  denotes the north pole of  $\mathbb{S}^2$ . In particular,

$$\text{length } \gamma_v = \pi;$$

that is,  $\gamma$  spends time  $\pi$  on every visit to  $\text{Star}_v$ .

After leaving  $\text{Star}_v$ , the local geodesic  $\gamma$  has to enter another simplex, say  $\sigma'$ . Since  $\tau$  is flag, the simplex  $\sigma'$  has a vertex  $v'$  not joined to  $v$  by an edge; that is

$$\text{Star}_v \cap \text{Star}_{v'} = \emptyset$$



The same argument as above shows that  $\gamma$  spends time  $\pi$  on every visit to  $\text{Star}_{v'}$ . Therefore the total length of  $\gamma$  is at least  $2\cdot\pi$ .  $\square$

**3.5.6. Exercise.** Assume that a spherical polyhedral space  $\mathcal{P}$  admits a triangulation  $\tau$  such that all dihedral angles of all simplices are at least  $\frac{\pi}{2}$ . Show that  $\mathcal{P}$  is a CAT[1] space if  $\tau$  is flag.

**The space of trees.** The following construction is given by Louis Billera, Susan Holmes and Karen Vogtmann in [11].

Let  $\mathcal{T}_n$  be the set of all metric trees with one initial vertex and  $n$  end vertices labeled by  $a^1, \dots, a^n$ . To describe one tree in  $\mathcal{T}_n$  we may fix a topological trivalent tree  $\tau$  with end vertices  $a^1, \dots, a^n$ , initial vertex of degree 2, and all other vertices of degree 3; and prescribe

the lengths of  $2 \cdot n - 3$  edges. If the length of an edge is 0, we assume that this edge degenerates; such a tree can be also described using a different topological tree  $\tau'$ . The subset of  $\mathcal{T}_n$  corresponding to the given topological tree  $\tau$  can be identified with the octant

$$\{ (x_1, \dots, x_{2 \cdot n - 3}) \in \mathbb{R}^{2 \cdot n - 3} \mid x_i \geq 0 \}.$$

Equip each such subset with the metric induced from  $\mathbb{R}^{2 \cdot n - 3}$  and consider the length metric on  $\mathcal{T}_n$  induced by these metrics.

**3.5.7. Exercise.** *Show that  $\mathcal{T}_n$  with the described metric is a CAT[0] space.*

## 3.6 Cubical complexes

The definition of a cubical complex mostly repeats the definition of a simplicial complex, with simplices replaced by cubes.

Formally, a *cubical complex* is defined as a subcomplex of the unit cube in the Euclidean space of large dimension; that is, a collection of faces of the cube such that together with each face it contains all its sub-faces. Each cube face in this collection will be called a *cube* of the cubical complex.

Note that according to this definition, any cubical complex is finite, that is, contains a finite number of cubes.

The union of all the cubes in a cubical complex  $\mathcal{Q}$  will be called its *underlying space*. A homeomorphism from the underlying space of  $\mathcal{Q}$  to a topological space  $\mathcal{X}$  is called a *cubulation of  $\mathcal{X}$* .

The underlying space of a cubical complex  $\mathcal{Q}$  will be always considered with the length metric induced from  $\mathbb{R}^N$ . In particular, with this metric, each cube of  $\mathcal{Q}$  is isometric to the unit cube of the same dimension.

It is straightforward to construct a triangulation of the underlying space of  $\mathcal{Q}$  such that each simplex is isometric to a Euclidean simplex. In particular the underlying space of  $\mathcal{Q}$  is a Euclidean polyhedral space.

The link of each cube in a cubical complex admits a natural all-right triangulation; each simplex corresponds to an adjusted cube.

**Cubical analog of a simplicial complex.** Let  $\mathcal{S}$  be a finite simplicial complex and  $\{v_1, \dots, v_N\}$  be the set of its vertices.

Consider  $\mathbb{R}^N$  with the standard basis  $\{e_1, \dots, e_N\}$ . Denote by  $\square^N$  the standard unit cube in  $\mathbb{R}^N$ ; that is

$$\square^N = \{ (x_1, \dots, x_N) \in \mathbb{R}^N \mid 0 \leq x_i \leq 1 \text{ for each } i \}.$$

Given a  $k$ -dimensional simplex  $\langle v_{i_0}, \dots, v_{i_k} \rangle$  in  $\mathcal{S}$ , mark the  $(k+1)$ -dimensional faces in  $\square^N$  (there are  $2^{N-k}$  of them) which are parallel to the coordinate  $(k+1)$ -plane spanned by  $e_{i_0}, \dots, e_{i_k}$ .

Note that the set of all marked faces of  $\square^N$  forms a cubical complex; it will be called the *cubical analog* of  $\mathcal{S}$  and will be denoted as  $\square_{\mathcal{S}}$ .

**3.6.1. Proposition.** *Let  $\mathcal{S}$  be a finite connected simplicial complex and  $\mathcal{Q} = \square_{\mathcal{S}}$  be its cubical analog. Then the underlying space of  $\mathcal{Q}$  is connected and the link of any vertex of  $\mathcal{Q}$  is isometric to  $\mathcal{S}$  equipped with the spherical right-angled metric.*

*In particular, if  $\mathcal{S}$  is a flag complex then  $\mathcal{Q}$  is a locally CAT[0] and therefore its universal cover  $\tilde{\mathcal{Q}}$  is a CAT[0] space.*

*Proof.* The first part of the proposition follows from the construction of  $\square_{\mathcal{S}}$ .

If  $\mathcal{S}$  is flag, then by Flag condition (3.5.5) the link of any cube in  $\mathcal{Q}$  is a CAT[1] space. Therefore, by cone the construction (Exercise 2.1.2)  $\mathcal{Q}$  is a locally CAT[0] space. It remains to apply the globalization theorem (3.3.1).  $\square$

From Proposition 3.6.1, it follows that the cubical analog of any flag complex is aspherical. The following exercise states that the converse also holds, see [26, 5.4].

**3.6.2. Exercise.** *Show that a finite simplicial complex is flag if and only if its cubical analog is aspherical.*

## 3.7 Exotic aspherical manifolds

By the globalization theorem (3.3.1), any CAT[0] proper length space is contractible. Therefore all complete, locally CAT[0] proper length spaces are *aspherical*; that is, they have contractible universal covers. This observation can be used to construct examples of aspherical spaces.

Let  $\mathcal{X}$  be a proper topological space. Recall that  $\mathcal{X}$  is called *simply connected at infinity* if for any compact set  $K \subset \mathcal{X}$  there is a bigger compact set  $K' \supset K$  such that  $\mathcal{X} \setminus K'$  is path connected and any loop which lies in  $\mathcal{X} \setminus K'$  is null-homotopic in  $\mathcal{X} \setminus K$ .

Recall that path connected spaces are not empty by definition. Therefore compact spaces are not simply connected at infinity.

The following example was constructed by Michael Davis in [25].

**3.7.1. Proposition.** *For any  $m \geq 4$  there is a closed aspherical  $m$ -dimensional piecewise linear manifold whose universal cover is not simply connected at infinity.*

In particular, the universal cover of this manifold is not homeomorphic to the  $m$ -dimensional Euclidean space.

The proof requires the following lemma.

**3.7.2. Lemma.** *Let  $\mathcal{S}$  be a finite flag complex,  $\mathcal{Q} = \square_{\mathcal{S}}$  be its cubical analog and  $\tilde{\mathcal{Q}}$  be the universal cover of  $\mathcal{Q}$ .*

*Assume  $\tilde{\mathcal{Q}}$  is simply connected at infinity. Then  $\mathcal{S}$  is simply connected.*

*Proof of Lemma 3.7.2.* Assume  $\mathcal{S}$  is not simply connected. Choose a shortest noncontractible circle  $\gamma: \mathbb{S}^1 \rightarrow \mathcal{S}$  formed by the edges of  $\mathcal{S}$ .

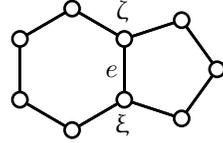
Note that  $\gamma$  forms a 1-dimensional subcomplex of  $\mathcal{S}$  which is a closed local geodesic. Denote by  $G$  the subcomplex of  $\mathcal{Q}$  which corresponds to  $\gamma$ .

Fix a vertex  $v \in G$ ; let  $G_v$  be the connected component of  $G$  containing  $v$ . Let  $\tilde{G}$  be a connect component of the inverse image of  $G_v$  in  $\tilde{\mathcal{Q}}$  for the universal cover  $\tilde{\mathcal{Q}} \rightarrow \mathcal{Q}$ . Fix a point  $\tilde{v} \in \tilde{G}$  in the inverse image of  $v$ .

Note that

❶  $\tilde{G}$  forms a convex set in  $\tilde{\mathcal{Q}}$ .

Indeed, according to Proposition 3.6.1,  $\tilde{\mathcal{Q}}$  is CAT[0]. By Exercise 2.2.12, it is sufficient to show that  $\tilde{G}$  is locally convex in  $\tilde{\mathcal{Q}}$ , or equivalently,  $G$  is locally convex in  $\mathcal{Q}$ .



Note that the latter can only fail if  $\gamma$  passes through two vertices, say  $\xi$  to  $\zeta$  in  $\mathcal{S}$ , which are joined by an edge not in  $\gamma$ ; denote this edge by  $e$ .

Each edge of  $\mathcal{S}$  has length  $\frac{\pi}{2}$ . Therefore each of two circles formed by  $e$  and an arc of  $\gamma$  from  $\xi$  to  $\zeta$  is shorter than  $\gamma$ . Moreover, at least one of them is noncontractible since  $\gamma$  is not. That is,  $\gamma$  is not a shortest noncontractible circle — a contradiction.  $\triangle$

Further, note that  $\tilde{G}$  is homeomorphic to the plane, since  $\tilde{G}$  is a 2-dimensional manifold without boundary which by the above is CAT[0] and hence is contractible.

Denote by  $C_R$  the circle of radius  $R$  in  $\tilde{G}$  centered at  $\tilde{v}$ . All  $C_R$  are homotopic to each other in  $\tilde{G} \setminus \{\tilde{v}\}$  and therefore in  $\tilde{\mathcal{Q}} \setminus \{\tilde{v}\}$ .

Note that the map  $\tilde{\mathcal{Q}} \setminus \{\tilde{v}\} \rightarrow \mathcal{S}$  which returns the direction of  $[\tilde{v}x]$  for any  $x \neq \tilde{v}$ , maps  $C_R$  to a circle homotopic to  $\gamma$ . Therefore  $C_R$  is not contractible in  $\tilde{\mathcal{Q}} \setminus \{\tilde{v}\}$ .

In particular,  $C_R$  is not contractible in  $\tilde{\mathcal{Q}} \setminus K$  if  $K \supset \tilde{v}$ . If  $R$  is large, the circle  $C_R$  lies outside of any compact set  $K'$  in  $\tilde{\mathcal{Q}}$ . It follows that  $\tilde{\mathcal{Q}}$  is not simply connected at infinity, a contradiction.  $\square$

**3.7.3. Exercise.** Under the assumptions of the Lemma 3.7.2, for any vertex  $v$  in  $\mathcal{S}$  the complement  $\mathcal{S} \setminus \{v\}$  is simply connected.

*Proof of Proposition 3.7.1.* Let  $\Sigma^{m-1}$  be an  $(m-1)$ -dimensional smooth homology sphere which is not simply connected and bounds a contractible smooth compact  $m$ -dimensional manifold  $\mathcal{W}$ .

For  $m \geq 5$  the existence of such  $(\mathcal{W}, \Sigma)$  follows from [37]. For  $m = 4$  it follows from the construction in [40].

Pick any smooth triangulation  $\tau$  of  $W$  and let  $\mathcal{S}$  be the resulting subcomplex that triangulates  $\Sigma$ .

We can assume that  $\mathcal{S}$  is flag; otherwise pass to the barycentric subdivision of  $\tau$  and apply Exercise 3.5.2.

Let  $\mathcal{Q} = \square_{\mathcal{S}}$  be the cubical analog of  $\mathcal{S}$ .

By Proposition 3.6.1,  $\mathcal{Q}$  is a homology manifold. It follows that  $\mathcal{Q}$  is a piecewise linear manifold with a finite number of singularities at its vertices.

Removing a small contractible neighborhood  $V_v$  of each vertex  $v$  in  $\mathcal{Q}$ , we can obtain a piecewise linear manifold  $\mathcal{N}$  whose boundary is formed by several copies of  $\Sigma$ . To each copy of  $\Sigma$ , glue a copy of  $\mathcal{W}$  along its boundary. This results in a closed piecewise linear manifold  $\mathcal{M}$  which is homotopically equivalent to  $\mathcal{Q}$ .

Indeed, since both  $V_v$  and  $W$  are contractible, the identity map of their common boundary  $\Sigma$  can be extended to a homotopy equivalence relative to the boundary between  $V_v$  and  $W$ . Therefore the identity map on  $\mathcal{N}$  extends to homotopy equivalences  $f: \mathcal{Q} \rightarrow \mathcal{M}$  and  $g: \mathcal{M} \rightarrow \mathcal{Q}$ .

Finally, by the Lemma 3.7.2, the universal cover  $\tilde{\mathcal{Q}}$  of  $\mathcal{Q}$  is not simply connected at infinity.

The same holds for the universal cover  $\tilde{\mathcal{M}}$  of  $\mathcal{M}$ . The latter follows since the constructed homotopy equivalences  $f: \mathcal{Q} \rightarrow \mathcal{M}$  and  $g: \mathcal{M} \rightarrow \mathcal{Q}$  lift to *proper maps*  $\tilde{f}: \tilde{\mathcal{Q}} \rightarrow \tilde{\mathcal{M}}$  and  $\tilde{g}: \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{Q}}$ ; that is, for any compact sets  $A \subset \tilde{\mathcal{Q}}$  and  $B \subset \tilde{\mathcal{M}}$ , the inverse images  $\tilde{g}^{-1}(A)$  and  $\tilde{f}^{-1}(B)$  are compact.  $\square$

## 3.8 Comments

In the globalization theorem (3.3.1) properness can be weakened to completeness, see our book [4] and the references therein.

The condition on polyhedral  $\text{CAT}[\kappa]$  spaces given in Theorem 3.4.2 might look easy to use, but in fact, it is hard to check even in the very simple cases. For example the description of those coverings of  $\mathbb{S}^3$  branching at three great circles which are  $\text{CAT}[1]$  requires quite a bit of work; see [23] — try to guess the answer before reading.

Another example is the space  $\mathcal{B}_4$  which is the universal cover of  $\mathbb{C}^4$  infinitely branching in the complex planes  $z_i = z_j$  with the induced length metric. So far it is not known if  $\mathcal{B}_4$  is CAT[0]. This space could be helpful to study the braid group on 4 strings; read [44] by Dmitri Panov and Anton Petrunin for more on it. This circle of questions is closely related to the generalization of the flag condition (3.5.5) to the spherical simplices with few acute dihedral angles.

