RESTRICTIONS ON THE GEOMETRY AT INFINITY OF NONNEGATIVELY CURVED MANIFOLDS

Luis Guijarro and Vitali Kapovitch

February 22, 1994

Abstract. We will prove that some positively curved Alexandrov spaces do not appear as ideal boundaries of complete manifolds of nonnegative curvature.

1. Introduction and basic results.

One of the most fruitful approaches to the study of open manifolds arises from the understanding of their geometry at infinity. This is, for example, the underlying idea in the proof of Mostow's rigidity theorem, and in many of the results dealing with Hadamard manifolds.

The structure of noncompact manifolds with complete Riemannian metrics of nonnegative curvature became fairly well understood after the work of Cheeger, Gromoll and Meyer in the seventies. The Soul theorem provides a good description of the differentiable structure of such manifolds. Namely, there exists a compact totally geodesic submanifold S, the soul, embeddded in M, whose normal bundle is diffeomorphic to the ambient space M ([CG]; for the differentiability part check [Gr] or [Po]).

However, it was only in the eighties that Gromov introduced an analogue of the ideal boundary for this class of manifolds ([BGS]). His approach (which is explicitly detailed in section 3 of this paper) consists in introducing a metric in the space of rays, where we have identified previously those rays that have not grown apart fast enough. In a series of exercises included in the same reference, he outlined some of the main consequences that metric properties of $M(\infty)$ have in M and vice versa.

A further development was carried out by Kasue ([K]), who provided explicit proofs of most of the statements made by Gromov, together with a natural extension of the definition of ideal boundary to a bigger class of manifolds, namely those with asymptotically nonnegative curvature. In this paper, though, we won't deal with this broader class and will remain within the nonnegative curvature bound. For this case, Shioya provided a very readable introduction to the concept of ideal boundary in his paper [Shi].

A new motivation for the study of this object, is its relation to collapsing problems under a lower curvature bound. Given a convergent sequence (under the Gromov-Hausdorff topology) of manifolds of a fixed dimension n with sectional

1991 *Mathematics Subject Classification.* Primary 53C20.
Key words and phrases. Ideal boundary, Sharafutdinov retraction, soul.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX curvatures bounded below, we'll say that this sequence collapses if its Gromov-Hausdorff limit has Hausdorff dimension smaller than n. According to [GP], the limit in this case is an Alexandrov space of curvature bounded below. Up till now, no other restrictions on the limit are known. In particular, it is not known whether or not any Alexandrov space can be obtained in this way.

The collapsing phenomenon occurs naturally when one considers a pointed sequence formed by rescaling of a nonnegatively curved open manifold by positive constants approaching 0. As we'll see later, the limit in this case is a euclidean cone over $M(\infty)$, and because of the simplicity of elements of the sequence, this setting appears to be a good starting point to understand collapsing to singular spaces in general.

It is then natural to ask what kind of properties $M(\infty)$ must satisfy. From the above characterization, it's easy to conclude that $M(\infty)$ is an Aleksandrov space with curvature bounded below by 1 ([BGP], or section 3). Up to now, no other restrictions were known. The purpose of this paper will be to prove the following

Main Theorem. *There are Aleksandrov spaces of positive curvature that never appear as ideal boundaries of nonnegatively curved manifolds.*

The essential tool in our proof of this result is

Main Lemma. *Let M^n be an open complete Riemannian manifold of nonnegative curvature. If the ideal boundary is a connected Riemannian manifold, then there exists a locally trivial fibration $f : S^k \longrightarrow M(\infty)$.*

These results may be of some importance for the general problem of collapsing under lower curvature bound in view of the following observation due to Perelman. Suppose an Alexandrov space X^m is a Gromov-Hausdorff limit of a sequence of Riemannian manifolds M_i^n with sectional curvatures bounded below. Then for any fixed $x \in X$ we can find appropriate sequence of scalars $\lambda_i \xrightarrow[i\to\infty]{} 0$, such that

$$(\frac{1}{\lambda_i}M_i^n, x_i) \xrightarrow[i\to\infty]{} (C\Sigma_x X, *),$$

where $\Sigma_x X$ denotes the space of directions at x, and $C\Sigma_x X$ is a euclidean cone over $\Sigma_x X$. Note that here the lower curvature bound for $\frac{1}{\lambda_i}M_i^n$ converges to 0.

Conjecture. *Under the above assumptions $\Sigma_x X$ should satisfy the same kind of restrictions as the ones stated in the Main Theorem for ideal boundaries of nonnegatively curved manifolds.*

If true, this conjecture would imply that there are Alexandrov spaces that can never appear as limits of manifolds satisfying a lower curvature bound.

The present paper is organized as follows: In section 2 we include some facts about complete nonnegatively curved manifolds, as well as some elementary consequences of them that will be required to start the proof of the Main Lemma. Section 3 contains definitions and results about the ideal boundary, together with some examples; the only new material in this part is another characterization of $M(\infty)$. Section 4 contains the proof of the Main Lemma, while section 5 includes the proof of the Main Theorem, as well as some splitting results when the ideal boundary is a sphere.

The authors would like to thank Karsten Grove and Grisha Perelman for their help during the elaboration of the present work.

2. On level subsets of the Cheeger–Gromoll exhaustion.

In this section we will collect some previously known results about complete manifolds of nonnegative curvature. From now on, we will reserve M to denote a nonnegatively curved complete open manifold, and S to denote its soul.

2.1 Totally convex sets.

Definition. A nonempty subset C of M will be called totally convex if for any two points $p, q \in C$ and any geodesic $c : [0, l] \to M$ from p to q, we have $c[0, l] \subset C$

In their proof of the soul theorem, Cheeger and Gromoll constructed an exhaustion of M by compact totally convex sets. These are sublevel sets of the function obtained by taking the supremum of Busemann functions corresponding to rays starting from a fixed point p in M.

To be more explicit, suppose $\gamma : [0, \infty) \to M$ is a ray, (i.e. a geodesic satisfying $d(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in [0, \infty)$) and define

$$b_\gamma(x) = \lim_{t\to\infty} \{t - d(x, \gamma(t)\} \qquad x \in M$$

It was proved in [CG] that b_γ is a well defined function whose convexity is guaranteed by the nonnegativity of the curvature. Let's fix now a point $p \in M$ lying in the soul, and denote by $\mathcal{R}_p$ the set of rays in M with initial point p.

Define $b : M \to R$ as

$$b(x) = \sup_{\gamma\in\mathcal{R}_p} b_\gamma(x) \qquad x \in M$$

b is a well defined convex function in M, and if $C_t = \{x | b(x) \leq t\}$ then $\{C_t\}_{t\geq 0}$ satisfy:

(1) Each C_t is a totally convex compact set
(2) C_0 has empty interior
(3) $\dim C_t = \dim M \qquad \forall t > 0$
(4) $\bigcup_{t>0} C_t = M$
(5) $s < t$ implies $C_s \subsetneqq C_t$, and $C_s = \{x \in C_t | d(x, \delta C_t) \geqq t - s\}$

It was also proved in the same place that each C_s, $s > 0$, has the structure of an imbedded n-dimensional submanifold of M with smooth totally geodesic interior and (possibly nonsmooth) boundary.

2.2 The Sharafutdinov retraction. In a continuation of the work of Cheeger and Gromoll, Sharafutdinov proved the following result:

Theorem 2.2 [Sha]**.**

(1) *If C is a sublevel set of a convex function defined in M, then there exists a strong deformation retraction from M to C which is distance nonincreasing.*
(2) *There exists a distance nonincreasing strong deformation retraction from M to S.*

Remark. From now on, we'll denote by $sh : M \to S$ a map satisfying the second part of the theorem

The first part is an analogue of the Busemann-Feller theorem for convex sets in the euclidean space. Its proof passes by constructing integral curves for the generalized gradient of a convex (possibly nonsmooth) function. The second part is proved using the above together with the Cheeger-Gromoll exhaustion. The interested reader can find more details in [Sha] and [Yi].

