MAT 257Y  Solutions to Practice Term Test 1

(1) Find the partial derivatives of the following functions
(a) \( f(x, y, z) = \sin(x \sin(y \sin z)) \)
(b) \( f(x, y, z) = x^y z^2 \)

Solution
(a) \( \frac{\partial f}{\partial x}(x, y, z) = (\cos(x \sin(y \sin z))) (\sin(y \sin z)) \)
\( \frac{\partial f}{\partial y}(x, y, z) = (\cos(x \sin(y \sin z))) (x \cos(y \sin z)) \sin z \)
\( \frac{\partial f}{\partial z}(x, y, z) = (\cos(x \sin(y \sin z))) (x \cos(y \sin z)) y \cos z \)
(b) First, we rewrite \( f(x, y, z) \) as \( f(x, y, z) = (e^{\ln x})^y z^2 = e^{(\ln x)y z^2} \)
\( \frac{\partial f}{\partial x}(x, y, z) = (e^{(\ln x)y z^2}) \frac{y z^2}{x} = (x y z^2) \frac{y z^2}{x} \)
\( \frac{\partial f}{\partial y}(x, y, z) = (e^{(\ln x)y z^2}) (\ln x) z^2 = (x y z^2) (\ln x) z^2 \)
\( \frac{\partial f}{\partial z}(x, y, z) = (e^{(\ln x)y z^2}) (\ln x) y(2z) = (x y z^2) (\ln x) y(2z) \)

(2) give an example of a nonempty set \( A \) such that the set of limit points of \( A \) is the same as the set of boundary points of \( A \).

Solution
Let \( A = S^1 = \{x \in \mathbb{R}^2 | |x| = 1 \} \). Then \( A = \text{Lim} A = \text{br}(A) \).

(3) Let \( A, B \subset \mathbb{R}^n \) be compact.
Prove that the set \( A + B = \{a + b | a \in A, b \in B \} \) is compact.

Solution
Consider the map \( f: \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) given by \( f(x, y) = x + y \). This map is linear and hence continuous. By construction, \( A + B = f(A \times B) \). \( A \times B \) is compact as a product of two compact sets and hence \( A + B = f(A \times B) \) is also compact as an image of a compact set under a continuous map.
(4) show that the intersection of arbitrary collection of closed sets is closed.

Solution

Let \( \{A_\alpha\}_{\alpha \in I} \) be a collection of closed sets in \( \mathbb{R}^n \).
Let \( U_\alpha = \mathbb{R}^n \setminus A_\alpha \). Then \( U_\alpha \) is open.

We have

\[
\mathbb{R}^n \setminus \bigcap_\alpha A_\alpha = \bigcup_\alpha U_\alpha
\]

is open as a union of open sets. Hence \( \bigcap_\alpha A_\alpha \) is closed.

(5) show that \( f: \mathbb{R}^n \to \mathbb{R}^m \) is continuous if and only if \( f^{-1}(A) \) is closed for any closed \( A \subset \mathbb{R}^m \).

Solution

Let \( f \) be continuous.

Suppose \( A \subset \mathbb{R}^m \) is closed. Then \( \mathbb{R}^m \setminus A \) is open.
By continuity of \( f \) this implies that \( f^{-1}(\mathbb{R}^m \setminus A) \) is open. It’s easy to see that \( f^{-1}(\mathbb{R}^m \setminus A) = \mathbb{R}^n \setminus f^{-1}(A) \).

hence \( f^{-1}(A) \) is closed. The reverse implication is proved similarly.

(6) Let \( \mathbb{R}^{n^2} \) be the space of all \( n \times n \) matrices. Consider the map \( f: \mathbb{R}^{n^2} \to \mathbb{R}^{n^2} \) given by the formula

\[
f(A) = A \cdot A^T.
\]

Here \( A^T \) means the transpose of \( A \).

Show that \( f \) is differentiable everywhere and compute \( df(A) \).

\textit{Hint:} use that \( df(A)(X) = D_X f(A) \).

Solution

First observe that \( f \) is clearly differentiable because its components are polynomials in entries of \( A \). to compute \( df(A) \) we use the fact that for differentiable maps \( df(A)(X) = D_X f(A) \).

By definition
\[ D_x f(A) = \lim_{t \to 0} \frac{f(A + tX) - f(A)}{t} = \lim_{t \to 0} \frac{(A + tX)(A + tX)^T - AA^T}{t} \]

\[ = \lim_{t \to 0} \frac{AA^T + tX A^T + tA X^T + t^2XX^T - AA^T}{t} = X A^T + AX^T \]

Therefore \( df(A)(X) = X A^T + AX^T \).

(7) Let \( f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) be given by the formula
\[
 f_1(x, y) = x + y + y^3 + 1, \quad f_2(x, y) = x e^y + 2
\]
Show that there exists an open set \( U \) containing \((0, 0)\) such that \( f : U \to f(U) \) is a bijection and \( f^{-1} \) is differentiable on \( f(U) \) and compute \( df^{-1}(1, 2) \).

**Solution**

Clearly \( f \) is differentiable everywhere. we compute
\[
 \frac{\partial f_1}{\partial x}(x, y) = 1, \quad \frac{\partial f_1}{\partial y}(x, y) = 1 + 3y^2, \quad \frac{\partial f_2}{\partial x}(x, y) = e^y, \quad \frac{\partial f_2}{\partial y}(x, y) = xe^y
\]
Therefore

\[
 [df(0, 0)] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\]

This matrix has \( \det = -1 \neq 0 \). \( f(0, 0) = (1, 2) \). hence, by the inverse function theorem, there exists an open set \( U \) containing \((0, 0)\) such that \( f : U \to f(U) \) is a bijection and \( f^{-1} \) is differentiable on \( f(U) \) and

\[ [df^{-1}(1, 2)] = [df(0, 0)]^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \]

(8) Let \( f(x, y) = x^y \) be defined on \( U = \{(x, y) | x > 0\} \).
Verify that
\[
 \frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)
\]

**Solution**
First we rewrite \( f(x, y) = e^{(\ln x)y} \). we compute 
\[
\frac{\partial f}{\partial x}(x, y) = e^{(\ln x)y} \frac{y}{x}, \quad \frac{\partial f}{\partial y}(x, y) = e^{(\ln x)y} \ln x. \]
Hence 
\[
\frac{\partial^2 f}{\partial x \partial y}(x, y) = e^{(\ln x)y} \frac{y^2}{x} \ln x + e^{(\ln x)y} \frac{1}{x} \quad \text{and} \quad 
\frac{\partial^2 f}{\partial y \partial x}(x, y) = e^{(\ln x)y} \ln x \frac{y}{x} + e^{(\ln x)y} \frac{1}{x}. \]
Thus 
\[
\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) \]