The construction used in the proof of Proposition 3.7.1 admits a number of interesting modifications, some of which are discussed in the survey [26] by Michael Davis.

The following Proposition was proved by Fredric Ancel, Michael Davis and Craig Guilbault in [8].

**3.8.1. Proposition.** *Given  $m \geq 5$ , there is a Euclidean polyhedral space  $\mathcal{P}$  such that:*

- a)  $\mathcal{P}$  is homeomorphic to a closed  $m$ -dimensional manifold.
- b)  $\mathcal{P}$  is locally CAT[0].
- c) The universal cover of  $\mathcal{P}$  is not simply connected at infinity.

There are no 3-dimensional examples of that type, see [48] by Dale Rolfsen. In [54], Paul Thurston conjectured that the same holds in the 4-dimensional case.

*Proof.* Apply Exercise 3.7.3 to the barycentric subdivision of the simplicial complex  $\mathcal{S}$  provided by Exercise 3.8.2.  $\square$

**3.8.2. Exercise.** *Given an integer  $m \geq 5$ , construct a finite  $(m-1)$ -dimensional simplicial complex  $\mathcal{S}$  such that Cone  $\mathcal{S}$  is homeomorphic to  $\mathbb{E}^m$  and  $\pi_1(\mathcal{S} \setminus \{v\}) \neq 0$  for some vertex  $v$  in  $\mathcal{S}$ .*

A similar construction was used by Michael Davis, Tadeusz Januszkiewicz and Jean-François Lafont in [28] to construct a closed smooth 4-dimensional manifold  $M$  with universal cover  $\tilde{M}$  diffeomorphic to  $\mathbb{R}^4$  such that  $M$  admits a polyhedral metric which is locally CAT[0], but does not admit a Riemannian metric with nonpositive sectional curvature. Another example of that type was constructed by Stephan Stadler, see [52].

It is noteworthy that any complete, simply connected Riemannian manifold with nonpositive curvature is homeomorphic to the Euclidean space of the same dimension. In fact, by the globalization theorem (3.3.1) the exponential map at a point of such a manifold is a homeomorphism. In particular, there is no Riemannian analog of Proposition 3.8.1. Moreover, according to Stone's theorem, see [53, 27], there

is no piecewise linear analog of the proposition; that is the homeomorphism to a manifold in Proposition 3.8.1 can not be made piecewise linear.

The construction described in this chapter also leads to the so-called *hyperbolization* procedure, a flexible tool to construct aspherical spaces; a good survey on the subject is given by Ruth Charney and Michael Davis in [24].

All the topics discussed above link Alexandrov geometry with the fundamental group. The theory of *hyperbolic groups*, a branch of *geometric group theory*, introduced by Mikhael Gromov, could be considered as a further step in this direction, see [31].



# Chapter 4

## Subsets

In this chapter we give a partial answer to the following question. *Which subsets of Euclidean space, equipped with their induced length-metrics, form CAT[0] spaces?*

### 4.1 Motivating examples

Consider three subgraphs of different quadric surfaces:

$$\begin{aligned} A &= \{ (x, y, z) \in \mathbb{E}^3 \mid z \leq x^2 + y^2 \}, \\ B &= \{ (x, y, z) \in \mathbb{E}^3 \mid z \leq -x^2 - y^2 \}, \\ C &= \{ (x, y, z) \in \mathbb{E}^3 \mid z \leq x^2 - y^2 \}. \end{aligned}$$

**4.1.1. Question.** *Which of the sets  $A$ ,  $B$  and  $C$ , if equipped with the induced length metric, forms a CAT[0] space and why?*

The answer is given below, but it is instructive to think about this question before reading further.

**A.** No,  $A$  is not CAT[0].

The boundary  $\partial A$  is the paraboloid described by  $z = x^2 + y^2$ ; in particular it bounds an open convex set in  $\mathbb{E}^3$  whose complement is  $A$ . The closest point projection of  $A \rightarrow \partial A$  is short (Exercise 2.2.13). It follows that  $\partial A$  is a convex set in  $A$  equipped with its induced length metric.

Therefore if  $A$  is CAT[0] then so is  $\partial A$ . The latter is not true:  $\partial A$  is a smooth convex surface, and has strictly positive curvature by the Gauss formula.

**B.** Yes,  $B$  is CAT[0].

Evidently  $B$  is a convex closed set in  $\mathbb{E}^3$ . Therefore the length metric on  $B$  coincides with Euclidean metric and CAT[0] comparison holds.

**C.** Yes,  $C$  is CAT[0], but the proof is not as easy as before. We give a sketch here; a complete proof of a more general statement is given in Section 4.3.

Consider the one-parameter family of sets

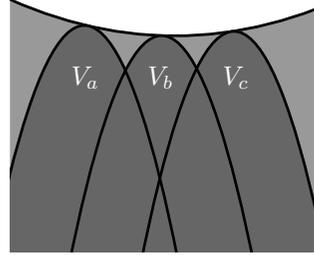
$$V_t = \{ (x, y, z) \in \mathbb{E}^3 \mid z \leq x^2 - y^2 - 2 \cdot (x - t)^2 \}.$$

Each set  $V_t$  is the subgraph of a paraboloid tangent to  $\partial C$  along the curve  $(t, y, t^2 - y^2)$ . Thus  $V_t$  is closed and convex, and

$$C = \bigcup_t V_t.$$

Further note that

- ❶ *If  $a < b < c$  then  $V_b \supset V_a \cap V_c$ .*



The inclusion ❶ makes it possible to apply Reshetnyak gluing theorem 2.3.1 recursively, in order to show that any finite union of  $V_t$  forms a CAT[0] space. By approximation, the CAT[0] comparison holds for any 4 points in  $C$ ; that is,  $C$  is CAT[0].

**Remark.** The set  $C$  is not convex, but it is two-convex as defined in the next section. As you will see, two-convexity of subsets is closely related to the inheritance of an upper curvature bound by a subset.

## 4.2 Two-convexity

The following definition is closely related to the one given by Mikhael Gromov in [32, §½], see also [43].

**4.2.1. Definition.** *We say that a subset  $K \subset \mathbb{E}^m$  is two-convex if the following condition holds for any plane  $W \subset \mathbb{E}^m$ : If  $\gamma$  is a simple closed curve in  $W \cap K$  which is null-homotopic in  $K$ , then it is null-homotopic in  $W \cap K$ , in particular the disc in  $W$  bounded by  $\gamma$  lies in  $K$ .*

Note that two-convex sets do not have to be connected or simply connected. The following two propositions follow immediately from the definition.

**4.2.2. Proposition.** *Any subset in  $\mathbb{E}^2$  is two-convex.*

**4.2.3. Proposition.** *The intersection of an arbitrary collection of two-convex sets in  $\mathbb{E}^m$  is two-convex.*

**4.2.4. Proposition.** *The interior of any two-convex set in  $\mathbb{E}^m$  is a two-convex set.*

*Proof.* Let  $K \subset \mathbb{E}^m$  be two-convex and let  $W$  be a 2-plane. Fix a closed simple plane curve  $\gamma$  in  $W \cap \text{Int } K$  which is contractible in the interior of  $K$ .

Since  $K$  is two-convex, the plane disc  $D$  bounded by  $\gamma$  lies in  $K$ . The same holds for the translations of  $D$  by small vectors. Therefore  $D$  lies in the interior of  $K$ . Hence the statement follows.  $\square$

**4.2.5. Definition.** *Given a subset  $K \subset \mathbb{E}^m$ , define its two-convex hull (briefly,  $\text{Conv}_2 K$ ) as the intersection of all two-convex subsets containing  $K$ .*

Note that by Proposition 4.2.3, the two-convex hull of any set is two-convex. Further, by Proposition 4.2.4, the two-convex hull of an open set is open.

The next proposition describes closed two-convex sets with smooth boundary.

**4.2.6. Proposition.** *Let  $K \subset \mathbb{E}^m$  be a closed subset.*

*Assume that the boundary of  $K$  forms a smooth hypersurface  $S$ . Consider the unit normal vector field  $\nu$  on  $S$  that points outside of  $K$ . Denote by  $k_1 \leq \dots \leq k_{m-1}$  the scalar fields of principal curvatures on  $S$  with respect to  $\nu$  (by this we mean that if  $K$  is convex then  $k_1 \geq 0$ ).*

*Then  $K$  is two-convex if and only if  $k_2(p) \geq 0$  for any point  $p \in S$ . Moreover, if  $k_2(p) < 0$  at some point  $p$ , then Definition 4.2.1 fails for some curve  $\gamma$  forming a triangle in an arbitrary neighborhood of  $p$ .*

The proof is taken from [32, § $\frac{1}{2}$ ], but we added some details.

*Proof; “Only-if” part.* If  $k_2(p) < 0$  for some  $p \in S$ , consider the plane  $W$  through  $p$  spanned by the first two principal directions. Choose a small triangle in  $W$  which surrounds  $p$  and move it slightly in the direction of  $\nu(p)$ . We get a triangle  $[xyz]$  which is null-homotopic in  $K$ , but the solid triangle  $\Delta = \text{Conv}\{x, y, x\}$  bounded by  $[xyz]$  does not lie in  $K$  completely. Therefore  $K$  is not two-convex.

*“If” part.* Recall that a smooth function  $f: \mathbb{E}^m \rightarrow \mathbb{R}$  is called *strongly convex* if its Hessian is positive definite at each point.

Suppose  $f: \mathbb{E}^m \rightarrow \mathbb{R}$  is a smooth strongly convex function such that the restriction  $f|_S$  is a Morse function. Note that a generic smooth strongly convex function  $f: \mathbb{E}^m \rightarrow \mathbb{R}$  has this property.

For a critical point  $p$  of  $f|_S$ , the outer normal vector  $\nu(p)$  is parallel to the gradient  $\nabla_p f$ ; we say that  $p$  is a *positive critical point* if  $\nu(p)$  and  $\nabla_p f$  point in the same direction, and *negative* otherwise. If  $f$  is generic, then we can assume that the sign is defined for all critical points; that is,  $\nabla_p f \neq 0$  for any critical point  $p$  of  $f|_S$ .

Since  $k_2 \geq 0$  and  $f$  is strongly convex, the negative critical points of  $f|_S$  have index at most 1.

Given a real value  $s$ , set

$$K_s = \{x \in K \mid f(x) < s\}.$$

Assume  $\varphi_0: \mathbb{D} \rightarrow K$  is a continuous map of the disc  $\mathbb{D}$  such that  $\varphi_0(\partial\mathbb{D}) \subset K_s$ . By the Morse Lemma, there is a homotopy  $\varphi_t: \mathbb{D} \rightarrow K$  rel  $\partial\mathbb{D}$  such that  $\varphi_1(\mathbb{D}) \subset K_s$ .

Indeed, we can construct a homotopy  $\varphi_t: \mathbb{D} \rightarrow K$  which decreases the maximum of  $f \circ \varphi$  on  $\mathbb{D}$  until the maximum occurs at a critical point  $p$  of  $f|_S$ . This point cannot be negative, otherwise its index would be at least 2. If this critical point is positive then it is easy to decrease the maximum a little by pushing the disc from  $S$  into  $K$  in the direction of  $-\nabla f_p$ .

Consider a closed curve  $\gamma: \mathbb{S}^1 \rightarrow K$  which is null-homotopic in  $K$ . Note that the distance function

$$f_0(x) = |\text{Conv } \gamma - x|_{\mathbb{E}^m}$$

is convex. Therefore  $f_0$  can be approximated by smooth strongly convex functions  $f$  in general position. From above, there is a disc in  $K$  with boundary  $\gamma$  which lies arbitrary close to  $\text{Conv } \gamma$ . Since  $K$  is closed, the statement follows.  $\square$

Note that the “if” part proves a somewhat stronger statement. Namely, any plane curve  $\gamma$  (*not necessary simple*) which is contractible in  $K$  is also contractible in the intersection of  $K$  with the plane of  $\gamma$ . The latter condition does not hold for the complement of two planes in  $\mathbb{E}^4$ , which is two-convex by Proposition 4.2.3; see also Exercise 4.5.3 below. The following proposition shows that there are no such open examples in  $\mathbb{E}^3$ .

**4.2.7. Proposition.** *Let  $\Omega \subset \mathbb{E}^3$  be an open two-convex subset. Then for any plane  $W \subset \mathbb{E}^3$ , any closed curve in  $W \cap \Omega$  which is null-homotopic in  $\Omega$  is also null-homotopic in  $W \cap \Omega$ .*

In the proof we use Papakyriakopoulos’ loop theorem. It states that for any 3-dimensional manifold  $M$  with nonempty boundary  $\partial M$ , if  $f: (D^2, \partial D^2) \rightarrow (M, \partial M)$  is a continuous map from disc  $D$  such that  $f|\partial D^2$  is not null-homotopic in  $\partial M$ , then there is an embedding

$h: (D^2, \partial D^2) \rightarrow (M, \partial M)$  with the same property. A proof can be found in [34].

*Proof.* Fix a closed plane curve  $\gamma$  in  $W \cap \Omega$  which is null-homotopic in  $\Omega$ . Suppose  $\gamma$  is not contractible in  $W \cap \Omega$ .

Let  $\varphi: D \rightarrow \Omega$  be a map of the disk that bounds  $\gamma$ .

Since  $\Omega$  is open we can first change  $\varphi$  slightly so that  $\varphi(x) \notin W$  for  $1 - \varepsilon < |x| < 1$  for some small  $\varepsilon > 0$ . By further changing  $\varphi$  slightly we can assume that it is transversal to  $W$  and agrees with the previous map near  $\partial D$ .

This means that  $\varphi^{-1}(W) \cap \text{Int } D$  consists of finitely many simple closed curves which cut  $D$  into several components. Consider one of the “innermost” components. It is a disk  $D'$  with boundary  $c'$ . Then  $\varphi(c')$  is a closed curve in  $W$ .

Note that by construction  $\varphi(D')$  completely lies in one of the two half-spaces with boundary  $W$ . We denote that half-space by  $H$ . If  $\varphi(c')$  is not contractible in  $W \cap \Omega$ , then applying the loop theorem to  $M^3 = H \cap \Omega$  we conclude that there exists a *simple* closed curve  $\gamma' \subset \Omega \cap W$  which is not contractible in  $\Omega \cap W$  but is contractible in  $\Omega \cap H$ . This contradicts two-convexity of  $\Omega$ .

Hence  $\varphi(c')$  is contractible in  $W \cap \Omega$ . Therefore  $\varphi$  can be changed in a small neighborhood  $U$  of  $D'$  so that the new map  $\hat{\varphi}$  maps  $U$  to one side of  $W$ . In particular, the set  $\hat{\varphi}^{-1}(W)$  consists of the same curves as  $\varphi^{-1}(W)$  with the exception of  $c'$ .

