(1) Prove that the canonical map $\mathbb{S}^n \to \mathbb{RP}^n$ is smooth with respect to the canonical smooth structures on $\mathbb{S}^n$ and $\mathbb{RP}^n$ defined in class.

(2) Let U be the open unit ball in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ be smooth. Prove that there exist smooth functions $g_1, \dots, g_n\colon U \to \mathbb{R}$ such that

$$f(x) = f(0) + \sum_{i=1}^{n} x_i g(x)$$

(3) LetM^n be a smooth n-dimensional manifold and let $p \in M$. Let $\mathcal{F}\colon C^\infty(M) \to \mathbb{R}$ be a linear functional satisfying

$$\mathcal{F}(fg) = \mathcal{F}(f) \cdot g(p) + \mathcal{F}(g) \cdot f(p)$$

for any $f, g \in C^\infty(M)$.

   (a) Suppose $f, g \in C^\infty(M)$ are such that $f = g$ on a open set containing p.
   Prove that $\mathcal{F}(f) = \mathcal{F}(g)$. This means that the functional $\mathcal{F}$ is local.

   (b) prove that there exists $v \in T_pM$ such that $\mathcal{F}(f) = D_v f$ for any $f \in C^\infty(M)$.

(4) prove that $O(n)$ is a manifold. Compute its dimension, its Lie algebra and find the formula for $[X, Y]$ for $X, Y \in \mathfrak{o}(n)$ where $\mathfrak{o}(n) = T_eO(n)$ is the Lie algebra of $O(n)$.

(5) Let $G = S^3$ be the 3-sphere with the Lie group structure coming from the identification of S^3 with unit quaternions $S^3 \subset H$.

   (a) Show that $\mathfrak{g} = T_eG = span(i, j, k)$

   (b) Compute $[i, j], [i, k]$ and $[j, k]$.

   (c) Let $v \in T_eG$ such that $|v| = 1$. Prove that $exp(tv) = \cos t + (\sin t) \cdot v$ where exp is the Lie group exponential in G.

(6) Let G be a Lie group. Let $f\colon G \to G$ be given by $f(g) = g^{-1}$. Prove that $df(e)(v) = -v$ for any $v \in T_eG$.

(7) Let G be a Lie group and X be aleft invariant vector field on G. Prove that the integral flow of X is defined for all t.

(8) Prove that $SL(2, \mathbb{R})$ does not admit a biinvariant Riemannian metric.