(1) Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold. Let ω be an l-form on M. recall that ω is called smooth if it can be extended to a smooth form on an open set containing M.

a) Prove that ω is smooth if and only if it's locally smooth. Here a form on M is locally smooth if for every $p \in M$ there exists open subset $U \subset \mathbb{R}^n$ containing p such that $\omega|_{M\cap U}$ is smooth.
Hint: use partition of unity.

b) Prove that ω is smooth if and only if for any smooth tangent fields $V_1(x), \dots V_l(x)$ on M the function $\omega(V_1(x), \dots V_l(x))$ is smooth in x.
Hint: For the if direction: by a) it's enough to argue locally. Extend local coordinates on M to a local diffeomorphism between open sets in $\mathbb{R}^n$, look at the form in those local coordinates and extend it there.

Solution

a) for every $p \in M$ let U_p be an open set in $\mathbb{R}^n$ such that ω admits a smooth extension ω_p to U_p. Let ϕ_i be a partition of unity subordinate to $\{U_p\}_{p\in M}$. For each i we have that $supp(\phi_i) \subset U_i = U_{p_i}$ for some $p_i \in M_i$. Let ω_i be a smooth extension of ω to U_i. Then $\phi_i \cdot \omega_i$ has compact support contained in U_i. Therefore we can extend it by zero to be a smooth form on $\mathbb{R}^n$. We'll still denote that extension by $\phi_i \cdot \omega_i$
Define $\bar{\omega} = \sum_{i=1}^{\infty} \phi_i \omega_i$. It is then easy to see that $\bar{\omega}$ is smooth on $U = \cup_i U_i$ and $\bar{\omega}|_M = \omega$

b) It's obvious that if ω is smooth then $\omega(V_1(x), \dots V_l(x))$ for any smooth vector fields $V_1(x), \dots V_l(x)$.
Let's prove the opposite implication. By part a) it's enough to show that ω is locally smooth. Let $p \in M$ and let $f\colon V \to M$ be a local parameterization coming from the definition of a manifold such that $V \subset \mathbb{R}^k$ (or $V \subset \mathbb{H}^k$ if M has boundary) is open and $p = f(0)$. Then by a result from class f can be extended to a diffeomorphism $F\colon W \to U$ where $U \subset \mathbb{R}^n, W \subset \mathbb{R}^n$ are open and $V' \times \{0\} = W \cap \mathbb{R}^k \times \{0\}$ contains 0. We will show that $\eta = f^*(\omega)$ can be extended to a smooth form on an open set containing 0. Since η is a n l-form on $\mathbb{R}^k$ and we can write it in coordinates as $\eta(x) = \sum_I \eta_I(x) dx_I$ where $I = (i_1 < \ldots < i_l)$ with $1 \le i_1, i_l \le k$.
Since push forward of a smooth vector field under a diffeomorphism is smooth we know that $\eta_I(x) = \eta(e_{i_1}, \dots, e_{i_l})(x)$ is smooth in $x \in V'$ that means that $\eta_I(x)$ can be extended to a smooth function on some open $W' \subset \mathbb{R}^n$ containing 0. Doing it for each I we get a smooth extension $\bar{\eta}$ of η to an open set in $\mathbb{R}^n$ containing 0. Finally, $(F^{-1})^*(\bar{\eta})$ will be a smooth extension of ω to an open set in $\mathbb{R}^n$ containing p.