Repeating this process several times we reduce the problem to the case where  $\varphi^{-1}(W) \cap \text{Int } D = \emptyset$ . This means that  $\varphi(D)$  lies entirely in one of the half-spaces bounded by  $W$ .

Again applying the loop theorem, we obtain a simple closed curve in  $W \cap \Omega$  which is not contractible in  $W \cap \Omega$  but is contractible in  $\Omega$ . This again contradicts two-convexity of  $\Omega$ . Hence  $\gamma$  is contractible in  $W \cap \Omega$  as claimed.  $\square$

### 4.3 Sets with smooth boundary

In this section we characterize the subsets with smooth boundary in  $\mathbb{E}^m$  that form CAT[0] spaces.

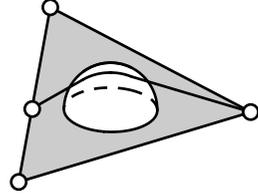
**4.3.1. Smooth two-convexity theorem.** *Let  $K$  be a closed, simply connected subset in  $\mathbb{E}^m$  equipped with the induced length metric. Assume  $K$  is bounded by a smooth hypersurface. Then  $K$  is CAT[0] if and only if  $K$  is two-convex.*

This theorem is a baby case of the main result in [1], which is briefly discussed at the end of the chapter.

*Proof.* Denote by  $S$  and by  $\Omega$  the boundary and the interior of  $K$  respectively. Since  $K$  is connected and  $S$  is smooth,  $\Omega$  is also connected.

Denote by  $k_1(p) \leq \dots \leq k_{m-1}(p)$  the principle curvatures of  $S$  at  $p \in S$  with respect to the normal vector  $\mathbf{v}(p)$  pointing out of  $K$ . By Proposition 4.2.6,  $K$  is two-convex if and only if  $k_2(p) \geq 0$  for any  $p \in S$ .

*“Only-if” part.* Assume  $K$  is not two-convex. Then by Proposition 4.2.6, there is a triangle  $[xyz]$  in  $K$  which is null-homotopic in  $K$ , but the solid triangle  $\Delta = \text{Conv}\{x, y, z\}$  does not lie in  $K$  completely. Evidently the triangle  $[xyz]$  is not thin in  $K$ ; that is,  $K$  is not CAT[0].



*“If” part.* Since  $K$  is simply connected, by the globalization theorem (3.3.1) it suffices to show that any point  $p \in K$  admits a CAT[0] neighborhood.

If  $p \in \text{Int } K$  then it admits a neighborhood isometric to a subset of  $\mathbb{E}^m$ . Hence the statement follows.

Fix  $p \in S$ . Assume that  $k_2(p) > 0$ . Fix a sufficiently small  $\varepsilon > 0$  and set  $K' = K \cap \overline{B}[p, \varepsilon]$ . Let us show that

❶  $K'$  is a CAT[0] space.

Consider the coordinate system with the origin at  $p$  and the principle directions and  $\mathbf{v}(p)$  as the coordinate directions. For small  $\varepsilon > 0$ , the set  $K'$  can be described as a subgraph

$$K' = \{ (x_1, \dots, x_m) \in \overline{B}[p, \varepsilon] \mid x_m \leq f(x_1, \dots, x_{m-1}) \}.$$

Moreover, since  $\varepsilon$  is small and  $k_2(p) > 0$ , for any fixed  $s \in [-\varepsilon, \varepsilon]$  the function

$$(x_2, \dots, x_{m-1}) \mapsto f(s, x_2, \dots, x_{m-1})$$

is concave for  $|x_i| < 2 \cdot \varepsilon$  for each  $i$

Fix a negative real value  $\lambda < k_1(p)$ . Given  $s \in (-\varepsilon, \varepsilon)$ , consider the set

$$V_s = \{ (x_1, \dots, x_m) \in K' \mid x_m \leq f(x_1, \dots, x_{m-1}) + \lambda \cdot (x_1 - s)^2 \}.$$

Note that the function

$$(x_1, \dots, x_{m-1}) \mapsto f(x_1, \dots, x_{m-1}) + \lambda \cdot (x_1 - s)^2$$

is concave near the origin. Since  $\varepsilon$  is small, we can assume that  $V_s$  forms a convex set in  $\mathbb{E}^m$  for any  $s$ .

Further note that

$$K' = \bigcup_{s \in [-\varepsilon, \varepsilon]} V_s$$

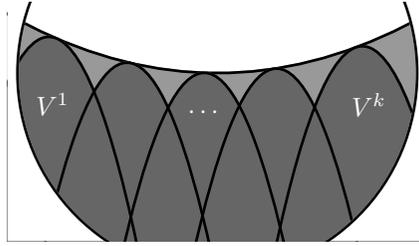
and

- ❷ If  $a < b < c$  then  $V_b \supset V_a \cap V_c$ .

Given an array of values  $s^1 < \dots < s^k$  in  $[-\varepsilon, \varepsilon]$ , set  $V^i = V_{s^i}$  and consider the unions

$$W^i = V^1 \cup \dots \cup V^i$$

equipped with the induced length metric.



Note that the array  $(s^n)$  can be chosen in such a way that  $W^k$  is arbitrarily Hausdorff close to  $K'$ .

By Proposition 2.1.1, in order to prove ❶, it is sufficient to show the following:

- ❸ All  $W^i$  form CAT[0] spaces.

This claim is proved by induction. Base:  $W^1 = V^1$  is CAT[0] as a convex subset in  $\mathbb{E}^m$ .

Step: Assume that  $W^i$  is CAT[0]. According to ❷

$$V^{i+1} \cap W^i = V^{i+1} \cap V^i.$$

Moreover, this set forms a convex set in  $\mathbb{E}^m$  and therefore it is a convex set in  $W^i$  and in  $V^{i+1}$ . By Reshetnyak's gluing theorem,  $W^{i+1}$  is CAT[0]. Hence the claim follows.  $\triangle$

Note that we have proved the following:

- ❹  $K$  is a CAT[0] space if  $K$  is strongly two-convex, that is,  $k_2(p) > 0$  at any point  $p \in S$ .

It remains to show that  $p$  admits a CAT[0] neighborhood in the case  $k_2(p) = 0$ .

Choose a coordinate system  $(x_1, \dots, x_m)$  as above, so that the  $(x_1, \dots, x_{m-1})$ -coordinate hyperplane is the tangent subspace to  $S$  at  $p$ .

Fix  $\varepsilon > 0$  so that a neighborhood of  $p$  in  $S$  is the graph

$$x_m = f(x_1, \dots, x_{m-1})$$

of a function  $f$  defined on the open ball  $B$  of radius  $\varepsilon$  centered at the origin in the  $(x_1, \dots, x_{m-1})$ -hyperplane. Fix a smooth positive strongly convex function  $\varphi: B \rightarrow \mathbb{R}_+$  such that  $\varphi(x) \rightarrow \infty$  as  $x$  approaches the boundary  $B$ . Note that for  $\delta > 0$ , the subgraph  $K_\delta$  defined by the inequality

$$x_m \leq f(x_1, \dots, x_{m-1}) - \delta \cdot \varphi(x_1, \dots, x_{m-1})$$

is strongly two-convex. By **4**,  $K_\delta$  is CAT[0].

Finally as  $\delta \rightarrow 0$ , the closed  $\varepsilon$ -neighborhoods of  $p$  in  $K_\delta$  converge to the closed  $\varepsilon$ -neighborhood of  $p$  in  $K$ . That is,  $\varepsilon$ -neighborhoods of  $p$  are CAT[0].  $\square$

## 4.4 Open plane sets

In this section we consider inheritance of upper curvature bounds by subsets of the Euclidean plane.

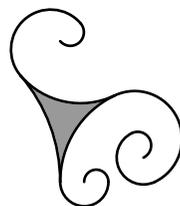
**4.4.1. Theorem.** *Let  $\Omega$  be an open simply connected subset of  $\mathbb{E}^2$ . Equip  $\Omega$  with its induced length metric and denote its completion by  $K$ . Then  $K$  is CAT[0].*

The assumption that the set  $\Omega$  is open is not critical; instead one can assume that the induced length metric takes finite values at all points of  $\Omega$ . We sketch the proof from the note [12] by Richard Bishop and leave the details to be finished as an exercise. A generalization of this result is proved by Alexander Lytchak and Stefan Wenger [39, Proposition 12.1]; this paper also contains a far reaching application.

*Sketch of proof.* It is sufficient to show that any triangle in  $K$  is thin, as defined in 2.2.1.

Note that  $K$  admits a length-preserving map to  $\mathbb{E}^2$  which extends the embedding  $\Omega \hookrightarrow \mathbb{E}^2$ . Therefore each triangle  $[xyz]$  in  $K$  can be mapped to the plane in a length-preserving way. Since  $\Omega$  is simply connected, any open region, say  $\Delta$ , that is surrounded by the image of  $[xyz]$  lies completely in  $\Omega$ .

Note that in each triangle  $[xyz]$  in  $K$ , the sides  $[xy]$ ,  $[yz]$  and  $[zx]$  intersect each other along a geodesic starting at a common vertex, possibly a one-point geodesic. In other words, every triangle in  $K$  looks like the one in the diagram.



Indeed, assuming the contrary, there will be a simple diangle in  $K$ , that is, two minimizing geodesics with common ends but no other common points. The image of this diangle in the plane must bound an open region of  $\Omega$  and both sides of the region must be concave in the plane, since otherwise one could shorten the sides by pushing them into  $\Omega$ . Evidently, there is no such plane diangle with concave sides, a contradiction.

Note that it is enough to consider only simple triangles  $[xyz]$ , that is, triangles whose sides  $[xy]$ ,  $[yz]$  and  $[zx]$  intersect each other only at the common vertices. If this is not the case, chopping the overlapping part of sides reduces to the injective case.

Again, the open region, say  $\Delta$ , bounded by the image of  $[xyz]$  has concave sides in the plane, since otherwise one could shorten the sides by pushing them into  $\Omega$ . It remains to solve the exercise below.  $\square$

**4.4.2. Exercise.** *Assume  $S$  is a closed plane region whose boundary is a triangle  $T$  with concave sides in the plane. Equip  $S$  with the induced length metric. Show that the triangle  $T$  is thin in  $S$ .*

Here is a spherical analog of Theorem 4.4.1, which can be proved along the same lines. It will be used in the next section.

**4.4.3. Exercise.** *Let  $\Theta$  be an open connected subset of the unit sphere  $\mathbb{S}^2$  which does not contain a closed hemisphere. Equip  $\Theta$  with the induced length metric. Let  $\tilde{\Theta}$  be a metric covering of  $\Theta$  such that any closed curve in  $\tilde{\Theta}$  of length smaller than  $2\pi$  is contractible.*

*Show that the completion of  $\tilde{\Theta}$  is a CAT[1] space.*

## 4.5 Open space sets

The material below was inspired by work of Samuel Shefel, and developed by Anton Petrunin.

Shefel seems very intrigued by the survival of metric properties under affine transformation. To describe an instance of such phenomena, note that two-convexity survives under affine transformations of a Euclidean space. Therefore, as a consequence of the smooth two-convexity theorem (4.3.1), the following holds.

**4.5.1. Corollary.** *Let  $K$  be closed connected subset of Euclidean space equipped with the induced length metric. Assume  $K$  is bounded by a smooth hypersurface and is a CAT[0] space. Then any affine transformation of  $K$  is also CAT[0].*

By Corollary 4.5.4, an analogous statement holds for sets bounded by Lipschitz surfaces in the three-dimensional Euclidean space. In higher dimensions this is no longer true, see Exercise 4.5.13.

Here is the main theorem of this section.

**4.5.2. Two-convexity theorem.** *Let  $\Omega$  be a connected open set in  $\mathbb{E}^3$ . Equip  $\Omega$  with the induced length metric and denote by  $K$  the completion of the universal metric cover of  $\Omega$ . Then  $K$  is a CAT[0] space if and only if  $\Omega$  is two-convex.*

The following exercise shows that the analogous statement does not hold in higher dimensions.

**4.5.3. Exercise.** *Let  $\Pi_1, \Pi_2$  be two planes in  $\mathbb{E}^4$  intersecting at a single point. Consider the universal cover  $K$  of the space obtained by infinitely branching  $\mathbb{E}^4$  along  $\Pi_1$  and  $\Pi_2$ , and equip  $K$  with the induced length metric.*

*Show that  $K$  is a CAT[0] space if and only if  $\Pi_1 \perp \Pi_2$ .*

Before coming to the proof of the two-convexity theorem, let us formulate a few corollaries. The following corollary is a generalization of the smooth two-convexity theorem (4.3.1) for three-dimensional Euclidean space.

**4.5.4. Corollary.** *Let  $K$  be a closed subset in  $\mathbb{E}^3$  bounded by a Lipschitz hypersurface. Then  $K$  with the induced length metric is a CAT[0] space if and only if the interior of  $K$  is two-convex and simply connected.*

*Proof.* Set  $\Omega = \text{Int } K$ . Since  $K$  is simply connected and bounded by a surface,  $\Omega$  is also simply connected.

Apply the two-convexity theorem to  $\Omega$ . Note that the completion of  $\Omega$  equipped with the induced length metric is isometric to  $K$  with the induced length metric. Hence the result follows.  $\square$

The following corollary is the main statement in Shefel's original paper [51]. In order to formulate it, we need yet one more definition.

Let  $U$  be an open set in  $\mathbb{R}^2$ . A continuous function  $f: U \rightarrow \mathbb{R}$  is called *saddle* if for any linear function  $\ell: \mathbb{R}^2 \rightarrow \mathbb{R}$ , the difference  $f - \ell$  does not have local maxima or local minima in  $U$ . Equivalently, the

open subgraph and epigraph of  $f$

$$\begin{aligned} & \{ (x, y, z) \in \mathbb{E}^3 \mid z < f(x, y), (x, y) \in U \}, \\ & \{ (x, y, z) \in \mathbb{E}^3 \mid z > f(x, y), (x, y) \in U \} \end{aligned}$$

are two-convex.

**4.5.5. Corollary.** *Let  $f: \mathbb{D} \rightarrow \mathbb{R}$  be a Lipschitz function which is saddle in the interior of the closed unit disc  $\mathbb{D}$ . Then the graph*

$$\Gamma = \{ (x, y, z) \in \mathbb{E}^3 \mid z = f(x, y) \},$$

*equipped with induced length metric is a CAT[0] space.*

*Proof.* Since the function  $f$  is Lipschitz, its graph  $\Gamma$  with the induced length metric is bi-Lipschitz equivalent to  $\mathbb{D}$  with the Euclidean metric.

Consider the sequence of sets

$$K_n = \{ (x, y, z) \in \mathbb{E}^3 \mid z \leq f(x, y) \pm \frac{1}{n}, (x, y) \in \mathbb{D} \}.$$

Note that each  $K_n$  is closed, simply connected. Directly from the definition, it follows that  $K$  is also two-convex. Moreover the boundary of  $K_n$  forms a Lipschitz surface.

