(1) (15 pts) Give the definitions of the following notions.
(a) an open set in $\mathbb{R}^n$;
(b) a boundary point of a set $A \subset \mathbb{R}^n$;
(c) a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ differentiable at a point $p$;
(d) a directional derivative of a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ at a point $p$.

Solution

(a) A set $U \subset \mathbb{R}^n$ is called open if for every $p = (p_1, p_2, \ldots, p_n) \in U$ there exists $\epsilon > 0$ such that the rectangle $I = (p_1 - \epsilon, p_1 + \epsilon) \times (p_1 - \epsilon, p_2 + \epsilon) \times \ldots \times (p_n - \epsilon, p_n + \epsilon)$ is contained in $U$.
(b) a point $p$ is called a boundary point of $A$ if for any $\epsilon > 0$ there exist $a \in B(p, \epsilon) \cap A$ and $b \in B(p, \epsilon) \cap A^c$
(c) a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at a point $p$ if there exists a linear map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that
\[
\lim_{h \to 0} \frac{f(p + h) - f(p) - T(h)}{|h|} = 0
\]
(d) Let $X \in \mathbb{R}^n$ and let $g(t) = f(p + tX)$. Then $D_X f(p) = g'(0)$ if it exists is called the directional derivative of $f$ at $p$ in the direction $X$.

(2) (15 pts) Find the partial derivatives of the following functions
(a) $f(x, y) = \int_x^y g(t) dt$
Hint: put $F(x, y) = \int_x^y g(t) dt$ and express $f$ as a composition.
(b) $f(x, y) = \ln((\sin(x + y^2))^{\cos 2x})$
Solution

(a) put $F(x, y) = \int_x^y g(t)dt$. By the fundamental theorem of calculus we have

$$\frac{\partial F}{\partial x}(x, y)(x, y) = -g(x) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y)(x, y) = g(y)$$

We also have that $f(x, y) = F(x, F(x, y))$. Therefore, by the chain rule we have

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial F}{\partial x}(x, F(x, y)) \frac{\partial x}{\partial x}(x, y) + \frac{\partial F}{\partial y}(x, F(x, y)) \frac{\partial F}{\partial x}(x, y) =$$

$$= -g(x) \cdot 1 + g(F(x, y)) \cdot (-g(x)) = -g(x) - g\left(\int_x^y g(t)dt\right)g(x)$$

Similarly,

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial F}{\partial x}(x, F(x, y)) \frac{\partial x}{\partial y}(x, y) + \frac{\partial F}{\partial y}(x, F(x, y)) \frac{\partial F}{\partial y}(x, y) =$$

$$= -g(x) \cdot 0 + g(F(x, y))g(y) = g\left(\int_x^y g(t)dt\right)g(y)$$

(b) First we simplify $f(x, y) = \ln((\sin(x+y^2))^{\cos 2x}) = (\cos 2x) \ln(\sin(x+y^2))$

Then we compute

$$\frac{\partial f}{\partial x}(x, y) = -2(\sin 2x) \ln(\sin(x+y^2)) + (\cos 2x)\frac{1}{\sin(x+y^2)} \cos(x+y^2)$$

$$\frac{\partial f}{\partial y}(x, y) = (\cos 2x)\frac{1}{\sin(x+y^2)} \cos(x+y^2)(2y)$$

(3) (20 pts) Let $f: R^2 \to R$ be given by the formula

$$f(x, y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Show that $f(x, y)$ is continuous at $(0, 0)$.

(b) Show that $f$ has partial derivatives at $(0, 0)$. 
(c) Does $f$ has directional derivatives at $(0, 0)$ in all directions?

(d) Show that $f$ is not differentiable at $(0, 0)$.

**Solution**

(a) we rewrite $f(x, y) = y\frac{x^2}{x^2 + y^2}$. Clearly $|\frac{x^2}{x^2 + y^2}| \leq 1$ and $\lim_{(x, y) \to (0, 0)} y = 0$.

Therefore $\lim_{(x, y) \to (0, 0)} f(x, y) = 0$.

(b) by definition of $f$ we see that $f(x, 0) = 0$ and $f(0, y) = 0$. therefore $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$.