2.3 Perelman's rigidity results. In his proof of the Cheeger-Gromoll conjecture, Perelman established the following fundamental fact:

Theorem 2.3 [P]**.**

(1) Let M be a complete nonnegatively curved manifold with soul S. Let $\alpha : [0,\infty] \to S$ be a geodesic, and $u \in N_pS$ a normal vector to the soul. Let U be the vector field along α obtained by parallel transport of u. Then

$$R(t,s) = exp_{\alpha(t)} sU(t) \qquad t \in [0,a], s \in [0,\infty)$$

is a flat rectangle totally geodesically immersed in M.
(2) $sh(exp_p tu) = p \qquad \forall t \in [0,\infty)$
(3) $sh : M \to S$ is a locally trivial fibration with C^1 fibers

From (2), it follows that $sh : M \to S$ is unique. We'll refer to it as the Sharafutdinov map, or the Sharafutdinov projection.

2.4 Some consequences from the structure of C_t.

Metrically. Let $p \in M$, $sh(p) \in S$ its Sharafutdinov projection, and let $\beta : [0,l] \to M$ a minimal geodesic joining $sh(p)$ and p with $\beta'(0) = u$. For any geodesic $\alpha : (-\infty,\infty) \to S$, consider the (infinite in one side) rectangle given by $R(t,s)$.

By [P], we know that $\bar{\alpha}(t) = R(t,l)$ is a geodesic in M, and since it stays at bounded distance of S, it is entirely contained in a compact set. The convexity of each b_γ and the definition of $\bar{\alpha}$ over all $\mathbb{R}$, implies that

$$b_\gamma(\bar{\alpha}(t)) = \text{constant} \qquad \forall t \in R$$

We get then the following

Lemma 2.4.1. *In the above situation, if $p \in \delta C_t$, then $\bar{\alpha}(R) \subseteq \delta C_t$*

Proof. Just use the definition of δC_t (C_t) as (sub)level set of $b(x) = \sup_{\gamma \in \mathcal{R}_q} b_\gamma(x)$ for $q = sh(p)$ $\square$

Let $o \in M$ be any point on the soul and define

$$T(t) = \sup_{y \in \delta C_T} d(o,y)$$

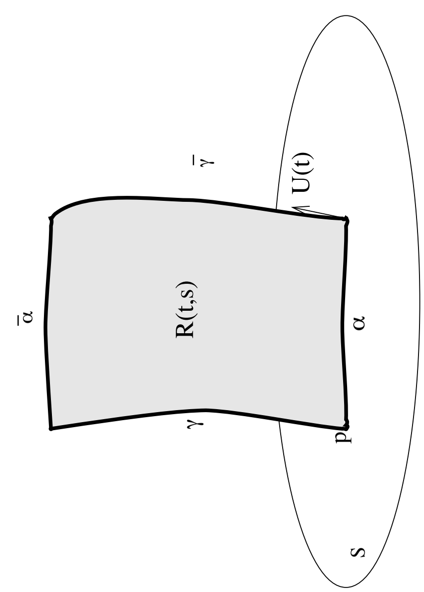

FIGURE 1. Perelman rigidity theorem

the circumradius of δC_t at the point o. Its inradius, $\inf_{y \in \delta C_T} d(o, y)$, is clearly equal to t by the construction of C_t. Let $b : M \to \mathbb{R}$ be the exhaustion function constructed in section 2. From $\lim_{x\to\infty} \frac{b(x)}{d(x,o)} = 1$, and the definition of δC_t it follows that

$$\lim_{t\to\infty} \frac{T(t)}{t} = 1$$

Lemma 2.4.2. *Let $x \in \delta C_t$, and let $v \in T_x M$ be any unit tangent vector, exterior to C_t. Let $\gamma : [o, l] \to M$ be any shortest geodesic between x and o, such that $\gamma(0) = x, \gamma(l) = o$, where $l = d(o, x)$. Then $\angle v\dot\gamma(0) \geq \pi/2 - \kappa(t)$, where $\kappa(t) \to 0$ as $t \to \infty$.*

Proof. Suppose $\angle v\dot\gamma(0) = \pi/2 - \alpha$. Since C_t is totally convex, we have that $\exp_x(sv) \notin C_t$ for any $s > 0$. Let us put $s = l \cdot \sin(\alpha)$. Then by the hinge version of the Toponogov's comparison theorem, we have $d(y, o) \leq d(\bar y, \bar o)$. Here, $y = \exp_x(l \sin(\alpha)v)$ and $\bar x, \bar y, \bar o$ are such points in R^2 that $d(\bar x, \bar o) = d(x, o) = l, d(\bar x, \bar y) = d(x, y) = l \cdot \sin(\alpha)$ and $\angle \bar y \bar x \bar o = \pi/2 - \alpha$. Then we clearly have $\angle \bar x \bar y \bar o = \pi/2$, hence $d(\bar o, \bar y) = l \cdot \cos(\alpha)$. So we get $d(y, o) \leq d(\bar o, \bar y) = l \cdot \cos(\alpha)$. But on the other hand, $y \notin C_t$, therefore $d(y, o) \geq t$, and we have $t \leq d(y, o) \leq l \cdot \cos(\alpha)$. Hence $\cos(\alpha) \geq t/l \geq t/T(t)$ and $\alpha \leq \arccos \frac{t}{T(t)} = \kappa(t)$, with $\kappa(t) \to 0$ as $t \to \infty$. $\square$

Remark. *From the proof of Lemma 2.4.2, we also get that $diam(o'_x) \leq \kappa(t)$, where $o'_x = \{\xi \in \Sigma_x M | \xi$ is a direction of a shortest geodesic between x and $o\}$.*

Topologically. It is now possible to determine the topological structure of δC_t.

Proposition 2.4.3. *Let M^n be a complete open manifold of nonnegative sectional curvature. Let S be a soul of M with $dim S = k$. Then $\delta C_t \cap sh^{-1}(s)$ with its induced topology is homeomorphic to S^{n-k-1} for any $s \in S$.*

Proof. According to [Gr] we know that d_S, the distance function to the soul, doesn't have any critical points outside S. Moreover if $x \in M \backslash S$ and $v \in T_x M$ is a gradient direction of d_S then v is tangent to the fiber of the Sharafutdinov retraction containing x. Indeed by the first variation argument we know that v is characterized by the property

$$\angle v S'_x = max_{w \in \Sigma_x M} \angle w S'_x$$

where S'_x is the set of unit vectors tangent to minimal geodesics joining x to the soul S. But by Perelman's Theorem (2.3) we have that $S'_x = (sh(x))'_x$ which lies in the tangent space to the fiber $sh^{-1}(sh(x))$. From this it clearly follows that v is also tangent to this fiber. By the remark after Lemma 2.4.2 $diam(S'_x) \xrightarrow[x\to\infty]{} 0$ hence $\angle v(x) S'_x \xrightarrow[x\to\infty]{} \pi$.

Using a partition of unity we can construct a vectorfield W so that coincides with the radial vector field for S close to S, $d'_S(W(x)) > 0$ for all $x \in M \backslash S$, $\angle v(x) W(x) \xrightarrow[x\to\infty]{} 0$ and therefore $d'_S(W(x)) > 0 \xrightarrow[x\to\infty]{} 1$. By $d'_S(W(x))$ we mean the one sided directional derivative of d_S in the direction $W(x)$. Now denote $S^t = \{x \in M | d_S(x) = t\}$. By a standard argument, for any $t_1 < t_2$ we can construct a homeomorphism $\Psi_{t_1,t_2} : S^{t_1} \to S^{t_2}$ using our vector field W. Clearly for small t we have that S^t is smooth and for any $s \in S$ $S^t \cap sh^{-1}(s)$ is homeomorphic to a standard S^{n-k-1}. By the construction of Ψ we have that it moves points along fibers of the Sharafutdinov retraction and hence for any $t_1 < t_2$ we have

$$S^{t_1} \cap sh^{-1}(s) \approx_{hom} S^{t_2} \cap sh^{-1}(s)$$

Since for small t this is known to be a sphere we obtain that for all $t > 0$ and $s \in S$, $S^t \cap sh^{-1}(s)$ is homeomorphic to a standard S^{n-k-1}. On the other hand by Lemma 2.4.2 we also have that for big t, $\angle v(x)\Sigma\delta C_t$ is very close to $\pi/2$ for all $x \in \delta C_t$. Thus $b'(v(x)) \xrightarrow[x\to\infty]{} 1$ where b is a Cheeger-Gromoll exhaustion function and once again, $b'(W(x))$ is a directional derivative. The construction of W implies that $b'(W(x)) \xrightarrow[x\to\infty]{} 1$. Let t be sufficiently big so that $T(t)/t - 1$ and $b'(W(x)) - 1$ are small. Using the flow of W we can construct a homeomorphism of δC_t onto $S^{T(t)}$. By the choice of W this homeomorphism moves points inside fibers of the Sharafudinov map, so we get that for any $s \in S$, $\delta C_t \cap sh^{-1}(s)$ is homeomorphic to $S^{T(t)} \cap sh^{-1}(s)$ which we already showed to be topologically a sphere. Thus we finally conclude that $\delta C_t \cap sh^{-1}(s)$ is homeomorphic to S^{n-k-1} $\square$.

