(1) Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Consider the function $f(x, y) = \frac{x^p}{p} + \frac{y^q}{q} - xy$ defined on the set $A = \{x \geq 0, y \geq 0\}$. Assuming that f has a minimum on A find it.

(2) Let $M(2, 2)$ be the space of 2×2 matrices identified with $\mathbb{R}^4$. Let $f\colon M(2, 2) \to M(2, 2)$ be given by $f(A) = A^2$. Let $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that $f(A_0) = Id$. Is it true that there is an open set U containing A_0 such that f is 1-1 on U, $f(U)$ is open and f^{-1} is differentiable on $f(U)$?

(3) Finish the proof of the following statement from class.

   Let $f\colon GL(n, R) \to GL(n, R)$ be given by $f(A) = A^{-1}$. Let A_0 be the identity matrix. Prove that $df_{A_0}(X) = -X$ for any $n \times n$ matrix X.

(4) Give a careful proof of the following statement from class. Let $f(r, \theta) = (r \cos\theta, r \sin\theta)$ be defined on $U = \{1 < r < 2, 0 < \theta < 2\pi\}$. Prove that
   (a) f is $1 - 1$ on U;
   (b) $f(U) = V = \{1 < x^2 + y^2 < 4\} \backslash \{(x, 0) \text{ with } 1 < x < 2\}$;
   (c) $g = f^{-1}\colon V \to U$ is C^1;
   (d) $||dg_p||$ is bounded on V but g is not L-Lipschitz on V for any L.

Extra Credit Problem (to be written up and submitted separately)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function such which is continuous in each variable, i.e. for every $x \in \mathbb{R}$ the function $y \mapsto f(x, y)$ is continuous and for every $y \in \mathbb{R}$ the function $x \mapsto f(x, y)$ is continuous.

(1) Give an example of such an f which is not everywhere continuous on $\mathbb{R}^2$;

(2) Suppose we also assume in addition that for any compact set $K \subset \mathbb{R}^2$ its image $f(K)$ is compact. Prove that f is continuous on $\mathbb{R}^2$.