Solutions to Term Test 3 Practice Test 2

(1) Give the following definitions
(a) a k-tensor on a vector space $V$.
(b) A $C^r$-manifold without a boundary in $\mathbb{R}^n$.

Solution
(a) A map $T: \underbrace{V \times V \ldots \times V}_{k \text{ times}} \rightarrow \mathbb{R}$ is a k-tensor on $V$ if it’s linear in every variable.
(b) A set $M \subset \mathbb{R}^n$ is a $k$-dimensional $C^r$-manifold without a boundary if for every point $p \in M$ there exists a set $U \subset M$ which is open in $M$, an open subset $V \subset \mathbb{R}^k$ and a $C^r$ map $f: V \rightarrow \mathbb{R}^n$ such that
(i) $f(V) = U$ and $f: V \rightarrow U$ is 1-1 and onto;
(ii) $\text{rank}[df_x] = k$ for any $x \in V$;
(iii) $f^{-1}: U \rightarrow V$ is continuous.

(2) Let $e_1, e_2, e_3, e_4$ be a basis of a 4-dimensional space $V$.
Let $\omega = \text{Alt}(e_1^* \otimes e_2^* + e_3^* \otimes e_4^*)$ and $\eta = 2e_2^* + e_3^*$.
Find $(\omega \wedge \eta)(e_2, e_3, e_4)$.

Solution
We have $\omega = \frac{111}{21}(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*)$. Hence $\omega \wedge \eta = \frac{1}{2}(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*) \wedge (2e_2^* + e_3^*) = \frac{1}{2}(e_1^* \wedge e_2^* \wedge 2e_2^* + e_1^* \wedge e_2^* \wedge e_3^*) + e_3^* \wedge e_4^* \wedge 2e_2^* + e_3^* \wedge e_4^* \wedge e_3^*) = \frac{1}{2}e_1^* \wedge e_2^* \wedge e_3^* + e_2^* \wedge e_3^* \wedge e_4^*.
Therefore $\omega \wedge \eta(e_2, e_3, e_4) = \frac{1}{2}e_1^* \wedge e_2^* \wedge e_3^*(e_2, e_3, e_4) + e_2^* \wedge e_3^* \wedge e_4^*(e_2, e_3, e_4) = 0 + 1 = 1.$

(3) Let $V$ be an n-dimensional vector space. Let $k \geq 1$.
Find all $k$-tensors $T$ on $V$ such that both $T$ and $|T|$ are tensors.

Solution
If both $T$ and $|T|$ are tensors then we must have $|T|(-v_1, v_2, \ldots, v_k) = -|T|(v_1, ldots, v_k)$ for any $v_1, ldots, v_k$.

But $|T| \geq 0$ so this can only happen if $T \equiv 0$.

(4) (15 pts) Let $M \subset \mathbb{R}^3$ be given by \( \{x^2 + y^2 - z^2 = 0\} \cap \{x + 2y - z = 1\} \).

Show that $M$ is a manifold and compute its dimension.

**Solution**

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (x^2 + y^2 - z^2, x + 2y - z)$. Note that $f$ is clearly $C^\infty$ and $M = f^{-1}\{(0, 1)\}$.

We claim that $(0, 1)$ is a regular value of $f$. To see this suppose $f(x, y, z) = (0, 1)$. Then $[df_{(x,y,z)}] = \begin{pmatrix} 2x & 2y & -2z \\ 1 & 2 & -1 \end{pmatrix}$

The only way this matrix could have rank 1 if the first row is a multiple of the second one i.e. $x = \lambda, y = 2\lambda, z = \lambda$. Then we must have $0 = x^2 + y^2 - z^2 = \lambda^2 + (2\lambda)^2 - (\lambda)^2 = 4(\lambda)^2$ and hence $\lambda = 0$ and $x = y = z = 0$. This contradicts $x + 2y - z = 1$. Thus $[df_{(x,y,z)}]$ has rank=2. Therefore $(0, 1)$ is a regular value of $f$ and hence, $M$ is a $C^\infty$ manifold without boundary of dimension $3 - 2 = 1$.

(5) (15 pts) Let $U \subset \mathbb{R}^2$ be given by $\{0 < x^2 + 4y^2 < 1\}$ and $f(x, y) = \frac{1}{\sqrt{x^2 + 4y^2}}$.

Determine if $\int_{U}^{ext} f$ exists and if it does compute it.