Theorem 2.4.4. *$sh|_{\delta C_t} : \delta C_t \to S$ is a locally trivial fibration whose fibers are spheres.*

Proof. Follow the former proof in order to construct an homeomeorphism of δC_t to the boundary of a tubular neigbourhood S^r of S for small r that respects the Sharafutdinov map (we can do this since the vectorfield W was tangent to the Sharatfudinov fibers). Then use that from Perelman's theorem, $sh : S^r \to S$ is clearly a locally trivial fibration whose fibers are as desired $\square$.

3. Old and new characterizations of the ideal boundary.

In this section, we will provide the necessary background regarding the ideal boundary, as well as a new description of it (lemma 3.6), that will be needed for the proof of the main theorem.

In what follows, fix a point o and denote by $S_t(o)$ the metric sphere of radius t around o. Though it does not need to be smooth, Kasue proved that at least for large t, $S_t(o)$ is a Lipschitz hypersurface [K]. It is posible then to consider $S_t(o)$ with the intrinsic metric that it inherits as a subset of M; in other words, if we take a curve

$$\alpha : [0, a] \to S_t(o)$$

define its length, $L(\alpha)$, as

$$L(\alpha) = \sup_{0=t_0<t_1<\cdots<t_k=a} \sum_{i=0}^{k} d_M(c(t_i), c(t_{i+1}))$$

and define the intrinsic metric d_t of $S_t(o)$ as

$$d_t(x, y) = \inf_{\alpha}\{L(\alpha) | \alpha \text{ joins } x \text{ to } y\}$$

for $x, y \in S_t(o)$. If x, y belong to different connected components of S_t, define $d_t(x, y) = \infty$.

Consider now the set of *all* rays in M, $\mathcal{R}_M$, and let's introduce an equivalence relation $\backsim$ on it as follows:

$$\sigma \backsim \gamma \text{ if and only if } \lim_{t\to\infty} \frac{d(\sigma(t), \gamma(t))}{t} = 0$$

Define a distance d_∞ in this set of equivalence classes as

$$d_\infty(\sigma, \gamma) = \lim_{t\to\infty} \frac{d_t(\sigma \cap S_t(o), \gamma \cap S_t(o))}{t} \tag{3.1}$$

Theorem 3.1 [K, 2.1]. *The limit (3.1) exists and is independent of the base point* o

Definition. $(\mathcal{R}/\backsim, d_\infty)$ is called the ideal boundary of M. We will denote it by $M(\infty)$.

Remark. d_∞ is usually called the Tits metric of $M(\infty)$

Example 3.2. Let G be a closed subgroup of $SO(n)$ and consider the induced action of G on $\mathbb{R}^{n+1}$ by isometries. G acts diagonally in $SO(n)\times\mathbb{R}^{n+1}$ by isometries and the quotient M is an open riemannian manifold of nonnegative curvature. It can be proved that $M(\infty) = \mathbb{R}^{n+1}(\infty)/G = S^n/G$ [K]. In this form, we get any quotient of a sphere by an isometric action as an ideal boundary.

Instead of taking the set of all rays, $\mathcal{R}_M$, we could have restricted ourselves to rays with the same initial point p, $\mathcal{R}_p$. As before, identify rays σ, γ so that

$$\lim_{t\to\infty} \frac{d_t(\sigma(t), \gamma(t))}{t} = 0$$

and define

$$\bar{d}_\infty([\sigma], [\gamma]) = \lim_{t\to\infty} \frac{d_t(\sigma(t), \gamma(t))}{t}$$

Then we have

Proposition 3.3 [K]. *The inclusion* $(\mathcal{R}_p/\backsim, \bar{d}_\infty) \to (\mathcal{R}_M/\backsim, d_\infty)$ *is an isometry for any* $p \in M$

A different type of construction is posible by passing to the category of pointed metric spaces. Consider the pair formed by (M, o) and denote by $(\lambda M, o)$ the pointed metric space obtained by rescaling the metric of M by a factor of λ. This is a sequence of nonnegatively curved spaces and therefore preconvergent by Gromov's precompactness theorem. However, we can get a stronger consequence in this case.

Lemma 3.4. $(\lambda M, o) \xrightarrow[G-Hausdorff]{\lambda\to 0} C_{\bar{o}}(M(\infty))$ *where by* $C_{\bar{o}}(Y)$ *we denote the euclidean metric cone over* Y *with vertex* $\bar{o}$

Proof. This can be found in [Sh]; but because of posible difficulties in finding this reference, we'll include it here. We will use pairs $(\sigma(\infty), a)$ to denote points in

$C_{\bar{o}}(M(\infty))$. Fix $r > 0$, and let $B_r(o, \lambda M)$, $B_r(\bar{o}, C_{\bar{o}}(M(\infty))$ be metric balls of radius r around $o, \bar{o}$ respectively. According to the definition of pointed Gromov-Hausdorff convergence, we need to check that for any sequence $\lambda_n \to 0$, $B_r(o, \lambda_n M) \to B_r(\bar{o}, C_{\bar{o}}(M(\infty)))$. Equivalently, it is enough to see that for any $\epsilon > 0$, and for n sufficiently big, there are ϵ-nets $X^{(n)}$, X in $B_r(o, \lambda_n M)$, $B_r(\bar{o}, C_{\bar{o}}(M(\infty)))$ and bilipschitz maps $f_n : X^{(n)} \to X$ with $\mathrm{dil}(f_n)$ tending to one.

So start by taking an $\epsilon/100$-net X in $B_r(\bar{o}, C_{\bar{o}}(M(\infty)))$ with points $(\sigma_i(\infty), a_i)$. Let $X^{(n)}$ be the set of points $\sigma_i(a_i {\lambda_n}^{-1})$ in $B_r(o, \lambda_n M)$.

For any two rays σ and γ in M, and for any numbers $a, b \geq 0$, we have

$$\lim_{t\to\infty} \frac{d(\sigma(at), \gamma(bt))}{t} = \sqrt{a^2 + b^2 - 2ab \cos \min\{d_\infty(\sigma(\infty), \gamma(\infty), \pi\}} \tag{3.2}$$

This is proposition 2.2 from [Shi]. Note that the right side is the distance in $C_{\bar{o}}(M(\infty))$ between the points $(\sigma(\infty), a)$ and $(\gamma(\infty), b)$ thus implying that by choosing λ_n^{-1} big enough, we can approximate the relative distances of points in $X^{(n)}$ by those of corresponding points in X. This means that the natural map from $X^{(n)}$ to X has the right dilatation. It remains to see that every point of $B_r(o, \lambda_n M)$ is at distance no farther than ϵ from $X^{(n)}$. There exists a $T > 0$ so that for any $t > T$ and any x in $S_t(o)$, there is a ray σ with $\frac{r d(x, \sigma(t))}{t} < \epsilon/2$ [K]. Rescaling by a small λ_n, we can make sure that the diameter of $B_{\lambda_n T}(o, \lambda_n M)$ is smaller than $\epsilon/2$. Choose a point $(\sigma_i(\infty), a_i)$ at distance less than $\epsilon/10$ of $(\sigma(\infty), \lambda_n t)$. Then

$$\frac{d(x, \sigma_i(a_i \lambda_n^{-1}))}{\lambda_n^{-1}} \leq \frac{d(x, \sigma(t))}{\lambda_n^{-1}} + \frac{d(\sigma(t), \sigma_i(a_i \lambda_n^{-1}))}{\lambda_n^{-1}} < \epsilon$$

where the last inequality follows from 3.2 and Toponogov's theorem. This concludes the argument □.

Corollary 3.5. *$M(\infty)$ is an Alexandrov space of curvature bounded below by 1*

Proof. This is just an immediate consequence of proposition 4.2.3 in [BGP] □.

