**MAT 257Y** **Practice Final 1**

(1) Let $A \subset R^n$ be a rectangle and let $f\colon A \to \mathbb{R}$ be bounded. Let $P_1, P_2$ be two partitions of $A$ . Prove that $L(f, P_1) \leq U(f, P_2)$.

(2) Let $T\colon \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a 2-tensor on $\mathbb{R}^n$. Show that $T$ is differentiable at $(0,0)$ and compute $df(0,0)$.

(3) Let $\omega = \frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$ be a 2-form on $\mathbb{R}^3 \backslash (0,0,0)$.

Verify that $\omega$ is closed.

*Hint:* One way to simplify the computation is to write $\omega = f \cdot \tilde{\omega}$ where $f = \frac{1}{(x^2+y^2+z^2)^{3/2}}$ and $\tilde{\omega} = xdy \wedge dz + ydz \wedge dx + zdx$.

(4) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (e^{2y}, 2x+y))$ and let $\omega = x^2 y dx + y dy$.

Compute $f^*(d\omega)$ and $d(f^*(\omega))$ and verify that they are equal.

(5) Determine if $\int^{ext}_{0<x^2+y^2<1} \ln(x^2+y^2)$ exists and if it does compute it.

(6) Let $U, V$ be open in $\mathbb{R}^n$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous nonnegative function such that $\int^{ext}_U f$ and $\int^{ext}_V f$ exist.

Prove that $\int^{ext}_{U\cup V} f$ exists.

*Hint:* use compact exhaustions of $U$ and $V$ to construct a compact exhaustion of $U \cup V$.

(7) Let $F(x) = \int_{e^x}^{x^2} f(tx)dt$ where $f\colon \mathbb{R} \to \mathbb{R}$ is $C^1$.

Show that $F(x)$ is $C^1$ and find the formula for $F'(x)$.

(8) Let $x(t_1,t_2) = t_1 \cos t_2, y(t_1,t_2) = t_1^2 + e^{t_1 t_2}$. Let $f(x,y)$ be a differentiable function $f\colon \mathbb{R}^2 \to \mathbb{R}$. Let $g(t_1,t_2) = f(x(t_1,t_2), y(t_1,t_2))$. Express $\frac{\partial g}{\partial t_1}(1,0)$ and $\frac{\partial g}{\partial t_2}(1,0)$ in terms of partial derivatives of $f$.

(9) Mark true or false. Justify your answer.

Let $A, B$ be any subsets of $\mathbb{R}^n$. Then

(a) $bd(A) \subset Lim(A)$
(b) $Lim(A) \subset A$
(c) $bd(A \cap B) \subset bd(A) \cap bd(B)$.

(10) let $M^2 \subset \mathbb{R}^3$ be the torus of revolution obtained by rotating the circle $(x-2)^2 + z^2 = 1$ in the $xz$ plane around the $yz$ axis. Consider the orientation on $M$ induced by the normal field $N$ where $N(3,0,0) = (1,0,0)$.

Find $\int_M x dy \wedge dz$.

(11) Let $M \subset \mathbb{R}^n$ be an oriented manifold.

Prove that $\mathrm{vol}(M) = \int_M dV$ is positive.