Equip  $K_n$  with the induced length metric. By Corollary 4.5.4,  $K_n$  is a CAT[0] space. It remains to note that as  $n \rightarrow \infty$  the sequence of space  $K_n$  converges to  $\Gamma$  in the sense of Gromov–Hausdorff, and apply Proposition 2.1.1.  $\square$

Now we are back to the proof of the two-convexity theorem (4.5.2).

Recall that a subset  $P$  of  $\mathbb{E}^m$  is called a *polytope* if it can be presented as a union of a finite number of simplices. Similarly, a *spherical polytope* is a union of a finite number of simplices in  $\mathbb{S}^m$ .

Note that any polytope admits a finite triangulation. Therefore any polytope equipped with the induced length metric forms a Euclidean polyhedral space as defined in 3.4.1.

Let  $P$  be a polytope and  $\Omega$  its interior, both considered with the induced length metrics. Typically, the completion  $K$  of  $\Omega$  is isometric to  $P$ . However in general we only have a locally distance preserving map  $K \rightarrow P$ ; it does not have to be onto and it may not be injective. An example can be guessed from the picture. Nevertheless, is easy to see that  $K$  is always a polyhedral space.



**4.5.6. Lemma.** *The two-convexity theorem (4.5.2) holds if the set  $\Omega$  is the interior of a polytope.*

The statement might look obvious, but there is a hidden obstacle in the proof. The proof uses the following exercises.

**4.5.7. Exercise.** *Show that any closed path of length  $< 2\cdot\pi$  in the unit sphere  $\mathbb{S}^2$  lies in an open hemisphere.*

**4.5.8. Exercise.** *Assume  $\Omega$  is an open subset in  $\mathbb{E}^3$  which is not two-convex. Show that there is a plane  $W$  such that the complement  $W\setminus\Omega$  contains an isolated point.*

*Proof of Lemma 4.5.6.* The “only if” part can be proved in the same way as in the smooth two-convexity theorem (4.3.1) with additional use of Exercise 4.5.8.

*“If” part.* Assume that  $\Omega$  is two-convex. Denote by  $\tilde{\Omega}$  the universal metric cover of  $\Omega$ . Let  $\tilde{K}$  and  $K$  be the completions of  $\tilde{\Omega}$  and  $\Omega$  correspondingly.

The main step is to show that  $\tilde{K}$  is CAT[0].

Note that  $K$  is a polyhedral space and the covering  $\tilde{\Omega} \rightarrow \Omega$  extends to a covering map  $\tilde{K} \rightarrow K$  which might be branching at some vertices. For example, if

$$K = \{ (x, y, z) \in \mathbb{E}^3 \mid |z| \leq |x| + |y| \leq 1 \}$$

and  $p$  is the origin then  $\Sigma_p$ , the space of directions at  $p$ , is not simply connected and  $\tilde{K} \rightarrow K$  branches at  $p$ .

Fix a point  $\tilde{p} \in \tilde{K}\setminus\tilde{\Omega}$ ; denote by  $p$  the image of  $\tilde{p}$  in  $K$ . By the globalization theorem (3.3.1), it is sufficient to show:

❶ *A small neighborhood of  $\tilde{p}$  in  $\tilde{K}$  is CAT[0].*

Recall that  $\Sigma_{\tilde{p}} = \Sigma_{\tilde{p}}\tilde{K}$  denotes the space of directions at  $\tilde{p}$ . Note that a small neighborhood of  $\tilde{p}$  in  $\tilde{K}$  is isometric to an open set in the cone over  $\Sigma_{\tilde{p}}\tilde{K}$ . By Exercise 2.1.2, ❶ follows once we can show

❷  *$\Sigma_{\tilde{p}}$  is CAT[1].*

By rescaling, we can assume that every face of  $K$  which does not contain  $p$  lies at distance at least 2 from  $p$ . Denote by  $\mathbb{S}^2$  the unit sphere centered at  $p$ , set  $\Theta = \mathbb{S}^2 \cap \Omega$ . Note that  $\Sigma_p K$  is isometric to the completion of  $\Theta$  and  $\Sigma_{\tilde{p}}\tilde{K}$  is the completion of the regular metric covering  $\tilde{\Theta}$  of  $\Theta$  induced by the universal metric cover  $\tilde{\Omega} \rightarrow \Omega$ .

By Exercise 4.4.3, it remains to show the following:

❸ *Any closed curve in  $\tilde{\Theta}$  of length less than  $2\cdot\pi$  is contractible.*

Fix a closed curve  $\tilde{\gamma}$  of length  $< 2\cdot\pi$  in  $\tilde{\Theta}$ . Its projection  $\gamma$  in  $\Theta \subset \mathbb{S}^2$  has the same length. Therefore, by Exercise 4.5.7,  $\gamma$  lies in an

open hemisphere. Then for a plane  $\Pi$  almost passing through  $p$ , the central projection  $\gamma'$  of  $\gamma$  to  $\Pi$  is defined and lies in  $\Omega$ . Moreover, it is contractible in  $\Omega$ . From two-convexity of  $\Omega$  and Proposition 4.2.7, the curve  $\gamma'$  is contractible in  $\Pi \cap \Omega$ .

It follows that  $\gamma$  is contractible in  $\Theta$  and therefore  $\tilde{\gamma}$  is contractible in  $\tilde{\Theta}$ .  $\square$

The following proposition describes a construction which was essentially given by Samuel Shefel in [51]. It produces  $\text{Conv}_2 \Omega$  for an open set  $\Omega \subset \mathbb{E}^3$ .

**4.5.9. Proposition.** *Let  $\Pi_1, \Pi_2 \dots$  be an everywhere dense sequence of planes in  $\mathbb{E}^3$ . Given an open set  $\Omega$ , consider the recursively defined sequence of open sets  $\Omega = \Omega_0 \subset \Omega_1 \subset \dots$  such that  $\Omega_n$  is the union of  $\Omega_{n-1}$  and all the bounded components of  $\mathbb{E}^3 \setminus (\Pi_n \cup \Omega_{n-1})$ . Then*

$$\text{Conv}_2 \Omega = \bigcup_n \Omega_n.$$

*Proof.* Set

$$\textcircled{4} \quad \Omega' = \bigcup_n \Omega_n.$$

Note that  $\Omega'$  is a union of open sets, in particular  $\Omega'$  is open.

Let us show that

$$\textcircled{5} \quad \text{Conv}_2 \Omega \supset \Omega'.$$

Suppose we already know that  $\text{Conv}_2 \Omega \supset \Omega_{n-1}$ . Fix a bounded component  $\mathcal{C}$  of  $\mathbb{E}^3 \setminus (\Pi_n \cup \Omega_{n-1})$ . It is sufficient to show that  $\mathcal{C} \subset \text{Conv}_2 \Omega$ .

By Proposition 4.2.4,  $\text{Conv}_2 \Omega$  is open. Therefore, if  $\mathcal{C} \not\subset \text{Conv}_2 \Omega$ , then there is a point  $p \in \mathcal{C} \setminus \text{Conv}_2 \Omega$  lying at maximal distance from  $\Pi_n$ . Denote by  $W_p$  the plane through  $p$  parallel to  $\Pi_n$ .

Note that  $p$  lies in a bounded component of  $W_p \setminus \text{Conv}_2 \Omega$ . In particular it can be surrounded by a simple closed curve  $\gamma$  in  $W_p \cap \text{Conv}_2 \Omega$ . Since  $p$  lies at maximal distance from  $\Pi_n$ , the curve  $\gamma$  is null-homotopic in  $\text{Conv}_2 \Omega$ . Therefore  $p \in \text{Conv}_2 \Omega$ , a contradiction.

By induction,  $\text{Conv}_2 \Omega \supset \Omega_n$  for each  $n$ . Therefore  $\textcircled{4}$  implies  $\textcircled{5}$ .

It remains to show that  $\Omega'$  is two-convex. Assume the contrary; that is, there is a plane  $\Pi$  and a simple closed curve  $\gamma: \mathbb{S}^1 \rightarrow \Pi \cap \Omega'$  which is null-homotopic in  $\Omega'$ , but not null-homotopic in  $\Pi \cap \Omega'$ .

By approximation we can assume that  $\Pi = \Pi_n$  for a large enough  $n$  and that  $\gamma$  lies in  $\Omega_{n-1}$ . By the same argument as in the proof of Proposition 4.2.7 using the loop theorem, we can assume that there is

an *embedding*  $\varphi: \mathbb{D} \rightarrow \Omega'$  such that  $\varphi|_{\partial\mathbb{D}} = \gamma$  and  $\varphi(D)$  lies entirely in one of the half-spaces bounded by  $\Pi$ . By the  $n$ -step of the construction, the entire bounded domain  $U$  bounded by  $\Pi_n$  and  $\varphi(D)$  is contained in  $\Omega'$  and hence  $\gamma$  is contractible in  $\Pi \cap \Omega'$ .  $\square$

**4.5.10. Key lemma.** *The two-convex hull of the interior of a polytope in  $\mathbb{E}^3$  is also the interior of a polytope.*

*Proof.* Fix a polytope  $P$  in  $\mathbb{E}^3$ . Set  $\Omega = \text{Int } P$ . We may assume that  $\Omega$  is dense in  $P$ . Denote by  $F_1, \dots, F_m$  the facets (2-faces) of  $P$ . By subdividing  $F_i$  if necessary, we may assume that all  $F_i$  are convex.

Set  $\Omega' = \text{Conv}_2 \Omega$  and let  $P'$  be the closure of  $\Omega'$ . Further, for each  $i$ , set  $F'_i = F_i \setminus \Omega'$ . In other words,  $F'_i$  is the subset of the facet  $F_i$  which remains on the boundary of  $P'$ .

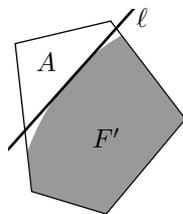
From the construction of the two-convex hull (4.5.9) we have:

⑥  $F'_i$  is a convex subset of  $F_i$ .

Further, since  $\Omega'$  is two-convex we obtain the following:

⑦ Each connected component of the complement  $F_i \setminus F'_i$  is convex.

Indeed, assume a connected component  $A$  of  $F_i \setminus F'_i$  fails to be convex. Then there is a supporting line  $\ell$  to  $F'_i$  touching  $F'_i$  at a single point in the interior of  $F_i$ . Then one could rotate the plane of  $F_i$  slightly around  $\ell$  and move it parallel to cut a “cap” from the complement of  $\Omega$ . The latter means that  $\Omega$  is not two-convex, a contradiction.  $\triangle$



From ⑥ and ⑦, we conclude

⑧  $F'_i$  is a convex polygon for each  $i$ .

Consider the complement  $\mathbb{E}^3 \setminus \Omega$  equipped with the length metric. By construction of the two-convex hull (4.5.9), the complement  $L = \mathbb{E}^3 \setminus (\Omega' \cup P)$  is locally convex; that is, any point of  $L$  admits a convex neighborhood.

Summarizing: (1)  $\Omega'$  is a two-convex open set, (2) the boundary  $\partial\Omega'$  contains a finite number of polygons  $F'_i$  and the remaining part is locally concave. It remains to show that (1) and (2) imply that  $\partial\Omega'$  is piecewise linear.

**4.5.11. Exercise.** *Prove the last statement.*  $\square$

*Proof of 4.5.2.* The “only if” part can be proved the same way as in the smooth two-convexity theorem (4.3.1) with additional use of Exercise 4.5.8.

*“If”-part.* Suppose  $\Omega$  is two-convex. We need to show that  $K$  is CAT[0].

Note that it is sufficient to show that CAT[0] comparison holds for any 4 points  $x^1, x^2, x^3, x^4 \in \Omega$ .

Fix  $\varepsilon > 0$ . Choose six broken lines connecting all pairs of points  $x^1, x^2, x^3, x^4$ , where the length of each broken line is at most  $\varepsilon$  bigger than the distance between its ends in the length metric on  $\Omega$ . Choose a polytope  $P$  in  $\Omega$  such that the interior  $\text{Int } P$  is simply connected and contains all six broken lines.

Denote by  $\Omega'$  the two-convex hull of the interior of  $P$ . According to Key Lemma (4.5.10),  $\Omega'$  is the interior of a polytope.

Equip  $\Omega'$  with the induced length metric. Consider the universal metric cover  $\tilde{\Omega}'$  of  $\Omega'$ . (The covering  $\tilde{\Omega}' \rightarrow \Omega'$  might be nontrivial; even though  $\text{Int } P$  is simply connected, its two-convex hull  $\Omega'$  might not be simply connected.) Denote by  $\tilde{K}'$  the completion of  $\tilde{\Omega}'$ .

By Lemma 4.5.6,  $\tilde{K}'$  is CAT[0].

Since  $\text{Int } P$  is simply connected, the embedding  $\text{Int } P \hookrightarrow \Omega'$  admits a lifting  $\iota: \text{Int } P \hookrightarrow \tilde{K}'$ . By construction,  $\iota$  almost preserves the distances between the points  $x^1, x^2, x^3, x^4$ ; namely

$$|\iota(x^i) - \iota(x^j)|_L \geq |x^i - x^j|_{\text{Int } P} \pm \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary and CAT[0] comparison holds in  $\tilde{K}'$ , we get that CAT[0] comparison holds in  $\Omega$  for  $x^1, x^2, x^3, x^4$ .

The statement follows since the quadruple  $x^1, x^2, x^3, x^4 \in \Omega$  is arbitrary.  $\square$

**4.5.12. Exercise.** Assume  $K \subset \mathbb{E}^m$  is a closed set bounded by a Lipschitz hypersurface. Equip  $K$  with the induced length metric. Show that if  $K$  is CAT[0] then  $K$  is two-convex.

The following exercise is analogous to Exercise 4.5.3. It provides a counterexample to the analog of Corollary 4.5.4 in higher dimensions.

**4.5.13. Exercise.** Let  $K = W \cap W'$ , where

$$W = \{ (x, y, z, t) \in \mathbb{E}^4 \mid z \geq -x^2 \}$$

and  $W' = \iota(W)$  for some motion  $\iota: \mathbb{E}^4 \rightarrow \mathbb{E}^4$ .

Show that  $K$  is always two-convex and one can choose  $\iota$  so that  $K$  with the induced length metric is not CAT[0].

**4.5.14. Exercise.** Let  $\mathcal{U}$  be a CAT[0] proper length space. Assume  $\tilde{\mathcal{U}} \rightarrow \mathcal{U}$  is a metric double covering branching along a geodesic. Show that  $\tilde{\mathcal{U}}$  is a CAT[0] space.