A similar approach was taken by Kasue, who considered the sequence of metric spaces $(S_t(o), d_t/t)$. He found that for any large t, there are maps

$$\Phi_{t,\infty} : S_t(o) \to M(\infty)$$

such that $\Phi_{t,\infty}(\gamma(t)) = [\gamma]$ for γ a ray, with

$$d_\infty(\Phi_{t,\infty}(x), \Phi_{t,\infty}(y)) \leq (1 + \kappa(t)) \frac{d_t(x, y)}{t}$$

and also

$$d_\infty(x_\infty, y_\infty) = \lim_{t\to\infty} \frac{d_t(\Phi_{t,\infty}^{-1}(x_\infty), \Phi_{t,\infty}^{-1}(y_\infty))}{t}$$

for x, y in S_t, x_∞, y_∞ in $M(\infty)$, and $\kappa(t) \to 0$ as $t \to \infty$.[1]

The last two inequalities just imply

$$\lim_{t\to 0} (S_t(o), d_t/t) = M(\infty)$$

which provides us with a fourth description of the ideal boundary.

The following lemma is on the crossroad of this and the previous section. It also represents the initial motivation for the proof of the main theorem:

[1] The above inequalities correspond to Proposition 2.2 from [K] after the corrections indicated in [D]

Lemma 3.6. *Let $(\delta C_t, d_t)$ be the boundary of C_t with its intrinsic metric. Then:*

1. *(Buyalo) $(\delta C_t, d_t)$ is an Alexandrov space of curvature bounded below by zero*
2. $\lim_{t\to\infty}(\delta C_t, d_t/t) = (M(\infty), d_\infty)$

Proof. The proof of (1) can be found in [Bu]. We'll provide only the proof of (2).

Given an arbitrary $\epsilon > 0$, let $X = \{\sigma_i(\infty)\}$ be an $\epsilon/100$-net in $M(\infty)$. Call $X^{T(t)}$ the subset of points $\{\sigma_i(T(t))\}$ in $S_{T(t)}(o)$, where $T(t)$ was defined in section 2.4. $X^{(t)}$ is defined in an analogous manner. By Kasue's results, we know that each of these subsets is an $\epsilon/10$ net in the normalized intrinsic metric of the corresponding metric spheres about o, which is almost Lipschitz equivalent to X. By the first part of theorem (2.2), there's a strong deformation retraction $\phi : M \to C_t$ that doesn't increase lenghts of curves. For any two points x_i, x_j in $X^{T(t)}$, let $\alpha_{i,j} : [0, l) \to S_{T(t)}$ be a curve realizing $d_{T(t)}(x_i, x_j)$. Then

$$d_{T(t)}(x_i, x_j) = l(\alpha_{i,j}) \geq l(\phi \circ \alpha_{i,j}) \geq d_{\delta C_t}(\phi(x_i), \phi(x_j)) \tag{3.3}$$

since $\phi \circ \alpha_{i,j}$ is entirely contained in δC_t. Conversely,

$$d_{\delta C_t}(\phi(x_i), \phi(x_j)) \geq 2(T(t) - t) + d_{T(t),t}(x_i, x_j) \tag{3.4}$$

where by $d_{T(t),t}$ we mean the intrinsic distance for the annulus $A_{T(t),t} = \bar{B}_{T(t)}(o) - B_t(o)$. After rescaling by t, (3.4) will read as

$$\frac{d_{T(t)}(x_i, x_j)}{t} \leq \frac{d_{\delta C_t}(\phi(x_i), \phi(x_j))}{t} + o(t) \tag{3.5}$$

since as t approaches ∞, $d_{T(t),S(t)}/t$ and $d_{T(t)}/t$ converge to the same limit. Combining (3.3) and (3.5) we can conclude that $X^{T(t)}$ and $\phi(X^{T(t)})$ are both ϵ-nets almost Lipschitz equivalent $\square$.

The proof of lemma 3.6 gives that ϕ can be used to construct a Hausdorff approximation from δC_t to $S_t(o)$ leaving invariant ray points. Composing with the Kasue's maps $\Phi_{t,\infty}$ we get then Hausdorff approximations from δC_t to $M(\infty)$ sending each $\gamma(t)$ to $\gamma(\infty)$ for γ a ray. We'll denote these maps again by $\Phi_{t,\infty}$. It will be clear from the context which one we are refering to.

4. Proof of the main lemma.

In this section, we will prove the following version of the Main Lemma, which is somewhat more general than the one stated in the Introduction.

Lemma 4.1. *Let M^n be a complete open manifold of nonnegative sectional curvature. Suppose $M(\infty)$ is connected and $(m, \delta(m))$-strained at each point, where $m = dim M(\infty)$ and $\delta(m)$ is sufficiently small. Then there is a locally trivial fibration $f : S^k \to M(\infty)$, where $k = n - dim S - 1$. Moreover, fibers of f are closed manifolds.*

We'll follow the following conventions. Throughout the proof, we will denote by $\kappa(t)$ and $\kappa(t, \delta)$ various constants such that $\kappa(t) \xrightarrow[t\to\infty]{} 0$ and $\kappa(t, \delta) \to 0$ as

$t \to \infty, \delta \to 0$. Moreover, here $\kappa(t,\delta)$ may depend on some additional parameters, but we require that $\mu(\delta) = \lim_{t\to\infty} \kappa(t,\delta)$ depends only on δ (and possibly dimension of $M(\infty)$) and $\mu(\delta) \xrightarrow[\delta\to 0]{} 0$.

The proof is organized as follows. First we show that infinitesimally δC_t looks more and more like a Riemannian manifold as $t \to \infty$. It was pointed out in [BGP] that provided the limit space is sufficiently regular, this condition implies that for any big t there is a locally trivial fibration of δC_t over the limit space . The next and most crucial step is to check that the restriction of this fibration to the intersection of δC_t with any fiber of Sharafutdinov retraction, is still a fibration.

4.2 Recall that by Lemma 3.6, $(\delta C_t, d_t)$ is an Alexandrov space of curv ≥ 0. Let us first show that $(\delta C_t, d_t)$ is $(n-1, \kappa(t))$-strained at each point. Here we use the following definition from [BGP]:

Definition. *Let X be a space of curv $\geq k$. A point $p \in X$ is called (n,δ)-strained if there exists points (a_i, b_i) for $i = 1, \ldots, n$, such that*

$$\tilde{\sphericalangle} a_i p b_i \geq \pi - \delta, \qquad \tilde{\sphericalangle} a_i p a_j \geq \pi/2 - \delta$$

$$\tilde{\sphericalangle} a_i p b_j \geq \pi/2 - \delta, \qquad \tilde{\sphericalangle} b_i p b_j \geq \pi/2 - \delta$$

for all $i \neq j$, where by $\tilde{\sphericalangle}$ we mean the corresponding comparison angle.

Let $o \in M$ be any point on the soul. As it was mentioned in the proof of Lemma 2.4.2, $\inf_{y\in\delta C_t} d(y,o) = t$. and for $T(t) = \sup_{y\in\delta C_t} d(y,o)$ then $T(t)/t \to 1$ as $t \to \infty$.

Now let $H \subset T_x M$ be any supporting hyperplane for δC_t at x. By Lemma 2.4.2, we clearly have $|\angle\dot{\gamma}(0)H - \pi/2| \leq \kappa_1(t)$. Note that also,

$$\Sigma_x \delta C_t = \delta(\Sigma_x C_t) \subset \Sigma_x M \simeq (S^{n-1}, can).$$

Take S_H to be the unit sphere in H. Then obviously, $S_H \simeq (S^{n-2}, can)$, and again using Lemma 2.4.2, we easily get that a metric projection from $\Sigma_x \delta C_t$ to S_H is a $\kappa_2(t)$ -Hausdorff approximation. Thus $d_{GH}(\Sigma_x \delta C_t, (S^{n-2}, can)) \to 0$ as $t \to \infty$ uniformly in $x \in \delta C_t$. This easily implies that δC_t is $(n-1, \kappa(t))$-strained at each point.