**Solution**

Let $U_n = \{\frac{1}{n^2} < x^2 + 4y^2 < 1\}$ where $n > 1$ be an open exhaustion of $U$. Then $U_n$ is rectifiable and $f$ is continuous and bounded on $U_n$. Therefore, $\int_{U_n} f$ exists for any $n > 1$ and hence $\int_{U_n}^{ext} f$ exists and $\int_{U_n}^{ext} f = \int_{U_n} f$. Let $V_n = U_n \backslash \{(x, 0)|x > 0\}$. Then $V_n$
is also open and rectifiable and hence \( \int_{V_n} f \) also exists. Since \( f \cdot \chi_{U_n} = f \cdot \chi_{V_n} \) except on a set of measure zero this means that \( \int_{V_n} f = \int_{U_n} f \).

To compute \( \int_{V_n} f \) we use the change of variables \((x, y) = g(r, \theta)\) given by \( x = r \cos \theta, y = \frac{1}{2} r \sin \theta \) where \( \frac{1}{n} < r < 1, 0 < \theta < 2\pi \). Then \( \det dg = \frac{r}{2} \) and

\[
\int_{V_n} f = \int_0^{2\pi} \int_{1/n}^{1} \frac{r}{2r} = \pi (1 - 1/n).
\]

We see that \( \lim_{n \to \infty} \int_{U_n} f = \lim_{n \to \infty} \pi(1 - 1/n) = \pi \). Since \( f > 0 \) on \( U \) this means that \( \int_{U_n}^{	ext{ext}} f = \int_{U_n} f = \pi \).

(6) (10 pts) Let \( v_1 = (1, 1, 0), v_2 = (-1, 0, 1), v_3 = (1, 1, 1) \) and \( w_1 = (0, 2, 0), w_2 = (1, 1, 0), w_3 = (-2, 1, 3) \) be two bases of \( \mathbb{R}^3 \). Do \((v_1, v_2, v_3)\) and \((w_1, w_2, w_3)\) have the same orientation?

**Solution**

Consider the standard basis \( e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1) \). The transition matrix from \((e_1, e_2, e_3)\) to \((v_1, v_2, v_3)\) has columns \( v_1, v_2, v_3 \), i.e. it’s given by

\[
A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.
\]

We compute \( \det A = 1 \). Hence, \((e_1, e_2, e_3)\) and \((v_1, v_2, v_3)\) have the same orientation.

Similarly, we compute that the transition matrix from \((e_1, e_2, e_3)\) to \((u_1, u_2, u_3)\) is

\[
B = \begin{pmatrix} 0 & 1 & -2 \\ 2 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}
\]

and \( \det B = -6 \). Hence, \((e_1, e_2, e_3)\) and \((u_1, u_2, u_3)\) have opposite orientations. Therefore, \((v_1, v_2, v_3)\) and \((w_1, w_2, w_3)\) have opposite orientations.

(7) Let \( M \subset \mathbb{R}^n \) be a manifold with boundary and \( N \subset \mathbb{R}^m \) be a manifold without boundary. Prove that \( M \times N \subset \mathbb{R}^{n+m} \) is a manifold with boundary.

**Solution**
Let $M$ be $k$-dimensional and $N$ be $l$-dimensional. First observe the following:

**Observation.** If $f: X_1 \to Y_1, g: X_2 \to Y_2$ are continuous then $f \times g: X_1 \times X_2 \to Y_1 \times Y_2$ is also continuous. Here by $f \times g$ we mean the map $f \times g(x_1, x_2) = f(x_1), g(x_2)$.

Let $N$ be $k$-dimensional and $M$ be $l$-dimensional. Let $p \in N, q \in M$. Let $f: V_1 \to N$ and $g: V_2 \to M$ be parameterizations coming from the definition of a manifold (with boundary). Here $V_1 \subset \mathbb{R}^k$ is open and $V_2 \subset \mathbb{H}^l$ is open and $p \in f(V_1)$ and $q \in g(V_2)$. Then the map $F = f \times g: V_1 \times V_2 \to N \times M \subset \mathbb{R}^{n+m}$ satisfies the definition of a manifold with boundary.

Indeed, $V_1 \times V_2$ is open in $\mathbb{R}^k \times \mathbb{H}^l = \mathbb{H}^{k+l}$, the map $F$ is obviously smooth, 1-1 and onto $f(V_1) \times g(V_2)$. The inverse map is continuous by the Observation above.

Lastly, $dF$ clearly has maximal rank everywhere since both $df$ and $dg$ do.