## 4.6 Comments

In [1] Stephanie Alexander, David Berg and Richard Bishop gave the exact upper bound on Alexandrov's curvature for the Riemannian manifolds with boundary. This theorem includes the smooth two-convexity theorem (4.3.1) as a partial case. Namely they show the following.

**4.6.1. Theorem.** *Let  $M$  be a Riemannian manifold with boundary  $\partial M$ . A direction tangent to the boundary will be called concave if there is a short geodesic in this direction which leaves the boundary and goes into the interior of  $M$ . A sectional direction (that is, a 2-plane) tangent to the boundary will be called concave if all the directions in it are concave.*

*Denote by  $\kappa$  an upper bound of sectional curvatures of  $M$  and sectional curvatures of  $\partial M$  in the concave sectional directions. Then  $M$  is locally CAT[ $\kappa$ ].*

**4.6.2. Corollary.** *Let  $M$  be a Riemannian manifold with boundary  $\partial M$ . Assume all the sectional curvatures of  $M$  and  $\partial M$  are bounded from above by  $\kappa$ . Then  $M$  is locally CAT[ $\kappa$ ].*

Under the name  $(n - 2)$ -convex sets, two-convex sets in  $\mathbb{E}^n$  were introduced by Mikhael Gromov in [32]. In addition to the inheritance of upper curvature bound by two-convex sets discussed in this chapter, these sets appear as the maximal open sets with vanishing curvature in Riemannian manifolds with non-negative or non-positive sectional curvature. This observation was made first by Sergei Buyalo for non-positive curvature [21, Lemma 5.8], and extended by Dmitri Panov and Anton Petrunin in [43].

Two-convex sets could be defined using homology instead of homotopy, as in the formulation of the Leftschetz theorem in [32, § $\frac{1}{2}$ ]. Namely, we can say that  $K$  is two-convex if the following condition holds: if a 1-dimensional cycle  $z$  has support in the intersection of  $K$  with a plane  $W$  and bounds in  $K$ , then it bounds in  $K \cap W$ .

The resulting definition is equivalent to the one used above. But unlike our definition it can be generalized to define  $k$ -convex sets in  $\mathbb{E}^m$  for  $k > 2$ . With this homological definition one can also avoid the use of Papakyriakopoulos' loop theorem, whose proof is quite involved. Nevertheless, we chose the definition using homotopies since it is easier to visualize.

We use our definition only for the open sets, for general sets one should be able to give a similar definition using an appropriate homology theory.

Say a surface  $S$  in  $\mathbb{E}^3$  is *saddle* if each connected bounded component which any plane cuts from  $S$  contains a point on the boundary of  $S$ . If  $S$  is locally described by a Lipschitz embedding, then this condition is equivalent to saying that  $S$  is two-convex.

**4.6.3. Shefel's conjecture.** *Any saddle surface in  $\mathbb{E}^3$  equipped with the length-metric is locally CAT[0].*

The conjecture is open even for the surfaces described by a bi-Lipschitz embedding of a disc. From another result of Shefel proved in [50], it follows that a saddle surface satisfies the isoperimetric inequality  $a \leq C \cdot \ell^2$  where  $a$  is the area of a disc bounded by a curve of length  $\ell$  and  $C = \frac{1}{3 \cdot \pi}$ . By a recent result of Alexander Lytchak and Stefan Wenger [39], Shefel's conjecture is equivalent to the isoperimetric inequality with the optimal constant  $C = \frac{1}{4 \cdot \pi}$ .

From Corollary 4.5.4, it follows that if there is a counterexample to Shefel's conjecture, then an arbitrary neighborhood of some point on the surface cannot be presented as a graph in any coordinate system.

Also note that a counterexample could not admit an approximation by smooth saddle surfaces. Examples of saddle surfaces which do not admit such approximation are not known so far.

Further, from the proof of Corollary 4.5.4, it follows that if a two-convex set  $S$  can be presented as an intersection of a nested sequence of simply connected open two-convex sets, then  $S$  equipped with the length-metric is locally CAT[0].

So far there is no known example of a saddle surface which cannot be presented this way. If such a surface  $S$  existed, then it would have a point  $p$  in its interior and an open set  $\Omega$  such that for any  $\varepsilon > 0$ , we have

$$\Omega \subset \text{Conv}_2[\Omega \cup B(p, \varepsilon)].$$

A subset  $K$  is called *strongly two-convex* if any null-homotopic circle  $\gamma: \mathbb{S}^1 \rightarrow K$  is also null-homotopic in  $K \cap \text{Conv}[\gamma(\mathbb{S}^1)]$ .

**4.6.4. Question.** *Is it true that any closed strongly two-convex set with dense interior bounded by a Lipschitz hypersurface in  $\mathbb{E}^m$  forms a CAT[0] space?*

For more on the subject, read [45] and [46] by Anton Petrunin and Stephan Stadler, where a variation of Shefel's conjecture with a CAT[0] ambient space is discussed, and also a related result of Shefel in dimension 2.



# Semisolutions

**Exercise 0.0.1.** Let  $\mathcal{X}$  be a 4-point metric space.

Fix a tetrahedron  $\Delta$  in  $\mathbb{R}^3$ . The vertices of  $\Delta$ , say  $x_0, x_1, x_2, x_3$ , can be identified with the points of  $\mathcal{X}$ .

Note that there is unique quadratic form  $W$  on  $\mathbb{R}^3$  such that

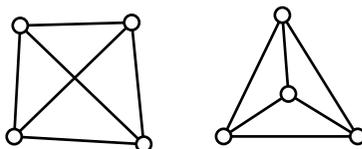
$$W(x_i - x_j) = |x_i - x_j|_{\mathcal{X}}^2$$

for all  $i$  and  $j$ ; here  $|x_i - x_j|_{\mathcal{X}}$  denotes the distance between  $x_i$  and  $x_j$  in  $\mathcal{X}$ .

By the triangle inequality,  $W(v) \geq 0$  for any vector  $v$  parallel to one of the faces of  $\Delta$ .

Note that  $\mathcal{X}$  is isometric to a 4-point subset in Euclidean space if and only if  $W(v) \geq 0$  for any vector  $v$  in  $\mathbb{R}^3$ .

Therefore, if  $\mathcal{X}$  is not of type  $\mathcal{E}_4$  then  $W(v) < 0$  for some vector  $v$ . From above, the vector  $v$  must be transversal to each of the 4 faces of  $\Delta$ . Therefore if we project  $\Delta$  along  $v$  to a plane transversal to  $v$  we see one of these two pictures.



Note that the combinatorics of the picture do not depend on the choice of  $v$ . Hence  $\mathcal{M}_4 \setminus \mathcal{E}_4$  is not connected.

It remains to show that if the combinatorics of the pictures for two spaces are the same then one can continuously deform one space into the other. This can be easily done by deforming  $W$  and applying a permutation of  $x_0, x_1, x_2, x_3$  if necessary.  $\square$

**Exercise 0.0.2.** The simplest proof we know requires the construction of tangent cones for Alexandrov spaces.

**Exercise 1.1.1.** This exercise is a basic introductory lemma on Gromov–Hausdorff distance [17, 7.3.30]. We present the proof of Travis Morrison.

Given any pair of point  $x_0, y_0 \in K$ , consider two sequences  $(x_n)$  and  $(y_n)$  such that  $x_{n+1} = f(x_n)$  and  $y_{n+1} = f(y_n)$  for each  $n$ .

Since  $K$  is compact, we can choose an increasing sequence of integers  $n_i$  such that both sequences  $(x_{n_i})_{i=1}^\infty$  and  $(y_{n_i})_{i=1}^\infty$  converge. In particular, both of these sequences are Cauchy; that is,

$$|x_{n_i} - x_{n_j}|_K, |y_{n_i} - y_{n_j}|_K \rightarrow 0 \text{ as } \min\{i, j\} \rightarrow \infty.$$

Since  $f$  is non-contracting, we get

$$|x_0 - x_{|n_i - n_j|}| \leq |x_{n_i} - x_{n_j}|.$$

It follows that there is a sequence  $m_i \rightarrow \infty$  such that

$$(*) \quad x_{m_i} \rightarrow x_0 \text{ and } y_{m_i} \rightarrow y_0 \text{ as } i \rightarrow \infty.$$

Set

$$\ell_n = |x_n - y_n|_K.$$

Since  $f$  is non-contracting,  $(\ell_n)$  is a non-decreasing sequence.

By (\*),  $\ell_{m_i} \rightarrow \ell_0$  as  $m_i \rightarrow \infty$ . It follows that  $(\ell_n)$  is a constant sequence.

In particular

$$|x_0 - y_0|_K = \ell_0 = \ell_1 = |f(x_0) - f(y_0)|_K$$

for any pair of points  $(x_0, y_0)$  in  $K$ . That is,  $f$  is distance preserving, in particular, injective.

From (\*), we also get that  $f(K)$  is everywhere dense. Since  $K$  is compact,  $f: K \rightarrow K$  is surjective. Hence the result follows.  $\square$

**Exercise 1.2.1.** A point in  $\mathbb{R} \times \text{Cone}\mathcal{U}$  can be described by a triple  $(x, r, p)$ , where  $x \in \mathbb{R}$ ,  $r \in \mathbb{R}_{\geq}$  and  $p \in \mathcal{U}$ . Correspondingly, a point in  $\text{Cone}[\text{Susp}\mathcal{U}]$  can be described by a triple  $(\rho, \varphi, p)$ , where  $\rho \in \mathbb{R}_{\geq}$ ,  $\varphi \in [0, \pi]$  and  $p \in \mathcal{U}$ .

The map  $\text{Cone}[\text{Susp}\mathcal{U}] \rightarrow \mathbb{R} \times \text{Cone}\mathcal{U}$  defined as

$$(\rho, \varphi, p) \mapsto (\rho \cdot \cos \varphi, \rho \cdot \sin \varphi, p)$$

is the needed isometry.  $\square$

**Exercise 1.4.3.** The following example is essentially due to Fedor Nazarov, see [42].

Consider the unit ball  $(B, d_0)$  in the space  $c_0$  of all sequences converging to zero equipped with the sup-norm.

Consider another metric  $d_1$  which is different from  $d_0$  by the conformal factor

$$\varphi(\mathbf{x}) = 2 + \frac{x_1}{2} + \frac{x_2}{4} + \frac{x_3}{8} + \dots,$$

where  $\mathbf{x} = (x_1, x_2, \dots) \in B$ . That is, if  $\mathbf{x}(t)$ ,  $t \in [0, \ell]$  is a curve parametrized by  $d_0$ -length then its  $d_1$ -length is

$$\text{length}_1 \mathbf{x} = \int_0^\ell \varphi \circ \mathbf{x}(t) \cdot dt.$$

Note that the metric  $d_1$  is bi-Lipschitz to  $d_0$ .

Assume  $\mathbf{x}(t)$  and  $\mathbf{x}'(t)$  are two curves which differ only in the  $m$ -th coordinate, say  $x_m(t)$  and  $x'_m(t)$ . Note that if  $x'_m(t) \leq x_m(t)$  for any  $t$  and the function  $x'_m(t)$  is locally 1-Lipschitz at all  $t$  such that  $x'_m(t) < x_m(t)$  then

$$\text{length}_1 \mathbf{x}' \leq \text{length}_1 \mathbf{x}.$$

Moreover this inequality is strict if  $x'_m(t) < x_m(t)$  for some  $t$ .

Fix a curve  $\mathbf{x}(t)$  parametrized by arc-length  $t \in [0, \ell]$  with respect to  $d_0$ . We can choose  $m$  large enough that  $x_m(t)$  is sufficiently close to 0 for any  $t$ . In particular, for some values  $t$ , we have  $y_m(t) < x_m(t)$ , where

$$y_m(t) = (1 - \frac{t}{\ell}) \cdot x_m(0) + \frac{t}{\ell} \cdot x_m(\ell) - \frac{1}{100} \cdot \min\{t, \ell - t\}.$$

Consider the curve  $\mathbf{x}'(t)$  as above with

$$x'_m(t) = \min\{x_m(t), y_m(t)\}.$$

Note that  $\mathbf{x}'(t)$  and  $\mathbf{x}(t)$  have the same end points and from above

$$\text{length}_1 \mathbf{x}' < \text{length}_1 \mathbf{x}.$$

That is, for any curve  $\mathbf{x}(t)$  in  $(B, d_1)$ , we can find a shorter curve  $\mathbf{x}'(t)$  with the same end points. In particular,  $(B, d_1)$  has no geodesics.  $\square$

**Exercise 1.4.7.** The following example is taken from [15].

Consider the following subset of  $\mathbb{R}^2$ :

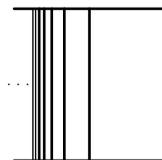
$$\mathcal{X} = ((0, 1] \times \{0, 1\}) \cup \left( \bigcup_{n \geq 1} \left\{ \frac{1}{n} \right\} \times [0, 1] \right)$$

Consider the induced length metric on  $\mathcal{X}$ . It is obviously locally compact and geodesic.

However, it is immediate to check that the metric completion

$$\bar{\mathcal{X}} = ([0, 1] \times \{0, 1\}) \cup \left( \bigcup_{n \geq 1} \left\{ \frac{1}{n} \right\} \times [0, 1] \right)$$

is neither.  $\square$



**Exercise 1.6.3.** Assume the contrary; that is

$$\angle[p_z^x] + \angle[p_z^y] < \pi.$$

By the triangle inequality for angles (1.6.2) we have

$$\angle[p_y^x] < \pi.$$

The latter contradicts the triangle inequality for the triangle  $[\bar{x}p\bar{y}]$ , where the points  $\bar{x} \in ]px]$  and  $\bar{y} \in ]py]$  are sufficiently close to  $p$ .  $\square$

**Exercise 1.8.2.** By the definition of the convergence

$$p \in A_\infty \iff \text{dist}_{A_n}(p) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The latter is equivalent to existence of a sequence  $p_n \in A_n$  such that  $|p_n - p| \rightarrow 0$  as  $n \rightarrow \infty$ ; or equivalently  $p_n \rightarrow p$ . Hence the first statement follows.

To show that the contrary does not hold, consider the alternating sequence of two distinct closed sets  $A, B, A, B, \dots$ ; note that it is not a converging sequence in the sense of Hausdorff. On the other hand, the set of all limit points is well defined and is the intersection  $A \cap B$ .  $\square$

*Remark.* The set  $\underline{A}_\infty$  of all limits of sequences  $p_n \in A_n$  is called the *lower closed limit* and the set  $\bar{A}_\infty$  of all partial limits of such sequences is called the *upper closed limit*.

Clearly  $\underline{A}_\infty \subset \bar{A}_\infty$ . If  $\underline{A}_\infty = \bar{A}_\infty$  then it is called the *closed limit* of  $A_n$ .

For the class of closed subsets of a proper metric space, closed limits coincide with limits in the sense of Hausdorff as we defined it.

All these notions were introduced by Felix Hausdorff in [36].