We will need the following trivial modification of the result of Yamaguchi, proved in [Y2]:

Theorem 4.3. *Let $M_i^n \xrightarrow[t\to\infty]{} M^m$ be a convergent sequence of Alexandrov spaces of curv $\geq k$, such that M is (m,δ)-strained at each point. Then for any ν-Hausdorff approximation $h_i : M_i \to M$, there exists an $f_i : M_i \to M$, such that f_i is $c\nu$ uniformly close to h_i, and that f_i is an $\varepsilon(\nu,\delta)$-almost Lipschitz submersion, with $\varepsilon(\nu,\delta) \to 0$ as $\nu, \delta \to 0$.*

Here we use the following definition from [Y2]:

Definition. *A map of two Alexandrov spaces $f : M \to M'$ is an ε-almost Lipschitz submersion if for any $p, q \in M$*

$$\left| \frac{d(f(p), f(q))}{d(p,q)} - \sin(\theta) \right| < \varepsilon,$$

where

$$\theta = \inf_{f(x)=f(p)} \angle qpx.$$

In our situation, we have $(\delta C_t, d_t/t) \xrightarrow[t\to\infty]{} M(\infty)$, where d_t is the induced inner metric on δC_t. Denote by $\bar{d}_t = d_t/t$. By the assumption of the theorem, $M(\infty)$ is (m,δ)-strained at each point. As an original Hausdorff approximation $h_t : (\delta C_t, \bar{d}_t) \xrightarrow[t\to\infty]{} M(\infty)$ we take an approximation, such that for any ray σ starting at o, we have $h_t(\sigma(t)) = [\sigma]$. For example, we can take h_t to be the Kasue map $\Phi_{t,\infty}$ mentioned in section 3. So the conditions of the Yamaguchi's theorem are satisfied, and thus there exists an $f_t : (\delta C_t, \bar{d}_t) \to M(\infty)$, which is an $\varepsilon(\nu(t),\delta)$-almost Lipschitz submersion. Here $\nu(t) = d_{GH}(\delta C_t/t, M(\infty))$. Let $s_0 \in S$ be any point on the soul. Then we have next

Lemma 4.4. *Under the above assumptions,*

$$f_t|_{\delta C_t \cap sh^{-1}(s_0)} : \delta C_t \cap sh^{-1}(s_0) \longrightarrow M(\infty)$$

is a locally trivial fibration for all large t.

Proof of Lemma 4.4. By assumption, $M(\infty)$ is (m,δ)-strained at each point, therefore there exist an $R > 0$, such that for any $p \in M(\infty)$ there is an (m,δ) -strainer $(a_i,b_i)_{i=1,\dots,m}$ at p, with $d_\infty(p,b_i) \geq R, d_\infty(p,a_i) \geq R$ for $i = 1,\dots,m$. Let us take t big enough that $\nu(t) \ll \min(R,\delta)$ and that $\varepsilon = \varepsilon(\nu(t),\delta)$, given by the Yamaguchi's theorem, is sufficientely small. Take any $\bar{p} \in \delta C_t \cap sh^{-1}(s_0)$, and put $p = f(\bar{p})$. Let $(a_i,b_i)_{i=1,\dots,m}$ be an (m,δ) -strainer at p, such that $d_\infty(p,b_i) \geq R$ and $d_\infty(p,a_i) \geq R$, for $i = 1,\dots,m$. Choose $\bar{a}_i \in f_t^{-1}(a_i), \bar{b}_i \in f_t^{-1}(b_i), i = 1,\dots,m$, to be any points in the fibers.

4.4.1 Since $\nu(t) \ll R$, we have that $(\bar{a}_i,\bar{b}_i)_{i=1,\dots,m}$ is an $(m,2\delta)$-strainer at $\bar{p}$. Let $k = dim S$, and let $v_1,\dots,v_k$ be some orthonormal basis in T_sS. Then clearly, for any sufficiently small r, collection $(s_i = \exp_{s_0}(rv_i), t_i = \exp_{s_0}(-rv_i))_{i=1,\dots,m}$ forms a (k,δ) -strainer at s_0. Let us choose such an r that also $r < \delta R$. Take $\gamma : [o,l] \to M$ to be any shortest geodesic between s_0 and $\bar{p}$, such that $\gamma(0) = s$ and $\gamma(l) = \bar{p}$, where $l = d(s_0,\bar{p})$. Let $\bar{v}_1,\dots,\bar{v}_k$ be the orthonormal basis in $T_{\bar{p}}M$, obtained as a result of the parallel transport of the basis $v_1,\dots,v_k$ along γ. Denote $\bar{\alpha}_i(\tau) = \exp_{\bar{p}}(\tau\bar{v}_i)$ and $\alpha_i(\tau) = \exp_{s_0}(\tau v_i)$.

By Lemma 2.4.1, $\bar{\alpha}_i(\tau)$ is a geodesic lying in δC_t, such that $sh(\bar{\alpha}_i(\tau)) = \alpha_i(\tau)$. So clearly, $(\bar{s}_i = \bar{\alpha}_i(r), \bar{t}_i = \bar{\alpha}_i(-r))$ is a (k,δ)-strainer at $\bar{p}$.

Now we got two sets of strainers at $\bar{p}$, namely : an $(m,2\delta)$-strainer $(\bar{a}_i,\bar{b}_i)_{i=1,\dots,m}$ and a (k,δ)-strainer $(\bar{s}_i,\bar{t}_i)_{i=1,\dots,k}$. Our next goal is to prove that together they form one $(m+k,\kappa(\delta,t))$-strainer.

Indeed, it is enough to show that $\tilde{\sphericalangle}\bar{a}_i\bar{p}\bar{s}_j \geq \pi/2 - \delta'$. All other inequalities are treated similarly. Suppose $\tilde{\sphericalangle}\bar{a}_i\bar{p}\bar{s}_j = \pi/2 - \beta$. We will use the next Lemma [BGP,L5.6]:

Lemma 4.4.2. *Let p,q,r,s be points in a space of curv $\geq k$, such that $d(q,s) < \delta \min\{d(p,q), d(r,q)\}$, and $\tilde{\sphericalangle}pqr > \pi - \delta_1$. Then*

$$|\tilde{\sphericalangle}pqs + \tilde{\sphericalangle}rqs - \pi| < 10\delta + 2\delta_1.$$

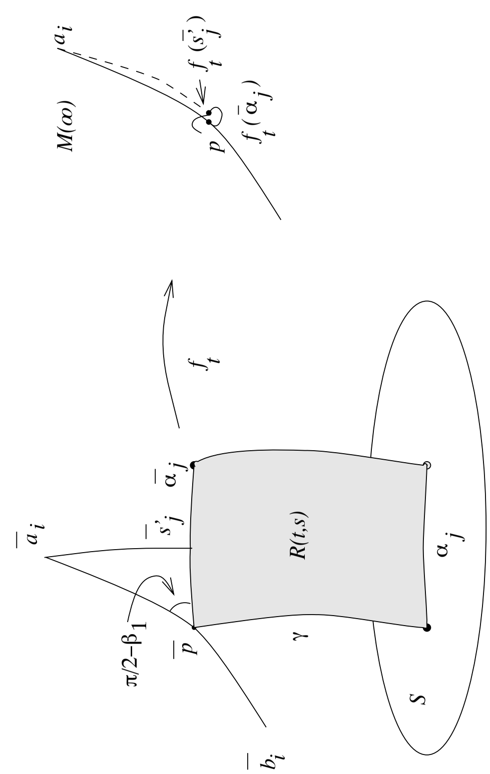

FIGURE 2

In particular, if there exist shortest geodesics, then

$$|\tilde{\sphericalangle} pqs - \sphericalangle pqs| < 20\delta + 4\delta_1,$$

and

$$|\tilde{\sphericalangle} rqs - \sphericalangle rqs| < 20\delta + 4\delta_1.$$

In our situation, we have

$$\tilde{\sphericalangle}\bar{a}_i\bar{p}\bar{b}_i \geq \pi - 2\delta, \bar{d}_t(\bar{p}\bar{s}_j) = r < \delta R < \delta \min\{\bar{d}_t(\bar{a}_i, \bar{p}), \bar{d}_t(\bar{b}_i, \bar{p})\}.$$

