Solutions to Term Test 3

(1) (13 pts) A k-tensor T on a vector space V is called *symmetric* if $T^\sigma = T$ for any $\sigma \in S_k$.

Prove that a 2-tensor T is symmetric if and only if $Alt(T) = 0$.

Solution

First note that $S_2 = \{e, (12)\}$ and $T^e = T$ for any tensor. For $\sigma_0 = (1,2)$ we have that $sign(\sigma_0) = -1$. Then $Alt(T) = \frac{1}{2}(T^e - T^{\sigma_0}) = \frac{1}{2}(T - T^{\sigma_0})$ so $Alt(T) = 0$ iff $T = T^{\sigma_0}$.

(2) (15 pts) Prove that $[0,1] \times [0,1] \subset \mathbb{R}^2$ is not a manifold with boundary.

Solution

Suppose $M = [0,1]\times[0,1]$ si a 2-manifold with boundary. Clearly, $(0,1) \times (0,1) \subset intM$ and $(0,1) \times \{0,1\} \cup \{0,1\} \times (0,1) \subset \partial M$. It's also easy to see that the vertices of $[0,1]^2$ can not belong to $intM$ so they must be in ∂M. Consider one of those vertices, say $p = (0,1)$. since $p \in \partial M$ there exists an open set $U \subset \mathbb{R}^2$ an open set $V \subset \mathbb{R}^2$ and a diffeomorphism $F\colon U \to V$ such that $F(U \cap M) = V \cap H^2$. Note that since boundary pf a manifold is well defined we must have that $F(\partial M) \subset \mathbb{R} \times \{0\}$. This means that $F(0,t) = (x(t), 0)$ and $F(t,0) = (0, \tilde{x}(t), 0$ for $t \geq 0$. this implies that $D_1F(0,0) = (x'(0), 0)$ and $D_2(0,0) = (\tilde{x}'(0), 0)$. Therefore DF_p is not invertible which contradicts the assumption that F is a diffeomorphism.

(3) (12 pts) Let $V = \mathbb{R}^3$. Let T be a 2-tensor on V given by $T(u,v) = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ 1 & 2 & -3 \end{pmatrix}$ Let $\mathcal{A}^2(\mathbb{R}^3)$ be the space of alternating 2-tensors on $\mathbb{R}^3$. Express T in the standard basis of $\mathcal{A}^2(\mathbb{R}^3)$.

Solution

The standard basis of $\mathcal{A}^2(\mathbb{R}^3)$ is given by $e_1^* \wedge e_2^*, e_1^* \wedge e_3^*, e_2^* \wedge e_3^*$. Then $T = T_{12} e_1^* \wedge e_2^* + T_{13} e_1^* \wedge e_3^* + T_{23} e_2^* \wedge e_3^*$ where $T_{ij} = T(e_i, e_j)$.

Plugging in we get $T_{12} = T(e_1, e_2) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & -3 \end{pmatrix} = -3.$

Similarly, $T_{13} = T(e_1, e_3) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -3 \end{pmatrix} = -2$ and $T_{23} = T(e_2, e_3) = \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -3 \end{pmatrix} = 1$

Hence, $T = -3e_1^* \wedge e_2^* - 2e_1^* \wedge e_3^* + e_2^* \wedge e_3^*$.

(4) (20 pts) Let $S = \{(x, y) \in R^2|$ such that $x^2 + \frac{y^2}{4} \leq 1, y \geq 0, -y/2 \leq x \leq y/2\}$.

Compute $\int_S y$.

Hint: use the appropriate change of variables.

Solution

Since S is rectifiable and $f(x, y) = y$ is continuous, $\int_S y$ exists and it is equal to $\int_U y = \int_U^{ext} y$ where $U = intS = \{(x, y) \in R^2|$ such that $x^2 + \frac{y^2}{4} < 1, y > 0, -y/2 < x < y/2\}$. using the change of variables $g(x, y) = (x, 2y)$ we see that

$\int_U^{ext} y = \int_V^{ext} 4y$ where $V = \{(x, y) \in R^2|$ such that $x^2 + y^2 < 1, y > 0, -y < x < y\}$. Making another change of variables $x = r \cos\theta, y = r \sin\theta$ we get

$\int_V^{ext} 4y = \int_W^{ext} 4r^2 \sin\theta$ where $W = \{(r, \theta) \in R^2|$ such that $0 < r < 1, \pi/4 < \theta < 3\pi/4\}$. By Fubini's theorem we get