**Exercise 1.9.4.** Fix a countable dense set of points  $\mathfrak{S} \subset \mathcal{X}_\infty$ . For each point  $x \in \mathfrak{S}$ , choose a sequence of points  $x_n \in \mathcal{X}_n$  such that  $x_n \xrightarrow{\rho} x$ .

Applying a diagonal procedure, we can pass to a subsequence of  $\mathcal{X}_n$  such that each of the constructed sequences  $\rho'$ -converge; that is,  $x_n \xrightarrow{\rho'} x'$  for some  $x' \in \mathcal{X}'_\infty$ .

This way we get a map  $\mathfrak{S} \rightarrow \mathcal{X}'_\infty$  defined as  $x \mapsto x'$ . Note that this map preserves the distances and therefore it can be extended to a distance preserving map  $\mathcal{X}_\infty \rightarrow \mathcal{X}'_\infty$ . Likewise we construct a distance preserving map  $\mathcal{X}'_\infty \rightarrow \mathcal{X}_\infty$ .

It remains to apply Exercise 1.1.1.  $\square$

**Exercise 2.1.2.** Given a point  $x \in \text{Cone}\mathcal{U}$ , denote by  $x'$  its projection to  $\mathcal{U}$  and by  $|x|$  the distance from  $x$  to the tip of the cone; if  $x$  is the tip then  $|x| = 0$  and we can take any point of  $\mathcal{U}$  as  $x'$ .

Let  $p, q, x, y$  be a quadruple in  $\text{Cone}\mathcal{U}$ . Assume that the spherical model triangles  $[\tilde{p}' \tilde{x}' \tilde{y}'] = \tilde{\Delta}(p'x'y')_{\mathbb{S}^2}$  and  $[\tilde{q}' \tilde{x}' \tilde{y}'] = \tilde{\Delta}(q'x'y')_{\mathbb{S}^2}$  are defined. Consider the following points in  $\mathbb{E}^3 = \text{Cone}\mathbb{S}^2$ :

$$\tilde{p} = |p| \cdot \tilde{p}', \quad \tilde{q} = |q| \cdot \tilde{q}', \quad \tilde{x} = |x| \cdot \tilde{x}', \quad \tilde{y} = |y| \cdot \tilde{y}'.$$

Note that  $[\tilde{p}\tilde{x}\tilde{y}] \stackrel{\text{iso}}{=} \tilde{\Delta}(pxy)_{\mathbb{E}^2}$  and  $[\tilde{q}\tilde{x}\tilde{y}] \stackrel{\text{iso}}{=} \tilde{\Delta}(qxy)_{\mathbb{E}^2}$ . Further note that if  $\tilde{z} \in [\tilde{x}\tilde{y}]$  then  $\tilde{z}' = \tilde{z}/|\tilde{z}|$  lies on the geodesic  $[\tilde{x}'\tilde{y}']$  in  $\mathbb{S}^2$ .

The CAT[1] comparison for  $|p' - q'|$  with  $\tilde{z}' \in [\tilde{x}'\tilde{y}']$  implies the CAT[0] comparison for  $|p - q|$  with  $\tilde{z} \in [\tilde{x}\tilde{y}]$ .

The converse statement also holds, where the CAT[0] comparison is applied to a quadruple of points  $s = a \cdot p, q, x, y$  with  $a \geq 0$  chosen so that the corresponding points  $\tilde{s} = a \cdot \tilde{p}, \tilde{q}, \tilde{x}, \tilde{y}$  lie in one plane.  $\square$

**Exercise 2.1.3.** Fix a quadruple

$$p = (p_1, p_2), \quad q = (q_1, q_2), \quad x = (x_1, x_2), \quad y = (y_1, y_2)$$

in  $\mathcal{U} \times \mathcal{V}$ . For the quadruple  $p_1, q_1, x_1, y_1$  in  $\mathcal{U}$ , construct two model triangles  $[\tilde{p}_1 \tilde{x}_1 \tilde{y}_1] = \tilde{\Delta}(p_1x_1y_1)_{\mathbb{E}^2}$  and  $[\tilde{q}_1 \tilde{x}_1 \tilde{y}_1] = \tilde{\Delta}(q_1x_1y_1)_{\mathbb{E}^2}$ . Similarly, for the quadruple  $p_2, q_2, x_2, y_2$  in  $\mathcal{V}$  construct two model triangles  $[\tilde{p}_2 \tilde{x}_2 \tilde{y}_2]$  and  $[\tilde{q}_2 \tilde{x}_2 \tilde{y}_2]$ .

Consider four points in  $\mathbb{E}^4 = \mathbb{E}^2 \times \mathbb{E}^2$

$$\tilde{p} = (\tilde{p}_1, \tilde{p}_2), \quad \tilde{q} = (\tilde{q}_1, \tilde{q}_2), \quad \tilde{x} = (\tilde{x}_1, \tilde{x}_2), \quad \tilde{y} = (\tilde{y}_1, \tilde{y}_2).$$

Note that the triangles  $[\tilde{p}\tilde{x}\tilde{y}]$  and  $[\tilde{q}\tilde{x}\tilde{y}]$  in  $\mathbb{E}^4$  are isometric to the model triangles  $\tilde{\Delta}(pxy)_{\mathbb{E}^2}$  and  $\tilde{\Delta}(qxy)_{\mathbb{E}^2}$ .

If  $\tilde{z} = (\tilde{z}_1, \tilde{z}_2) \in [\tilde{x}\tilde{y}]$  we have then  $\tilde{z}_1 \in [\tilde{x}_1\tilde{y}_1]$  and  $\tilde{z}_2 \in [\tilde{x}_2\tilde{y}_2]$  and

$$\begin{aligned} |\tilde{z} - \tilde{p}|_{\mathbb{E}^4}^2 &= |\tilde{z}_1 - \tilde{p}_1|_{\mathbb{E}^2}^2 + |\tilde{z}_2 - \tilde{p}_2|_{\mathbb{E}^2}^2, \\ |\tilde{z} - \tilde{q}|_{\mathbb{E}^4}^2 &= |\tilde{z}_1 - \tilde{q}_1|_{\mathbb{E}^2}^2 + |\tilde{z}_2 - \tilde{q}_2|_{\mathbb{E}^2}^2, \\ |p - q|_{\mathcal{U} \times \mathcal{V}}^2 &= |p_1 - q_1|_{\mathcal{U}}^2 + |p_2 - q_2|_{\mathcal{V}}^2. \end{aligned}$$

Therefore CAT[0] comparison for the quadruples  $p_1, q_1, x_1, y_1$  in  $\mathcal{U}$  and  $p_2, q_2, x_2, y_2$  in  $\mathcal{V}$  implies the CAT[0] comparison for the quadruples  $p, q, x, y$  in  $\mathcal{U} \times \mathcal{V}$ .  $\square$

**Exercise 2.1.4.** According to Lemma 1.4.4, it is sufficient to prove the existence of a midpoint for two given points  $x$  and  $y$ .

For each  $n$  choose a  $\frac{1}{n}$ -midpoint  $z_n$ ; that is, a point such that

$$|x - z_n|, |y - z_n| \leq \frac{1}{2} \cdot |x - y| + \frac{1}{n}.$$

From the CAT[0] comparison inequality for the quadruple  $x, y, z_n, z_m$  we have that  $|z_m - z_n| \rightarrow 0$  as  $n, m \rightarrow \infty$ ; that is,  $z_n$  is Cauchy and hence converges.

Since the space is complete the sequence  $z_n$  has a limit, say  $z$ , which is clearly a midpoint for the pair  $x$  and  $y$ .  $\square$

**Exercise 2.2.8.** Without loss of generality we can assume that  $\kappa = 1$ . Fix a sufficiently small  $0 < \varepsilon < \pi$ .

Recall that by Proposition 2.2.7, any local geodesic shorter than  $\pi$  in  $\mathcal{U}$  is a geodesic.

Consider a sequence of directions  $\xi_n$  of geodesics  $[pq_n]$ . We can assume that the distances  $|p - q_n|_{\mathcal{U}}$  are equal to  $\varepsilon$  for all  $n$ ; here we use that the geodesics are extendable as local geodesics and minimizing up to length  $\pi$ .

Since  $\mathcal{U}$  is proper, the sequence  $q_n$  has a partial limit, say  $q$ . It remains to note that the direction  $\xi$  of  $[pq]$  is the limit of directions  $\xi_n$ , assuming the latter is defined.  $\square$

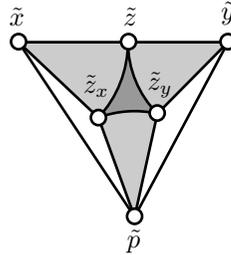
**Exercise 2.2.10.** By Alexandrov's lemma (1.5.1), there are nonoverlapping triangles

$$[\tilde{p}\tilde{x}\tilde{z}_x] \stackrel{iso}{=} [\tilde{p}\tilde{x}\tilde{z}]$$

and

$$[\tilde{p}\tilde{y}\tilde{z}_y] \stackrel{iso}{=} [\tilde{p}\tilde{y}\tilde{z}]$$

inside the triangle  $[\tilde{p}\tilde{x}\tilde{y}]$ .



Connect the points in each pair  $(\tilde{z}, \tilde{z}_x)$ ,  $(\tilde{z}_x, \tilde{z}_y)$  and  $(\tilde{z}_y, \tilde{z})$  with arcs of circles centered at  $\tilde{y}$ ,  $\tilde{p}$ , and  $\tilde{x}$  respectively. Define  $F$  as follows.

- ◊ Map  $\text{Conv}[\tilde{p}\tilde{x}\tilde{z}_x]$  isometrically onto  $\text{Conv}[\tilde{p}\tilde{x}\tilde{z}]$ ; similarly map  $\text{Conv}[\tilde{p}\tilde{y}\tilde{z}_y]$  onto  $\text{Conv}[\tilde{p}\tilde{y}\tilde{z}]$ .
- ◊ If  $x$  is in one of the three circular sectors, say at distance  $r$  from center of the circle, let  $F(x)$  be the point on the corresponding segment  $[pz]$ ,  $[xz]$  or  $[yz]$  whose distance from the lefthand endpoint of the segment is  $r$ .
- ◊ Finally, if  $x$  lies in the remaining curvilinear triangle  $\tilde{z}\tilde{z}_x\tilde{z}_y$ , set  $F(x) = z$ .

By construction,  $F$  satisfies the conditions.  $\square$

**Exercise 2.2.11.** For CAT[0], the statement follows from convexity of distance functions to points in  $\mathbb{E}^2$ , and thinness of triangles. For CAT[1], the statement follows from thinness of triangles and convexity of balls of radius  $< \pi/2$  in  $\mathbb{S}^2$ .

**Exercise 2.2.12.** Denote by  $K$  a closed, connected, locally convex set. Note that by Corollary 2.2.5,  $\text{dist}_K$  is convex in a neighborhood  $\Omega \supset K$ , that is, convex along any geodesic completely contained in  $\Omega$ .

Since  $K$  is locally convex, it is locally path connected. Since  $K$  is connected and locally path connected,  $K$  is path connected.

Fix two points  $x, y \in K$ . Let us connect  $x$  to  $y$  by a path  $\alpha: [0, 1] \rightarrow K$ . By Theorem 2.2.3, the geodesic  $[x\alpha(s)]$  is uniquely defined and depends continuously on  $s$ .

If  $[xy] = [x\alpha(1)]$  does not completely lie in  $K$  then there is a value  $s \in [0, 1]$  such that  $[x\alpha(s)]$  lies in  $\Omega$ , but does not completely lie in  $K$ . Therefore  $f = \text{dist}_K$  is convex along  $[x\alpha(s)]$ . Note that  $f(x) = f(\alpha(s)) = 0$  and  $f \geq 0$ , therefore  $f(x) = 0$  for any  $x \in [x\alpha(s)]$ ; that is,  $x \in K$ , a contradiction.  $\square$

**Exercise 2.2.13.** Since  $\mathcal{U}$  is proper, the set  $K \cap \overline{B}[p, R]$  is compact for any  $R < \infty$ . Hence the existence of at least one point  $p^*$  that minimizes the distance from  $p$  follows.

Assume  $p^*$  is not uniquely defined; that is, two distinct points in  $K$ , say  $x$  and  $y$ , minimize the distance from  $p$ . Since  $K$  is convex, the midpoint  $z$  of  $[xy]$  lies in  $K$ .

Note that  $|p - z| < |p - x| = |p - y|$ , a contradiction.

It remains to show that the map  $p \mapsto p^*$  is short, that is,

$$\bullet \quad |p - q| \geq |p^* - q^*|.$$

Note that

$$\angle[p^* p q^*] \geq \frac{\pi}{2} \quad \text{and} \quad \angle[q^* p p^*] \geq \frac{\pi}{2},$$

if the lefthand sides are defined.

Construct the model triangles  $[\tilde{p}\tilde{p}^*\tilde{q}^*]$  and  $[\tilde{p}\tilde{q}\tilde{q}^*]$  of  $[pp^*q^*]$  and  $[pqq^*]$  so that the points  $\tilde{p}^*$  and  $\tilde{q}$  lie on the opposite sides from  $[\tilde{p}\tilde{q}^*]$ .

By comparison and the triangle inequality for angles 1.6.2, we get

$$\angle[\tilde{p}^*\tilde{p}\tilde{q}^*] \geq \angle[p^*p q^*] \geq \frac{\pi}{2} \quad \text{and} \quad \angle[\tilde{q}^*\tilde{q}\tilde{p}^*] \geq \angle[q^*q p^*] \geq \frac{\pi}{2},$$

assuming the left hand sides are defined. Hence

$$|\tilde{p} - \tilde{q}| \geq |\tilde{p}^* - \tilde{q}^*|.$$

The latter is equivalent to  $\bullet$ .  $\square$

**Exercise 2.4.7.** By approximation, it is sufficient to consider the case when  $A$  and  $B$  have smooth boundary. If  $[xy] \cap A \cap B \neq \emptyset$  then  $z_0 \in [xy]$  and  $\dot{A}, \dot{B}$  can be chosen to be arbitrary half-spaces containing  $A$  and  $B$  respectively.

Suppose  $[xy] \cap A \cap B = \emptyset$ . Then  $z_0 \in \partial(A \cap B)$ . Let

$$C = \{z \in \mathbb{E}^m \mid f(z) \leq f(z_0)\}.$$

Note that  $C$  is compact, convex and has smooth boundary. Suppose  $z_0 \in \partial A \cap \text{Int } B$ . Then  $A$  and  $C$  touch at  $z_0$  and we can set  $\dot{A}$  to be the uniquely defined supporting half-space to  $A$  at  $z_0$  and  $\dot{B}$  to be any half-space containing  $B$ . The case  $z_0 \in \partial B \cap \text{Int } A$  is treated similarly.