So by applying Lemma 4.4.2, we get $|\tilde{\sphericalangle}\bar{a}_i\bar{p}\bar{s}_j - \sphericalangle\bar{a}_i\bar{p}\bar{s}_j| < 20\delta + 8\delta = 28\delta$. Hence, $\sphericalangle\bar{a}_i\bar{p}\bar{s}_j < \pi/2 - \beta + 28\delta = \pi/2 - \beta_1$, where $\beta_1 = \beta - 28\delta$. Let $\bar{s}'_j = \bar{\alpha}_j(\bar{d}_t(\bar{p}\bar{a}_i)\sin(\beta_1))$. Then by the Toponogov's comparison theorem, $\bar{d}_t(\bar{a}_i, \bar{s}'_j) \leq \bar{d}_t(\bar{p}\bar{a}_i)\cos(\beta_1)$. But f_t is an ε-almost Lipschitz submersion, so

$$\Big|\frac{d_\infty(f_t(\bar{a}_i), f_t(\bar{s}'_j))}{\bar{d}_t(\bar{a}_i, \bar{s}'_j)} - \sin(\theta)\Big| < \varepsilon(\delta, \nu(t)),$$

where $\nu(t)$ is a parameter of Hausdorff approximation and

$$\theta = \inf_{f_t(x)=f_t(\bar{a}_i)} \angle \bar{s}'_j\bar{a}_i x.$$

Combining these inequalities we obtain:

$$d_\infty(f_t(\bar{a}_i), f_t(\bar{s}'_j)) \leq (1+\varepsilon)\bar{d}_t(\bar{a}_i, \bar{s}'_j) \leq (1+\varepsilon)\cos(\beta_1)\bar{d}_t(\bar{p}\bar{a}_i). \tag{1}$$

On the other hand, it is easy to see that for each point $x \in \delta C_t$ and for any ξ -direction of some shortest geodesic from s_0 to x, there exists a ray σ in M starting at s_0, such that $\angle\dot{\sigma}(0)\xi \leq \kappa(t)$. In particular, we can choose such a σ for $x = \bar{p}$ and $\xi = \dot{\gamma}(0)$. Denote $s'_j = sh(\bar{s}'_j) = \alpha_j(\bar{d}_t(\bar{p}\bar{a}_i)\sin(\beta_1))$. Then by [CE,Th.8.10], the result of a parallel translation of $\dot{\sigma}(0)$ along α_j is a direction of some other ray σ'_j, with $\sigma \sim \sigma'_j$. Let $q = \sigma(t)$, $q'_j = \sigma'_j(t)$. Then obviously, $q, q'_j \in \delta C_t$, and by the choice of our original Hausdorff approximation, $h_t(q) = h_t(q'_j)$. By Theorem 2.3, the result of a parallel translation of $\dot{\gamma}(0)$ along α_j is a direction of some shortest geodesic γ_j between s'_j and $\bar{s}'_j$. From $|l/t - 1| < \kappa_1(t)$ and $\angle\dot{\gamma}_j(0)\dot{\sigma}_j(0) = \angle\dot{\gamma}(0)\dot{\sigma}(0) \leq \kappa_2(t)$, it follows that $d(q'_js'_j)/t \leq \kappa_3(t)$, and moreover $\bar{d}_t(q'_js'_j) \leq \kappa_4(t)$. Analogously, $\bar{d}_t(q\bar{p}) \leq \kappa_4(t)$. Therefore

$$\begin{aligned} d_\infty(h_t(\bar{p}), h_t(\bar{s}'_j)) &\leq d_\infty(h_t(\bar{p}), h_t(q)) + d_\infty(h_t(q), h_t(q'_j)) + d_\infty(h_t(q'_j), h_t(\bar{s}'_j)) \\ &\leq \kappa_4(t) + \nu(t) + 0 + \kappa_4(t) + \nu(t) = \kappa_5(t). \end{aligned}$$

Since $f_t : (\delta C_t, \bar{d}) \to M(\infty)$ is $c\nu$ -close to h_t, we get

$$\begin{aligned} d_\infty(f_t(\bar{p}), f_t(\bar{s}'_j)) &\leq d_\infty(f_t(\bar{p}), h_t(\bar{p})) + d_\infty(h_t(\bar{p}), h_t(\bar{s}'_j)) + d_\infty(h_t(\bar{s}'_j), f_t(\bar{s}'_j)) \\ &\leq c\nu(t) + \kappa_5(t) + c\nu(t) = \kappa_6(t), \end{aligned}$$

and finally $d_\infty(f_t(\bar p), f_t(\bar s'_j)) \le \kappa_6(t)$. Thus $f_t(\bar p)$ is very close to $f_t(\bar s'_j)$ (see fig. 2). Therefore $d_\infty(f_t(\bar s'_j), a_i) \ge d_\infty(p, a_i) - \kappa_6(t) \ge \bar d_t(\bar p, \bar a_i) - \kappa_7(t)$. On the other hand, by (1) we have $d_\infty(f_t(\bar a_i), f_t(\bar s'_j)) \le (1+\varepsilon)\cos(\beta_1)\bar d_t(\bar p, \bar a_i)$. Combining these two inequalities, we get $\bar d_t(\bar p, \bar a_i) - \kappa_7(t) \le (1+\varepsilon)\cos(\beta_1)\bar d_t(\bar p, \bar a_i)$. Hence,

$$\cos(\beta_1) \ge \frac{1-\kappa_8(t)}{1+\varepsilon(t,\delta)} = 1 - \kappa_9(t,\delta).$$

Consequently, $\beta_1 \le \arccos(1-\kappa_9(t,\delta)) = \kappa_{10}(t,\delta)$, and $\beta \le \kappa_{10}(t,\delta) + 28\delta = \delta'$, where $\delta' = \kappa(t,\delta)$. So we proved that the collection $(\bar a_i, \bar b_i)_{i=1,\dots,m}, (\bar s_i, \bar t_i)_{i=1,\dots,k}$ forms an $(m+k, \delta')$-strainer at $\bar p$. By **4.2**, we know that $\Sigma\delta C_t$ is $\kappa(t)$-Hausdorff close to a standard sphere, and it is clear that we can complete this strainer to a full $(n-1, \kappa(t,\delta))$-strainer $((\bar a_i, \bar b_i)_{i=1,\dots,m}, (\bar s_i, \bar t_i)_{i=1,\dots,k}, (\bar c_i, \bar d_i)_{i=1,\dots,l})$, where $k+m+l = n-1$.

Let $r_0 = \min(r, R, (\bar d_t(\bar c_j, \bar p), \bar d_t(\bar d_j, \bar p))_{j=1,\dots,l})$. Consider $\bar S_j = sh^{-1}(s_j) \cap B_\eta(\bar s_j)$ and $\bar T_j = sh^{-1}(t_j) \cap B_\eta(\bar t_j)$, where $\eta \ll \delta r_0$. Then by Theorem 2.3, we can find positive $\rho_0 \ll \delta r_0$, such that the following holds.

(2) For all $x \in sh^{-1}(s_0) \cap B_{\rho_0}(\bar p)$, $\bar d_t(x, S_j) = \bar d_t(\bar p, \bar s_j)$, for $j = 1, \dots, k$

(3) The map $s \mapsto (d(s,s_1), \dots, d(s,s_k))$ is a Bilipschitz homeomorphism of $B_{\rho_0}(s_o)$ onto a domain in R^n. Here the ball is taken as a metric ball in S. Condition (3) is guaranteed by [BGP,L5.7]. For the same reason, the map $\Phi : x \mapsto (d_\infty(x,a_1), \dots, d_\infty(x,a_m))$ is a homeomorphism of some $B_{\rho_1}(p)$ onto an open set in R^m.

Let $\rho = \min\{\rho_0, \rho_1\}$. Consider the map $F : B_\rho(\bar p) \longrightarrow R^{n-1}$, given by $F(x) = (\Phi(f_t(x)), \bar d_t(x, \bar c_1), \dots, \bar d_t(x, \bar c_l), \bar d_t(x, S_1), \dots, \bar d_t(x, S_k))$.

Next we will prove the following Sublemma:

Sublemma 4.5. *F is a homeomorphism of $B_\rho(\bar p)$ onto some open domain U in R^{n-1}.*

Proof. First we'll show that F is injective on a small neighbourhood of $\bar p$. Suppose $F(x) = F(y)$ for some $x, y \in B_\rho(\bar p)$. Let z be a midpoint of a shortest geodesic connecting x and y. We will obtain a contradiction by showing that the collection $((\bar a_i, \bar b_i)_{i=1,\dots,m}, (\bar s_i, \bar t_i)_{i=1,\dots,k}, (\bar c_i, \bar d_i)_{i=1,\dots,l}, (x,y))$ forms an $(n, \kappa(t,\delta))$-strainer at z. This is impossible since $\dim(\delta C_t) = n-1$.

