MAT 257Y **Practice Final 2**

1. Let $A \subset \mathbb{R}^n$ be a rectangle. Let $f\colon A \to \mathbb{R}$ be integrable. Let

$$f_+(x) = \begin{cases} f(x) \text{ if } f(x) \geq 0 \\ 0 \text{ if } f(x) < 0 \end{cases}$$

   Prove that f_+ is also integrable on A.

2. Mark True or False. **If true, give a proof. If false, give a counterexample.**
   (a) Let $S \subset \mathbb{R}^n$. If $bd(S)$ is rectifiable then S is rectifiable.
   (b) Let $A, B \subset \mathbb{R}^n$. Then $bd(A \cap B) = bd(A) \cap bd(B)$;
   (c) Let $A \subset \mathbb{R}^n$. Then $int(intA) = int(A)$
   (d) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If $A \subset \mathbb{R}^n$ is open then $f(A)$ is open.

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 \text{ if } (x,y) = (0,0) \\ \frac{2x^3+xy^2}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \end{cases}$$

   (a) Show that the partial derivatives $D_1 f(0,0), D_2 f(0,0)$ exist and compute them.
   (b) Is f differentiable at $(0,0)$? If yes, find $df_{(0,0)}$. If no, explain why not.
   Hint: use part a).

4. Let $F(x,y) = \int_x^y \sqrt{e^{tx} + 3y}\,dt$. Let $c = F(0,1)$.
   Show that near $(0,1)$ the level set $F(x,y) = c$ can be written as $y = g(x)$ for some differentiable function g and compute $g'(0)$.

5. Let η be an alternating k-tensor on a vector space V. Let $v_1, \dots v_k \in V$ be linearly dependent.
   Show that $\eta(v_1, \dots, v_k) = 0$.

6. Let $M^3 = \{(x,y,z) \in \mathbb{R}^3 |$ such that $1 \leq x^2+y^2+z^2 \leq 4\}$ with the orientation induced from $\mathbb{R}^3$.

Let $p = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Find a positive basis of $T_p\partial M$ with respect to the orientation of ∂M induced from M.

7. Let (X, d) be a metric space.
   (a) Let $p \in X$ be any point. Prove that $\{p\}$ is a closed subset of X.
   (b) Let $C \subset X$ be compact. Prove that C is closed. **You are not allowed to use any theorems about compact sets in the proof.**
8. Let $U = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 > 1\}$. Let $f(x, y) = \frac{y}{x^2+y^2}$.
   Determine if $\int_U^{ext} f$ exists and if it does compute it.
9. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(s, t) = (st, s + 2t)$ and let $\omega = \sin x dy$. Compute $f^*(d\omega)$ and $d(f^*\omega)$ and verify that they are equal.
10. Let $a, b > 0$ and Let $M \subset \mathbb{R}^2$ be the ellipse $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ with the orientation induced by the standard orientation on $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$.
    Find $\int_M (\cos x) y dx + (x + \sin(x)) dy$.