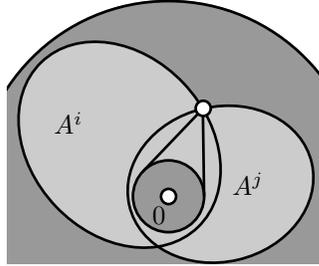
Finally, suppose  $z_0 \in \partial A \cap \partial B$ . Then set  $\dot{A}$  ( $\dot{B}$ ) to be the uniquely defined supporting half-space to  $A$  ( $B$ ) at  $z_0$  containing  $A$  ( $B$ ).  $\square$

**Exercise 2.4.8.** Fix  $r > 0$  and  $R < \infty$  so that

$$B(0, r) \subset A^i \subset B(0, R)$$

for each wall  $A^i$ .

It remains to note that all the intersections of the walls have  $\varepsilon$ -wide corners for



$$\varepsilon = 2 \cdot \arcsin \frac{r}{R}.$$

$\square$

**Exercise 2.4.9.** Note that there is  $R < \infty$  such that if  $X$  is an intersection of an arbitrary number of walls, then for any point  $p \in X$  there is an isometry of  $X$  which moves  $p$  to a point in  $B(0, R)$ .

The rest of the proof is the same as in Exercise 2.4.8.  $\square$

**Exercise 3.2.5.** Note that the existence of a null-homotopy is equivalent to the following. There are two one-parameter families of paths  $\alpha_\tau$  and  $\beta_\tau$ ,  $\tau \in [0, 1]$  such that

- $\diamond$  length  $\alpha_\tau$ , length  $\beta_\tau < \pi$  for any  $\tau$ .
- $\diamond$   $\alpha_\tau(0) = \beta_\tau(0)$  and  $\alpha_\tau(1) = \beta_\tau(1)$  for any  $\tau$ .
- $\diamond$   $\alpha_0(t) = \beta_0(t)$  for any  $t$ .
- $\diamond$   $\alpha_1(t) = \alpha(t)$  and  $\beta_1(t) = \beta(t)$  for any  $t$ .

By Corollary 3.2.3, the construction in Corollary 3.2.4 produces the same result for  $\alpha_\tau$  and  $\beta_\tau$ . Hence the result follows.  $\square$

**Exercise 3.3.5.** The following proof works for compact locally simply connected metric spaces; by uniqueness of geodesics (2.2.3) this class of spaces includes compact locally  $\text{CAT}[\kappa]$  length spaces. In this setting,

one assumes there is a nontrivial homotopy class of curves (shorter than  $2\pi$  in the CAT[1] case).

Consider a shortest noncontractible closed curve  $\gamma$  in the space. (Note that it exists.)

Assume that  $\gamma$  is not a geodesic circle, that is, there are two points  $p$  and  $q$  on  $\gamma$  such that the distance  $|p - q|$  is shorter than the lengths of the arcs, say  $\alpha_1$  and  $\alpha_2$ , of  $\gamma$  from  $p$  to  $q$ . Consider the products, say  $\gamma_1$  and  $\gamma_2$ , of  $[qp]$  with  $\alpha_1$  and  $\alpha_2$ . Note that

- ◊  $\gamma_1$  or  $\gamma_2$  is noncontractible,
- ◊  $\text{length } \gamma_1, \text{length } \gamma_2 < \text{length } \gamma$ ,

a contradiction. □

**Exercise 3.4.3.** Assume  $\mathcal{P}$  is not CAT[0]. Then by Theorem 3.4.2, a link  $\Sigma$  of some simplex contains a closed geodesic  $\alpha$  with length  $4\cdot\ell < 2\cdot\pi$ . Divide  $\alpha$  into two equal arcs  $\alpha_1$  and  $\alpha_2$  parametrized by  $[-\ell, \ell]$

Fix a small  $\delta > 0$  and consider two curves in  $\text{Cone } \Sigma$  written in polar coordinates as

$$\gamma_i(t) = (\alpha_i(\tan \frac{t}{\delta}), \sqrt{\delta^2 + t^2}).$$

Note that both form geodesics in  $\text{Cone } \Sigma$  and have common ends.

Finally note that a small neighborhood of the tip of  $\text{Cone } \Sigma$  admits an isometric embedding into  $\mathcal{P}$ . Hence the statement follows. □

**Advanced exercise 3.4.4.** It is sufficient to construct a polyhedral space  $\mathcal{P}$  homeomorphic to the 3-disc such that (1)  $\mathcal{P}$  is locally CAT[0] in its interior and (2) the boundary of  $\mathcal{P}$  is locally concave; in particular, each edge on the boundary of  $\mathcal{P}$  has angle at least  $\pi$ . Once  $\mathcal{P}$  is constructed, taking the double of  $\mathcal{P}$  along its boundary produces the needed metric on  $\mathbb{S}^3$ .

The construction of  $\mathcal{P}$  goes along the same lines as the construction of a Riemannian metric on the 3-disc with concave boundary and negative sectional curvature. This construction is given by Joel Hass in [35]. □

By the globalization theorem (3.3.1) the obtained metric on  $\mathbb{S}^3$  is not locally CAT[0]. This problem originated from a discussion in Oberwolfach between Brian Bowditch, Tadeusz Januszkiewicz, Dmitri Panov and Anton Petrunin.

**Exercise 3.5.2.** Checking the flag condition is straightforward once we know the following description of the barycentric subdivision.

Each vertex  $v$  of the barycentric subdivision corresponds to a simplex  $\Delta_v$  of the original triangulation. A set of vertices forms a simplex

in the subdivision if it can be ordered, say as  $v_1, \dots, v_k$ , so that the corresponding simplices form a nested sequence

$$\Delta_{v_1} \subset \dots \subset \Delta_{v_k}.$$

□

**Exercise 3.5.6.** The proof goes along the same lines as the proof of Flag condition 3.5.5. The only difference is that a geodesic may spend time *more than*  $\pi$  in the star of each vertex. □

**Exercise 3.5.7.** The space  $\mathcal{T}_n$  has a natural cone structure with the vertex formed by the completely degenerate tree — all its edges have zero length. Note that the space  $\Sigma$ , over which the cone is taken comes naturally with a triangulation with all-right spherical simplices.

It remains to check that the complex is flag and apply the flag condition (3.5.5) with Exercise 2.1.2. □

**Exercise 3.6.2.** If the complex  $\mathcal{S}$  is flag then its cubical analog  $\square_{\mathcal{S}}$  is locally CAT[0] and therefore aspherical.

Assume now that the complex  $\mathcal{S}$  is not flag. Extend it to a flag complex  $\mathcal{T}$  by gluing a simplex in every complete subgraph of its 1-skeleton.

Note that the cubical analog  $\square_{\mathcal{S}}$  is a proper subcomplex in  $\square_{\mathcal{T}}$ . Since  $\mathcal{T}$  is flag,  $\tilde{\square}_{\mathcal{T}}$ , the universal cover of  $\square_{\mathcal{T}}$ , is CAT[0].

Choose a minimal cube  $Q$  in  $\tilde{\square}_{\mathcal{T}}$  which is not present in  $\tilde{\square}_{\mathcal{S}}$ . By Exercise 2.2.12,  $Q$  is a convex set in  $\tilde{\square}_{\mathcal{T}}$ . The closest point projection  $\tilde{\square}_{\mathcal{T}} \rightarrow Q$  is a retraction. It follows that the boundary  $\partial Q$  is not contractible in  $\tilde{\square}_{\mathcal{T}} \setminus \text{Int } Q$  and therefore it is also not contractible in  $\tilde{\square}_{\mathcal{S}}$ . □

**Exercise 3.7.3.** The solution goes along the same lines as the proof of Lemma 3.7.2.

The only difference is that  $G$  is not a subcomplex of the cubical analog. It has to be made from the squares parallel to the squares of the cubical complex which meet the edges of the complex orthogonally at their midpoints. □

**Exercise 3.8.2.** In the proof we apply the following lemma from [29]; it follows from the disjoint discs property.

**Lemma.** *Let  $\mathcal{S}$  be a finite simplicial complex which is homeomorphic to an  $m$ -dimensional homology manifold for some  $m \geq 5$ . Assume that all vertices of  $\mathcal{S}$  have simply connected links. Then  $\mathcal{S}$  is a topological manifold.*

It is sufficient to construct a simplicial complex  $\mathcal{S}$  such that

- ◇  $\mathcal{S}$  is a closed  $(m - 1)$ -dimensional homology manifold;
- ◇  $\pi_1(\mathcal{S} \setminus \{v\}) \neq 0$  for some vertex  $v$  in  $\mathcal{S}$ ;
- ◇  $\mathcal{S} \sim \mathbb{S}^{m-1}$ ; that is,  $\mathcal{S}$  is homotopy equivalent to  $\mathbb{S}^{m-1}$ .

Indeed, assume such  $\mathcal{S}$  is constructed. Then the suspension  $\mathcal{R} = \text{Susp } \mathcal{S}$  is an  $m$ -dimensional homology manifold with a natural triangulation coming from  $\mathcal{S}$ . According to the lemma,  $\mathcal{R}$  is a topological manifold. According to the generalized Poincaré conjecture,  $\mathcal{R} \simeq \mathbb{S}^m$ ; that is  $\mathcal{R}$  is homeomorphic to  $\mathbb{S}^m$ . Since  $\text{Cone } \mathcal{S} \simeq \mathcal{R} \setminus \{s\}$  where  $s$  denotes a south pole of the suspension and  $\mathbb{E}^m \simeq \mathbb{S}^m \setminus \{p\}$  for any point  $p \in \mathbb{S}^m$  we get

$$\text{Cone } \mathcal{S} \simeq \mathbb{E}^m.$$

Let us construct  $\mathcal{S}$ . Fix an  $(m - 2)$ -dimensional homology sphere  $\Sigma$  with a triangulation such that  $\pi_1 \Sigma \neq 0$ . An example of that type exists for any  $m \geq 5$ ; a proof is given in [37].

Remove from  $\Sigma$  one  $(m - 2)$ -simplex. Denote the resulting complex by  $\Sigma'$ . Since  $m \geq 5$ , we have  $\pi_1 \Sigma = \pi_1 \Sigma'$ .

Consider the product  $\Sigma' \times [0, 1]$ . Attach to it the cone over its boundary  $\partial(\Sigma' \times [0, 1])$ . Denote by  $\mathcal{S}$  the resulting simplicial complex and by  $v$  the tip of the attached cone.

Note that  $\mathcal{S}$  is homotopy equivalent to the spherical suspension over  $\Sigma$ , which is a simply connected homology sphere and hence is homotopy equivalent to  $\mathbb{S}^{m-1}$ . Hence  $\mathcal{S} \sim \mathbb{S}^{m-1}$ .

The complement  $\mathcal{S} \setminus \{v\}$  is homotopy equivalent to  $\Sigma'$ . Therefore

$$\pi_1(\mathcal{S} \setminus \{v\}) = \pi_1 \Sigma' = \pi_1 \Sigma \neq 0.$$

That is,  $\mathcal{S}$  satisfies the conditions above. □

**Exercise 4.4.2.** By approximation, it is sufficient to consider the case when  $S$  has polygonal sides.

The latter case can be done by induction on the number of sides. The base case of triangle is evident.

To prove the induction step, apply Alexandrov's lemma (1.5.1) together with the construction in Exercise 2.2.10. □

**Exercise 4.4.3.** Unlike in the plane, the sphere contains diangular and triangular regions whose sides are concave in the sphere. To adapt the proof of Theorem 4.4.1 we use Exercise 4.5.7.

From Exercise 4.5.7, it follows that if such a concave diangle or triangle has perimeter smaller than  $2 \cdot \pi$  then it contains a closed hemisphere in its interior.

By the assumption,  $\Theta$  does not contain a closed hemisphere. Therefore the argument in the proof of Theorem 4.4.1 shows that any triangle in  $\tilde{\Theta}$  with perimeter less than  $2 \cdot \pi$  is spherically thin. □

**Exercise 4.5.3.** The space  $K$  forms a cone over the branched covering  $\Sigma$  of  $\mathbb{S}^3$  infinitely branching along two great circles.

If the planes are not orthogonal then the minimal distance between the circles is less than  $\frac{\pi}{2}$ . Assume that this distance is realized by a geodesic  $[\xi\zeta]$ . The broken line made by four liftings of  $[\xi\zeta]$  forms a closed local geodesic in  $\Sigma$ . By Proposition 2.2.7, (or Corollary 3.3.4)  $\Sigma$  is not CAT[1]. Therefore by Exercise 2.1.2,  $K$  is not CAT[0].

If the planes are orthogonal then the corresponding great circles in  $\mathbb{S}^3$  are formed by subcomplexes of a flag triangulation of  $\mathbb{S}^3$  with all-right simplicies. The branching cover is also flag. It remains to apply the flag condition 3.5.5.

*Comments.* In [23], Ruth Charney and Michael Davis gave a complete answer to the analogous question for three planes. In particular they show that if a covering space of  $\mathbb{E}^4$  branching at three planes through one origin is CAT[0] then these all are complex planes for some complex structure on  $\mathbb{E}^4$ .  $\square$

**Exercise 4.5.7.** Let  $\alpha$  be a closed curve in  $\mathbb{S}^2$  of length  $2 \cdot \ell$ .

Assume  $\ell < \pi$ . Let  $\check{\alpha}$  be a subarc of  $\alpha$  of length  $\ell$ , with endpoints  $p$  and  $q$ . Since  $|p - q| \leq \ell < \pi$ , there is a unique geodesic  $[pq]$  in  $\mathbb{S}^2$ . Let  $z$  be the midpoint of  $[pq]$ . We claim that  $\alpha$  lies in the open hemisphere centered at  $z$ . If not,  $\alpha$  intersects the boundary great circle in a point say  $r$ . Without loss of generality we may assume that  $r \in \check{\alpha}$ .

The arc  $\check{\alpha}$  together with its reflection in  $z$  form a closed curve of length  $2 \cdot \ell$  which passes through  $r$  and its antipodal point  $r'$ . Thus  $\ell = \text{length } \check{\alpha} \geq |r - r'| = \pi$ , a contradiction.  $\square$

*A solution via Crofton formula.* Let  $\alpha$  be a closed curve in  $\mathbb{S}^2$  of length  $\leq 2 \cdot \pi$ . We wish to prove  $\alpha$  is contained in a hemisphere in  $\mathbb{S}^2$ . By approximation it suffices to prove this for smooth curves  $\alpha$  of length  $< 2 \cdot \pi$  with transverse self-intersections.

Given  $v \in \mathbb{S}^2$ , denote by  $v^\perp$  the equator in  $\mathbb{S}^2$  with the pole at  $v$ . Further,  $\#X$  will denote the number of points in the set  $X$ .

Obviously, if  $\#(\alpha \cap v^\perp) = 0$  then  $\alpha$  is contained in one of the hemispheres determined by  $v^\perp$ . Note that  $\#(\alpha \cap v^\perp)$  is even for almost all  $v$ .