Step 1. For each $i = 1, \dots, l$, we have $\bar d_t(x, \bar c_i) = \bar d_t(y, \bar c_i)$. Clearly $\tilde\sphericalangle \bar c_i y \bar t_i \ge \pi - 2\delta$, hence by Lemma 4.4, $|\tilde\sphericalangle \bar c_i yx - \sphericalangle \bar c_i yx| \le 28\delta$. Analogously, $|\tilde\sphericalangle \bar c_i xy - \sphericalangle \bar c_i xy| \le 28\delta$. But in the same way, $|\tilde\sphericalangle \bar c_i yz - \sphericalangle \bar c_i yz| \le 28\delta$ and $|\tilde\sphericalangle \bar c_i xz - \sphericalangle \bar c_i xz| \le 28\delta$. Therefore $|\tilde\sphericalangle \bar c_i yx - \tilde\sphericalangle \bar c_i yz| \le 56\delta$, and $|\tilde\sphericalangle \bar c_i xy - \tilde\sphericalangle \bar c_i xz| \le 56\delta$. But $\tilde\sphericalangle \bar c_i xy = \pi/2 - \frac{1}{2}\tilde\sphericalangle x \bar c_i y$ and $\tilde\sphericalangle x \bar c_i y \le \delta$, since $\bar d_t(x,y) \ll \delta \bar d_t(x, \bar c_i)$. Hence, $|\tilde\sphericalangle \bar c_i xy - \pi/2| \le \delta$, and thus $|\tilde\sphericalangle \bar c_i xz - \pi/2| \le 56\delta + \delta = 57\delta$. But $\tilde\sphericalangle z \bar c_i x \le \delta$, by the same reason as above, and since $\tilde\sphericalangle \bar c_i zx = \pi - \tilde\sphericalangle z \bar c_i x - \tilde\sphericalangle \bar c_i xz$, we finally get

$$|\tilde\sphericalangle \bar c_i zx - \pi/2| \le 57\delta + \delta = 58\delta \tag{4}$$

Using the fact that $\operatorname{diam}(\bar S_i) \ll \delta \bar d_t(\bar s_i, \bar p)$, by the same argument, we obtain

$$|\tilde\sphericalangle \bar s_i zx - \pi/2| \le 100\delta \tag{5}$$

Step 2.We also have that $\Phi \circ f_t(x) = \Phi \circ f_t(y)$. Since Φ is $1-1$ on $B_{\rho_1}(p)$, we have that $f_t(x) = f_t(y)$. We also know that f_t is an ε-almost Lipschitz submersion, which implies

$$\Big|\frac{d_\infty(f_t(\bar a_i), f_t(x))}{\bar d_t(\bar a_i, x)} - \sin(\theta)\Big| < \varepsilon(\delta, \nu(t)),$$

where

$$\theta = \inf_{f_t(x)=f_t(q)} \sphericalangle \bar a_i q x.$$

Note that $\theta \le \sphericalangle \bar a_i y x$, since $f_t(x) = f_t(y)$. Also observe that

$$\Big|\frac{d_\infty(f_t(\bar a_i), f_t(x))}{\bar d_t(\bar a_i, x)} - 1\Big| \le \nu(t)/R = \kappa(t).$$

Thus $\sin(\theta) \ge 1 - \kappa(t) - \varepsilon(\delta, \nu(t))$, and hence $\theta \ge \pi/2 - \kappa(t,\delta)$. This in turn implies that $\sphericalangle \bar a_i y x \ge \pi/2 - \kappa(t,\delta)$. Analogously, $\sphericalangle \bar a_i x y \ge \pi/2 - \kappa(t,\delta)$. As in Step 1., we have that $|\tilde\sphericalangle \bar a_i y x - \sphericalangle \bar a_i x y| \le 28\delta$, and therefore $\tilde\sphericalangle \bar a_i y x \ge \pi/2 - \kappa(t,\delta)$, and analogously $\tilde\sphericalangle \bar a_i x y \ge \pi/2 - \kappa(t,\delta)$. But evidently, $\tilde\sphericalangle \bar a_i y x + \tilde\sphericalangle \bar a_i x y \le \pi$, and so $|\tilde\sphericalangle \bar a_i y x - \pi/2| \le \kappa(t,\delta)$ and $|\tilde\sphericalangle \bar a_i y x - \pi/2| \le \kappa(t,\delta)$. Now arguing as in Step 1., we obtain

$$|\tilde\sphericalangle \bar a_i y x - \pi/2| \le \kappa(t,\delta), \tag{6}$$

$$|\tilde\sphericalangle \bar a_i y x - \pi/2| \le \kappa(t,\delta) \tag{7}$$

Step 3. Combining (4), (5), (6) and (7), we get that the collection $((\bar a_i, \bar b_i)_{i=1,\dots,m}, (\bar s_i, \bar t_i)_{i=1,\dots,k}, (\bar c_i, \bar d_i)_{i=1,\dots,l}, (x,y))$ forms an $(n, \kappa(t,\delta))$-strainer at z. But this is impossible since δC_t is $(n-1)$-dimensional [BGP]. So we have that F is injective in $B_\rho(\bar p)$.

Now recall that according to [CG], δC_t is topologically a manifold. By the Invariance of Domain Theorem, this implies that F is a homeomorphism of $B_\rho(\bar p)$ onto some open domain U in R^{n-1}. $\square$

4.6 Next we will check that $x \in sh^{-1}(s_0) \cap B_\rho(\bar p)$ iff $F(x) \in U$ and $\bar d_t(x, S_j) = \bar d_t(\bar p, \bar s_j), j = 1, \dots, k$. One implication is true by (2). Now suppose $\bar d_t(x, S_j) = \bar d_t(\bar p, \bar s_j), j = 1, \dots, k$. Then by Theorem 2.3, $sh(x)$ satisfies $d(sh(x), s_j) = d(s_0, s_j), j = 1, \dots, k$. Whence, by (3), we should have $sh(x) = s_0$.

This means that F maps $sh^{-1}(s_0) \cap B_\rho(\bar p)$ homeomorphically onto the intersection of U and the hyperplane $\{u_{m+l+j} = \bar d_t(\bar p, \bar s_j)\}_{j=1,\dots,k}$. Consequently, $\Phi \circ f_t : sh^{-1}(s_0) \cap B_\rho(\bar p) \longrightarrow R^m$ is a locally trivial fibration. Moreover, our argument shows that its fibers are locally Euclidean. Finally, Φ is a homeomorphism of B_{ρ_1} onto an open set in R^m and we get $f_t|_{sh^{-1}(s_0)\cap B_\rho(\bar p)} : sh^{-1}(s_0) \cap B_\rho(\bar p) \longrightarrow M(\infty)$ is also a locally trivial fibration. Hence, by the Siebenmann's theorem [S,Cor.6.14,Th.5.4],

$$f_t|_{\delta C_t \cap sh^{-1}(s_0)} : \delta C_t \cap sh^{-1}(s_0) \longrightarrow M(\infty)$$

is a locally trivial fibration. Moreover, by above, its fibers are locally Euclidean; hence they are topological manifolds without boundaries. $\square$

Remark. *Using the same argument as in the the proof of Sublemma 4.5, plus Siebenmann's theorem, we can prove the following observation, stated without a proof in [BGP]: Let* $f : M^m \to N^n$ *be an* ε *-almost Lipschitz submersion of Alexandrov spaces where* M *is* (m, δ)*-strained at each point, and* N *is* (n, δ)*-strained at each point, with* ε, δ *being sufficiently small. Then* f *is a locally trivial fibration.*

Now to finish the proof of Lemma 4.1 we only need to note that by Proposition 2.4.3, $\delta C_t \cap sh^{-1}(s)$ with the induced topology is homeomorphic to S^{m+l}, for any $s \in S$. □

5. Some applications of the main lemma.

The Main Lemma has some immediate consequences that we collect in this section.

Corollary 5.1. *Let* M^n *be a complete open manifold with sec* ≥ 0. *If* $M(\infty)$ *is m-dimensional and* (m, δ)*-strained at each point, it has some finite covering* N *satisfying one of the following conditions:*

(a) N *is homotopically equivalent to* S^m*,*
(b) N *is homotopically equivalent to* $\mathbb{CP}^{m/2}$ *or*
(c) N *has rational homology ring of* $\mathbb{HP}^{m/4}$.