$\int_W^{ext} 4r^2 \sin\theta = \int_W 4r^2 \sin\theta = \int_0^1 (\int_{\pi/4}^{3\pi/4} 4r^2 \sin\theta d\theta) dr = \frac{4\sqrt{2}}{3}$

(5) (15 pts) Let $M \subset \mathbb{R}^n$ be a k-dimensional C^r manifold with boundary and let $N \subset \mathbb{R}^m$ be an l-dimensional C^r manifold with boundary where $r \geq 1$. A map $f\colon M \to N$ is called a C^r diffeomorphism if f is C^r as a map from M to $\mathbb{R}^m$, f is a bijection from M to N and the inverse map $f^{-1}\colon N \to M$ is also C^r as a map from N to $\mathbb{R}^n$.

Prove that if $f\colon M \to N$ is a C^r diffeomorphism then $k = l$.

Hint: Look at the maps in local coordinates on M and N.

Solution

Let $p \in M, q = f(p) \in N$. Let $\phi\colon V \to M$ and $\psi\colon W \to N$ be local charts on M and N respectively where $V \subset \mathbb{H}^k, W \subset \mathbb{H}^l$ are open, $p = \phi(a)$ and $q = \psi(b)$. Then ψ^{-1} and ϕ^{-1} are smooth wehere defined. Therefore $h = \psi^{-1} \circ f \circ \phi\colon V' \to W'$ is smooth where $V' \subset V$ and $W' \subset W$ are open. Similarly $g = \phi^{-1} \circ f^{-1} \circ \psi\colon W' \to V'$ is also smooth. Note that $g = h^{-1}$. By the chain rule that means that $dh_a \circ dg_b = id$ and $dg_b \circ dh_a = id$. Therefore, $dh_a\colon \mathbb{R}^k \to \mathbb{R}^l$ is an isomorphism and hence $k = l$.

(6) (25 pts) **True or False. If True give a proof, if False give a counterexample.**
   (a) Let $M \subset \mathbb{R}^n$ be a manifold without boundary. Let $U \subset \mathbb{R}^n$ be open. Then $U \cap M$ is also a manifold without boundary.
   (b) Let $U \subset \mathbb{R}^n$ be a bounded open set, $f\colon U \to \mathbb{R}$ be continuous and bounded. Suppose $\int_U^{ext} f$ exists. Then $\int_U f$ exists.
   (c) If $M \subset \mathbb{R}^n$ is a manifold with boundary then $\partial M = bd(M)$.
   (d) Let $e_1, \dots, e_n$ be a basis of a vector space V and let $\sigma \in S_n$ be an even permutation, i.e. $sign(\sigma) = +1$. Then $e_1, \dots, e_n$ and $e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n)}$ have the same orientation.

Solution

(a) Let $M \subset \mathbb{R}^n$ be a manifold without boundary. Let $U \subset \mathbb{R}^n$ be open. Then $U \cap M$ is also a manifold without boundary.
True. Let $f\colon V \to M$ be a local parameterization coming from the definition of a manifold where $V \subset \mathbb{R}^k$ is open. Then $f\colon V \cap f^{-1}(U) \to M \cap U$ is a parameterization for an open subset of $M \cap U$.

(b) Let $U \subset \mathbb{R}^n$ be a bounded open set, $f\colon U \to \mathbb{R}$ be continuous and bounded. Suppose $\int_U^{ext} f$ exists. Then $\int_U f$ exists.
False. Let U be a bounded open set which is not rectifiable. and $f(x) \equiv 1$. Then $\int_U^{ext} f$ exists but $\int_U f$ does not.

(c) If $M \subset \mathbb{R}^n$ is a manifold with boundary then $\partial M = bd(M)$.

False. Let $M = [0,1] \times \{0\} \subset \mathbb{R}^2$. Then $\partial M = \{0,1\} \times \{0\}$ but $bd(M) = M$.

(d) Let $e_1, \ldots, e_n$ be a basis of a vector space V and let $\sigma \in S_n$ be an even permutation, i.e. $sign(\sigma) = +1$. Then $e_1, \ldots, e_n$ and $e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)}$ have the same orientation.
True. The transition matrix from $e_1, \ldots, e_n$ to $e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)}$ is given by P_σ. Be definition of the sign we have $\det P_\sigma = sign(\sigma) = 1$. Hence $\det P_\sigma > 0$ which means that $e_1, \ldots, e_n$ and $e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)}$ have the same orientation.