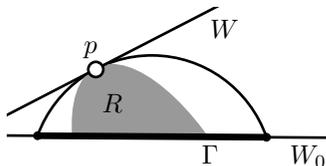
Therefore, if  $\alpha$  does not lie in a hemisphere then  $\#(\alpha \cap v^\perp) \geq 2$  for almost all  $v \in \mathbb{S}^2$ .

By Crofton's formula we have that

$$\begin{aligned} \text{length}(\alpha) &= \frac{1}{4} \cdot \int_{\mathbb{S}^2} \#(\alpha \cap v^\perp) \cdot \mathbf{d}_v \text{ area} \geq \\ &\geq 2 \cdot \pi. \end{aligned}$$

$\square$

**Exercise 4.5.8.** Since  $\Omega$  is not two-convex, by definition there is a simple closed curve  $\gamma$  which lies in the intersection of a plane  $W_0$  and  $\Omega$ , and is contractible in  $\Omega$  but not contractible in  $\Omega \cap W_0$ .



Let  $\varphi: \mathbb{D} \rightarrow \Omega$  be a disc which shrinks  $\gamma$ . Applying the loop theorem (arguing as in the proof of Proposition 4.2.7), we can assume that  $\varphi$  is an embedding and  $\varphi(\mathbb{D})$  lies on one side of  $W_0$ .

Let  $Q$  be the bounded domain cut from  $\mathbb{E}^3$  by  $\varphi(\mathbb{D})$  and  $W_0$ . By assumption it contains a point which is not in  $\Omega$ . Changing  $W_0, \gamma$  and  $\varphi$  slightly, we can assume that such a point lies in the interior of  $Q$ .

Fix a circle  $\Gamma$  in  $W_0$  which surrounds  $Q \cap W_0$ . Since  $Q$  lies in a half-space with boundary  $W_0$ , there is a smallest spherical dome with boundary  $\Gamma$  which includes the set  $R = Q \setminus \Omega$ . The dome has to touch  $R$  at some point  $p$ . The plane  $W$  tangent to the dome at  $p$  has the required property — the point  $p$  is an isolated point of the complement  $W \setminus \Omega$ .  $\square$

**Exercise 4.5.11.** Consider the surface  $S$  formed by the closure of the remaining part of the boundary. Note that the boundary  $\partial S$  of  $S$  is formed by some number of broken lines.

Assume  $S$  is not piecewise linear. Show that there is a line segment  $[pq]$  in  $\mathbb{E}^3$  tangent to  $S$  at  $p$  which has no common points with  $S$  except  $p$ .

Since  $S$  is locally concave, there is a local inner supporting plane  $\Pi$  at  $p$  which contains the segment  $[pq]$ .

Note that  $\Pi \cap S$  contains a segment  $[xy]$  passing through  $p$  with the ends at  $\partial S$ .

Show that there is a line segment of  $\partial S$  starting at  $x$  or  $y$  which lies in  $\Pi$ .

From the latter statement and local convexity of  $S$ , all points of  $[pq]$  sufficiently close to  $p$  lie in  $S$ , a contradiction.  $\square$

**Exercise 4.5.12.** Show that if  $K$  is not two-convex then there is a plane triangle  $\Delta$  whose sides lie completely in  $K$ , but its interior contains some points from the complement  $\mathbb{E}^m \setminus K$ .

It remains to note that  $\Delta$  is not thin in  $K$ .  $\square$

**Exercise 4.5.13.** Clearly the set  $W$  is two-convex. Therefore  $K$  is two-convex as an intersection of two-convex sets.

The set  $K$  on large scale looks like the complement of two 3-dimensional half-spaces in  $\mathbb{E}^4$ . More precisely, the Gromov-Hausdorff limit  $K'$  of  $\frac{1}{n} \cdot K$  as  $n \rightarrow \infty$  is the completion of the complement,

equipped with length metric, of the union of half-spaces in two 3-dimensional spaces in  $\mathbb{E}^4$  passing through 0. In particular  $K'$  is a cone over a space  $\Sigma$  which can be obtained as the completion of the sphere  $\mathbb{S}^3$  with two 2-dimensional hemispheres, say  $H_1$  and  $H_2$ , removed. The intersection of these hemispheres is typically formed by a geodesic segment, say  $[\xi\zeta]$ .

Consider hemispheres  $H_1$  and  $H_2$  for which  $[\xi\zeta]$  is orthogonal to the boundary spheres and has length less than  $\frac{\pi}{2}$ . Then the projection of a closed geodesic in  $\Sigma$  to  $\mathbb{S}^3$  is formed by a product of four copies of  $[\xi\zeta]$ . In particular there is a closed geodesic in  $\Sigma$  shorter than  $2\cdot\pi$ . Hence  $\Sigma$  is not CAT[1] and therefore  $K'$  and consequently  $K$  are not CAT[0].  $\square$

**Exercise 4.5.14.** Consider an  $\varepsilon$ -neighborhood  $A$  of the geodesic. Note that  $A_\varepsilon$  forms a convex set. By Reshetnyak's gluing theorem the doubling  $\mathcal{W}_\varepsilon$  of  $\mathcal{U}$  along  $A_\varepsilon$  is a CAT[0] space.

Next, consider the space  $\mathcal{W}'_\varepsilon$  obtained by doubly covering  $\mathcal{U}\setminus A_\varepsilon$  and gluing back  $A_\varepsilon$ .

Note that  $\mathcal{W}'_\varepsilon$  is locally isometric to  $\mathcal{W}_\varepsilon$ . That is, for any point  $p' \in \mathcal{W}'_\varepsilon$  there is a point  $p \in \mathcal{W}_\varepsilon$  such that the  $\delta$ -neighborhood of  $p'$  is isometric to the  $\delta$ -neighborhood of  $p$  for all small  $\delta > 0$ .

Further note that  $\mathcal{W}'_\varepsilon$  is simply connected since it admits a deformation retraction onto  $A_\varepsilon$ , which is contractible. By the globalization theorem,  $\mathcal{W}'_\varepsilon$  is CAT[0].

It remains to note that  $\tilde{\mathcal{U}}$  can be obtained as a limit of  $\mathcal{W}'_\varepsilon$  as  $\varepsilon \rightarrow 0$ , and apply Proposition 2.1.1.  $\square$

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# Bibliography

- [1] Alexander, S.; Berg, D.; Bishop, R., *Geometric curvature bounds in Riemannian manifolds with boundary*. Trans. Amer. Math. Soc., 339 (1993), 703–716.
- [2] Alexander, S.; Bishop, R., *The Hadamard–Cartan theorem in locally convex spaces*. Enseignement Math., 36 (1990), 309–320.
- [3] Alexander, S.; Bishop, R., *Curvature bounds for warped products of metric spaces* GAFA. (2004). 14, # 6, 1143–1181.
- [4] Alexander, S.; Kapovitch, V.; Petrunin, A., *Alexandrov geometry*, <http://www.math.psu.edu/petrunin/>
- [5] Alexandroff, A. The inner geometry of an arbitrary convex surface. C. R. (Doklady) Acad. Sci. URSS (N.S.) 32, (1941). 467–470.
- [6] Александров, А. Д., *Одна теорема о треугольниках в метрическом пространстве и некоторые ее приложения*, Тр. МИАН СССР, 38, 1951, 5–23
- [7] Alexandrow, A. D., *Über eine Verallgemeinerung der Riemannschen Geometrie*, Schriftenreihe Inst. Math. 1 (1957), p. 33–84.
- [8] Ancel, F.; Davis, M.; Guilbault, C., *CAT(0) reflection manifolds*. Geometric topology, 441–445.
- [9] Ballmann, W., *Singular spaces of non-positive curvature*, Sur les Groupes Hyperboliques d’après Mikhael Gromov, Chapter 10, Progr. Math. 83, Birkhäuser, Boston, 1990.
- [10] Ballmann, W., *Lectures on Spaces of Nonpositive Curvature*, with an appendix by Misha Brin, 1995.
- [11] Billera, L.; Holmes, S.; Vogtmann, K., *Geometry of the space of phylogenetic trees*. Adv. in Appl. Math. 27 (2001), no. 4, 733–767.
- [12] Bishop, R., *The intrinsic geometry of a Jordan domain*. Int. Electron. J. Geom. 1 (2008), no. 2, 33–39.
- [13] Blaschke, W., *Kreis und Kugel*. 1916.

- [14] Bowditch, B. H., *Notes on locally CAT(1) spaces* in “Geometric group theory”, de Gruyter (1995) 1–48.
- [15] Bridson, M., Haefliger, A., *Metric Spaces of Non-positive Curvature*, 1999.
- [16] Burago, D., *Hard balls gas and Alexandrov spaces of curvature bounded above*. Proceedings of the International Congress of Mathematicians, Vol. II, Doc. Math. 1998, Extra Vol. II, 289–298 =
- [17] Burago, D.; Burago, Yu.; Ivanov, S., *A course in metric geometry*. 2001.
- [18] Burago, D.; Ferleger, S.; Kononenko, A., *Uniform estimates on the number of collisions in semi-dispersing billiards*, Annals of Math.(2) 147 (1998), 695–708.
- [19] Burago, D.; Ferleger, S.; Kononenko, A., *Topological entropy of semi-dispersing billiards*. Ergodic Theory and Dynamical Systems 18.04 (1998), 791–805.
- [20] Busemann, H., *Spaces with non-positive curvature*. Acta Math. 80, (1948) 259–310.
- [21] Buyalo, S. V., *Volume and fundamental group of a manifold of nonpositive curvature*. Mathematics of the USSR-Sbornik 50, no. 1 (1985): 137–150.
- [22] Cartan, É., *Leçons sur la Géométrie des Espaces de Riemann*. 1928.
- [23] Charney, R.; Davis, M., *Singular Metrics of Nonpositive Curvature on Branched Covers of Riemannian Manifolds* American Journal of Mathematics, Vol. 115, No. 5 (Oct., 1993), pp. 929–1009
- [24] Charney, R.; Davis, M., *Strict hyperbolization*. Topology 34.2 (1995) 329–350.
- [25] Davis, M., *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*. Ann. of Math. (2) 117 (1983), no. 2, 293–324.
- [26] Davis, M., *Exotic aspherical manifolds*. Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001), 371–404, ICTP Lect. Notes, 9, 2002.
- [27] Davis, M.; Januszkiewicz, T., *Hyperbolization of polyhedra*. J. Differential Geom. 34 (1991), no. 2, 347–388.
- [28] Davis, M.; Januszkiewicz, T.; Lafont, J.-F., *4-dimensional locally CAT(0)-manifolds with no Riemannian smoothings*. Duke Math. J. 161 (2012), no. 1, 1–28.
- [29] Edwards, R., *The topology of manifolds and cell-like maps*. Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 111–127, 1980.

- [30] Frolík, Z., *Concerning topological convergence of sets*, Czechoslovak Math. J. 10 (1960), 168–180.
- [31] Gromov. M., *Hyperbolic groups*, Essays in Group Theory, Math. Sciences Research Institute Publications 8, 1987, 75–264.
- [32] Gromov, M., *Sign and geometric meaning of curvature*. 1994.
- [33] Hadamard, J., *Sur la forme des lignes géodésiques à l'infini et sur les géodésiques des surfaces réglées du second ordre*. Bull. Soc. Math. France 26 (1898), 195–216.
- [34] Hatcher, A., *Notes on Basic 3-Manifold Topology*, a draft available at <http://www.math.cornell.edu/~hatcher/>
- [35] Hass, J., *Bounded 3-manifolds admit negatively curved metrics with concave boundary*. J. Differential Geom. 40 (1994), no. 3, 449–459.
- [36] Hausdorff, F., *Grundzüge der Mengenlehre*, 1914.
- [37] Kervaire, M., *Smooth homology spheres and their fundamental groups*. Trans. Amer. Math. Soc. 144 (1969) 67–72.
- [38] Lebedeva, N.; Petrunin, A., *Local characterization of polyhedral spaces*. Geom. Dedicata 179 (2015), 161–168.
- [39] Lytchak, A.; Stefan W., *Isoperimetric characterization of upper curvature bounds*. arXiv:1611.05261 [math.DG]
- [40] Mazur, B., *A note on some contractible 4-manifolds*. Ann. of Math. (2) 73 (1961) 221–228.
- [41] von Mangoldt, H., *Ueber diejenigen Punkte auf positiv gekrümmten Flächen, welche die Eigenschaft haben, dass die von ihnen ausgehenden geodätischen Linien nie aufhören, kürzeste Linien zu sein*. J. Reine Angew. Math. (1881) Bd. 91. S. 23–53.
- [42] Nazarov, F., *Intrinsic metric with no geodesics*, MathOverflow <http://mathoverflow.net/q/15720>
- [43] Panov, D.; Petrunin, A., *Sweeping out sectional curvature*. Geom. Topol. 18 (2014), no. 2, 617–631.
- [44] Panov, D.; Petrunin, A., *Ramification conjecture and Hirzebruch's property of line arrangements*. to appear in Compositio Mathematica.
- [45] Petrunin, A., *Metric minimizing surfaces*. Electron. Res. Announc. Amer. Math. Soc. 5 (1999), 47–54.
- [46] Petrunin, A.; Stadler, S., *Metric minimizing surfaces revisited*, in preparation.
- [47] Reshetnyak, Yu. G., *On the theory of spaces of curvature not greater than  $K$* , Inextensible mappings in a space of curvature not greater than  $K$ , Siberian Math. Jour. 9 (1968), 683–689.

- [48] Rolfsen, D., *Strongly convex metrics in cells*. Bull. Amer. Math. Soc. 74 1968 171–175.
- [49] Rinow, W., *Die innere Geometrie der metrischen Raume*. 1961.
- [50] Шефель, С. З., *О седловых поверхностях ограниченной спрямляемой кривой*. Доклады АН СССР, 162 (1965) №2, 294–296.
- [51] Шефель, С. З., *О внутренней геометрии седловых поверхностей*. Сибирский математический журнал, 5 (1964), 1382–1396
- [52] Stadler, S., *An obstruction to the smoothability of singular nonpositively curved metrics on 4-manifolds by patterns of incompressible tori*, GAFA 25 (2015), no 5, 1575–1587
- [53] Stone, D., *Geodesics in piecewise linear manifolds*. Trans. Amer. Math. Soc. 215 (1976), 1–44.
- [54] Thurston, P., *CAT(0) 4-manifolds possessing a single tame point are Euclidean*. J. Geom. Anal. 6 (1996), no. 3, 475–494 (1997).
- [55] Wald, A., *Begründung einer Koordinatenlosen Differentialgeometrie der Flächen*, Ergebnisse eines mathematischen Kolloquium 6 (1935), 24–46.
- [56] Wijsman, R. A., *Convergence of sequences of convex sets, cones and functions. II*. Trans. Amer. Math. Soc. 123 (1966) 32–45.