The proof relies on the following theorem due to Browder [B]:

Theorem. *Let* $p : S^n \to B$ *be a fiber map, with base* B *and fiber* F *connected polyhedra,* $F, B \neq$ *point. Then we have the following possibilities:*

(a) F *is homotopically equivalent to* S^1*, and* B *is homotopically equivalent to* $\mathbb{CP}^m$*;*

(b) F *is homotopically equivalent to* S^7*, and* B *is homotopically equivalent to* S^8*; or*

(c) F *is homotopically equivalent to* S^3*, and* B *has rational homology ring of* $\mathbb{HP}^m$.

Proof of Corollary 5.1. By Lemma 4.1, there exists a locally trivial fibration $f : S^{n-k-1} \to M(\infty)$, whose fibers are closed manifolds. An easy lifting argument shows that there exists a finite covering $\Pi : N \to M(\infty)$ and a fibration $\tilde{f} : S^{n-k-1} \to N$, with connected fibers such that $\Pi \circ \tilde{f} = f$. Now since the fibers of $\tilde{f}$ are connected, we can apply Browder's theorem, which immediately gives us the conclusion of the Corollary. □

Proof of the Main Theorem. As an example of a positively curved Alexandrov space which can not be an ideal boundary, we can take any known positively curved Riemannian manifold, which does not satisfy any of conditions (a), (b) or (c) from 5.1. For instance, the Allof–Wallach examples (see[AW]), the Cayley plane and some flags manifolds with any smooth metric of positive curvature fit into this category. □

Remark. *We actually can rule out as ideal boundaries not only some manifolds, but also some singular spaces. Indeed, if X is any space known not to be a boundary, then the Splitting Theorem shows that a spherical suspension over X can not be an ideal boundary either* [K].

Corollary 5.2. *Let M^n be a complete simply connected open manifold with $sec \geq 0$. Suppose $M(\infty)$ is m-dimensional and (m, δ) -strained at each point. Suppose, in addition, that $M(\infty)$ is homeomorphic to S^m, and $m \neq 2, 4, 8$. Then M has an isometric splitting as $S \times R^{n-m}$, where S is a soul of M and R^{n-m} has a metric of nonnegative curvature, with $R^{n-m}(\infty) \cong M(\infty)$.*

Proof. By Lemma 4.1, there is a fibration $f : S^{n-k-1} \to M(\infty)$, where $k = dimS$. But $M(\infty)$ is homeomorphic to S^m, and $m \neq 2, 4, 8$. So by the standard theorem of homotopy theory, this implies $n - k - 1 = m$, and the map f is actually a homeomorphism. Therefore the normal holonomy group $H(s)$ of S in M is discrete, since $dimM(\infty) \leq n - k - 1 - dimH(s)$. But $\pi_1(S) = 0$ implies $H(s) = 0$, and by [St], M has an isometric splitting. □

Remark. *Note that condition $m \neq 2, 4, 8$ in the previous Corollary is relevant, as the following examples indicate. If we put in Example 3.2 $G = S^1$ acting on $\mathbb{R}^4 = \mathbb{C}^2$ by complex multiplication, we get the boundary $(G\backslash(G \times \mathbb{R}^4))(\infty) \approx \mathbb{CP}^1 \approx S^2$. If we put $G = Sp(1)$ acting on $\mathbb{R}^8 = \mathbb{H}^2$ by quaternionic multiplication, then $(G\backslash(G \times \mathbb{R}^8))(\infty) \approx \mathbb{HP}^1 \approx S^4$. Similarly, for $G = F_4$ acting on $\mathbb{R}^{16}$, we have $(G\backslash(G \times \mathbb{R}^{16}))(\infty) \approx S^8$.*

We can also provide some examples of spaces that may occur as ideal boundaries, when the soul is different from a point, but may not be ideal boundaries when the soul is a point. It is easy to construct an example with a lens space $L(p)$ as an ideal boundary. Say if we put in Example 3.2 $G = Z_p$ acting on $\mathbb{R}^4 = \mathbb{C}^2$ by complex multiplication, we get the boundary $(G\backslash(G \times \mathbb{R}^4))(\infty) \cong L(p)$.

On the other hand, we have the following:

Corollary 5.3. *Let M^n $(n > 2)$ be a complete open manifold with $sec \geq 0$, such that the soul is a point, and $M(\infty)$ is a Riemannian manifold. Then $\pi_1(M(\infty)) = 0$.*

Proof. Lemma 4.1 shows that in this case, $f_t : \delta C_t \to M(\infty)$ is a fibration, and δC_t is homeomorphic to a sphere. But for big t the fibers of f_t are very small and therefore path connected. Indeed, if a fiber F is not path connected, then we can find a very short path between two points, lying in different components of the fiber. Then the map f_t sends this path to a very short and therefore homotopic to a point loop. On the other hand, this path represents a nontrivial element in $\pi_1(\delta C_t, F)$, which contradicts the well known fact that any fibration map should induce an isomorphism of $\pi_1(\delta C_t, F)$ and $\pi_1(M(\infty))$. So the fibers are path connected, and using long exact homotopy sequence of a fibration, we immediately get that $\pi_1(M(\infty)) = 0$. □

References

[AW] S. Aloff and N. Wallach, *An infinite family of distinct 7–manifolds admitting positively curved Riemannian structures*, Bull. Amer. Math. Soc. **81**, 93-97.

[B] W. Browder, *Higher torsion in H-spaces*, Trans. Amer. Math. Soc. **108** (1963), 353–375.

[BGS] W. Ballman, M. Gromov and V. Schroeder, *Manifolds of Nonpositive Curvature*, Birkhäuser, Boston, 1985.

[BGP] Y. Burago, M. Gromov and G. Perelman, *A.D. Alexandrov's spaces with curvatures bounded from below* I, Uspechi Mat. Nauk **47** (1992), 3–51.

[Bu] S. V. Buyalo, *Shortest paths on convex hypersurfaces of a Riemannian space*, Zap. Nauchn. Sen. Leningrad. Otdel Mat. Inst. Steklov. (LOMI) **66** (1976), 114-132.

[CE] J. Cheeger, D. Ebin, *Comparison Theorems in Riemannian Geometry*, North Holland Publishing Co., 1975.

[CG] J. Cheeger, D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972), 413–443.

[D] G. Drees, *Asymptotically flat manifolds of nonnegative curvature*, Differential Geometry and its Applications **4** (1994), 77–90.

[GP] K. Grove, P. Petersen, *Manifolds near the boundary of existence*, Journal of Differential Geometry **33** (1991), 379-394.

[Gr] K. Grove, *Metric Differential Geometry*, Lecture Notes in Math. **1263** (1987), 171-227.

[K] A. Kasue, *A compactification of a manifold with asymptotically nonnegative curvature*, Ann. Scient. Éc. Norm. Sup. (4) **21** (1988), 593–622.

[P] G. Perelman, *Prrof of the Soul Conjecture of Cheeger and Gromoll*, Journal of Differential Geometry **40**, 209–212.

[Po] W.A. Poor, *Some results on nonnegatively curved manifolds*, J. Diff. Geometry **9**, 583–600.

[Sha] V. A. Sharafutdinov, *The Pogorelov-Klingerberg theorem for manifolds homeomorphic to R^n*, Sib. Math. Zh. **18** (1977), 915–925. (Russian)

[Sh] K. Shiohama, *An Introduction to the Geometry of Alexandrov Spaces*, Notes of the Series of Lectures held at the Seoul National University.

[Shi] T. Shioya, *Splitting theorems for nonnegatively curved open manifolds with large ideal boundary*, Math. Z. **212** (1993), 223–238.

[Sie] L. S. Siebenmann, *Deformation of homeomorphisms on stratified sets*, Comment. Math. Helv. **47** (1972), 123-163.

[St] M. Strake, *A splitting theorem for open nonnegatively curved manifolds*, Manuscripta Math. **61** (1988), 315-325.

[Y1] T. Yamaguchi, *Collapsing and pinching under lower curvature bound*, Ann. of Math. **133** (1991), 317–357.

[Y2] ———, *A convergence theorem in the geometry of Alexandrov space* (preprint).

[Yi] J. W. Yim, *Distance nonincreasing retraction on a complete open manifold of nonnegative sectional curvature*, Ann. Global Anal. Geom. **6**, 191-206.

Department of mathematics, University of Maryland, College Park, Maryland 20742

E-mail address: luis@lakisis.umd.edu; vitali@lakisis.umd.edu