(c) Does $f$ has directional derivatives at $(0, 0)$ in all directions?

For a direction $v = (v_1, v_2) \neq 0$ we compute

$$f((0, 0) + tv) = f(tv_1, tv_2) = \frac{t^3 v_1^2 v_2}{t^2 (v_1^2 + v_2^2)} = t \frac{v_1^2 v_2}{v_1^2 + v_2^2}.$$

This function is differentiable in $t$ with $f((0, 0) + tv)'(0) = \frac{v_1^2 v_2}{v_1^2 + v_2^2}$. By definition this means that $D_v f(0, 0)$ exists and is equal to $\frac{v_1^2 v_2}{v_1^2 + v_2^2}$.

**Answer:** Yes.

(d) Suppose $f$ is differentiable at $(0, 0)$. then the matrix $[df(0, 0)] = [\frac{\partial f}{\partial x}(0, 0), \frac{\partial f}{\partial y}(0, 0)] = [0, 0]$. By definition of differentiability this would mean that

$$\lim_{h \to 0} \frac{f(h) - f(0) - 0}{|h|} = 0$$

However along the line $(t, t)$ we have

$$\lim_{t \to 0} \frac{f(t, t) - f(0, 0)}{|t|} = \lim_{t \to 0} \frac{t^3}{2t^2} \neq 0$$

This is a contradiction which means that $f$ is not differentiable at $(0, 0)$.

(4) (10 pts) Show that a compact subset of $R^n$ is bounded.

**Solution**

Let $C \subset R^n$ be compact. Let $U_n = B(0, n)$ where $n = 1, 2, 3, \ldots$. Then $U_n$ is open and $\cup_n U_n = R^n$. 
Hence $C \subset \bigcup_n U_n$. By definition of compactness we can choose a finite subcover $U_{n_1}, \ldots, U_{n_k}$ still covering $C$. Let $m = \max_k n_k$. Then $C \subset U_m$ and hence it is bounded.

(5) (10 pts) let $f(x, y) = x^2 + 5y^2 - 4xy - 2y$. Find all possible points of minimum of $f(x, y)$.

Solution

$f$ is clearly differentiable everywhere. Its minimum can occur only at points where both partial derivatives vanish. we compute

$$\frac{\partial f}{\partial x}(x, y) = 2x - 4y \quad \frac{\partial f}{\partial y}(x, y) = 10y - 4x - 2$$

we solve

$$\begin{cases} 2x - 4y = 0 \\ 10y - 4x - 2 = 0 \end{cases} \begin{cases} x = 2 \\ y = 1 \end{cases}$$

Thus the only possible point of minimum is (2,1).

(6) (15 pts) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous.

Are the following statements true or false? Prove if true and give counterexamples if false.

(a) If $A \subset \mathbb{R}^n$ is closed and bounded then $f(A)$ is closed and bounded.

(b) If $A \subset \mathbb{R}^n$ is closed then $f(A)$ is closed.

(c) If $A \subset \mathbb{R}^n$ is bounded then $f(A)$ is bounded.

Solution

(a) True

If $A \subset \mathbb{R}^n$ is closed and bounded if and only if it’s compact and and image of a compact set under a continuous map is compact.

(b) False. let $f(x) = \arctan x$. Then $f([0, \infty)) = [0, \pi/2)$ is not closed.

(c) True

If $A$ is bounded it’s contained in a closed ball $B = \bar{B}(0, R) = \{x \in \mathbb{R}^n \text{ such that } |x| \leq R\}$ for some $R > 0$. Then $f(A) \subset f(B)$ but $B$ is compact.
hence $f(B)$ is also compact and in particular it is bounded.

(7) (15 pts) Let $GL(n, R)$ be the set of all $n \times n$ invertible matrices.

Show that $GL(n, R)$ is open in $R^{n^2}$.

**Solution**

Let $f: R^{n^2} \to R$ be given by $f(A) = \det(A)$. then $f$ is a polynomial in coordinate entries and hence is continuous. We know that $A$ is invertible if and only if $\det(A) \neq 0$.

Therefore $GL(n, R) = f^{-1}(((-\infty, 0) \cup (0, \infty))$ and hence is open as the preimage of an open set.