(1) (15 pts) Give the following definitions
(a) A k-tensor on a vector space $V$.

(b) Cross product on $\mathbb{R}^n$.

(c) A partition of unity on an open set $U$ subordinate to an open cover $U_\alpha$.

Solution

(a) A k-tensor on a vector space $V$ is a map $T: V \times \ldots \times V \to \mathbb{R}$ which is linear in every variable, i.e.

$$T(v_1, \ldots, v_{i-1}, av'_i + bv''_i, v_{i+1}, \ldots, v_k) = aT(v_1, \ldots, v_{i-1}, v'_i, v_{i+1}, \ldots, v_k) + bT(v_1, \ldots, v_{i-1}, v''_i, v_{i+1}, \ldots, v_k)$$

for any $i = 1, \ldots, k$.

(b) Cross product on $\mathbb{R}^n$. For $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ we define $v_1 \times \ldots \times v_n$ as the unique vector $v \in \mathbb{R}^n$ such that $\langle v, w \rangle = \det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{pmatrix}$ for any $w \in \mathbb{R}^n$.

(c) A partition of unity on an open set $U$ subordinate to an open cover $U_\alpha$ is a sequence of functions $\phi_i: \mathbb{R}^n \to \mathbb{R}$ with the following properties

(i) $\phi_i$ is $C^\infty$ for any $i$.

(ii) For any $i$ there exists $\alpha$ such that $\text{supp}(\phi_i) \subset U_\alpha$.

(iii) $\phi_i \geq 0$ for any $i$.

(iv) For any $p \in U$ there exists $\epsilon > 0$ such that $B(p, \epsilon)$ intersects only finitely many $\text{supp}(\phi_i)$.

(v) $\sum_{i=1}^\infty \phi_i = 1$ on $U$.

(2) (10 pts) Let $U \subset \mathbb{R}^n$ be open. Let $f, g: U \to \mathbb{R}$ be continuous and $|f| \leq g$. Suppose $\int_U^\text{ext} g$ exists.

Prove that $\int_U^\text{ext} f$ also exists.

Solution

Let $\phi_i$ be a partition of unity on $U$. Then by definition of extended integral,

$$\sum_{i=1}^\infty \int_U |g| \phi_i < \infty$$

Therefore

$$\sum_{i=1}^\infty \int_U |f| \phi_i \leq \sum_{i=1}^\infty \int_U |g| \phi_i < \infty$$

and hence $\int_U^\text{ext} f$ exists by the definition.

(3) (15 pts) Let $e_1, e_2, e_3, e_4$ be a basis of a 4-dimensional space $V$.

Let $\omega = \text{Alt}(e_1^* \otimes e_2^* + e_3^* \otimes e_4^*)$ and $\eta = 2e_2^* + e_3^*$.

Find $\omega \wedge \eta(e_2, e_3, e_4)$. 

Solution

We have $\omega = \frac{11}{24} (e_1^* \wedge e_2^* + e_3^* \wedge e_4^*)$. Hence $\omega \wedge \eta = \frac{1}{2} (e_1^* \wedge e_2^* + e_3^* \wedge e_4^*) \wedge (2e_2^* + e_3^*) = \frac{1}{2} (e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + 2e_2^* + e_3^* + e_4^* \wedge e_3^*) = \frac{1}{2} e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_3^* \wedge e_3^* = \frac{1}{2} e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_3^* \wedge e_3^*.$

Therefore $\omega \wedge \eta(e_2, e_3, e_4) = \frac{1}{2} e_1^* \wedge e_2^* \wedge e_3^*(e_2, e_3, e_4) + e_2^* \wedge e_3^* \wedge e_4^*(e_2, e_3, e_4) = 0 + 1 = 1.$

(4) (15 pts) Let $U = \{(x, y) \in \mathbb{R}^2\}$ such that $0 < x^2 + y^2/4 < 1, y > 0, -y/2 < x < y/2$.

Compute $\int_U y$.

Hint: use the appropriate change of variables.

Solution

First we make the change of variables $y = 2v, x = u$, i.e. $f(u, v) = u, 2v$. Then $\det[df] = 2$ and hence by the change of variables formula $\int_U y = \int_V 2 \cdot 2v = \int_V 4v$ where $V = \{(u, v) \in \mathbb{R}^2\}$ such that $0 < u^2 + v^2 < 1, y > 0, -v < u < v$. making another change of coordinates $u = r \cos \theta, v = r \sin \theta$ or $g(r, \theta) = (r \cos \theta, r \sin \theta)$ we see that $\det[df] = r$ and $\int_V 4v = \int_W 4r \cdot r \sin \theta = \int_W 4r^2 \sin \theta$ where $W = \{0 < r < 1, \pi/4 < \theta < 3\pi/4\}$. By Fubini’s theorem we compute

$$\int_W 4r^2 \sin \theta = \int_{\pi/4}^{3\pi/4} (\int_0^1 4r^2 \sin \theta dr) d\theta = \int_{\pi/4}^{3\pi/4} 4 \sin \theta d\theta = \frac{4}{3} \cos \theta \bigg|_{\pi/4}^{3\pi/4} = \frac{4\sqrt{2}}{3}$$

(5) (15 pts) Let $v_1, \ldots, v_{n-1}$ be vectors in $\mathbb{R}^n$.

Show that $v_1 \times v_2 \times \ldots \times v_{n-1} = 0$ if $v_1, \ldots, v_{n-1}$ are linearly dependent.

Solution

Let $v = v_1 \times v_2 \times \ldots \times v_{n-1}$. By definition,

$$\langle v, w \rangle = \det \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_{n-1} \\ w & & \end{pmatrix} = 0$$

because the rows of this matrix are linearly dependent.

Thus $\langle v, w \rangle = 0$ for any $w \in \mathbb{R}^n$. In particular for $w = v$ we get $0 = \langle v, v \rangle = |v|^2$ which means that $v = 0$.

(6) (15 pts) Let $e_1, e_2$ be a basis of a vector space $V$ of dimension 2. Let $T \in \mathcal{T}^2(V)$ be given by $e_1^* \otimes e_1^* + e_2^* \otimes e_2^*$. Prove that $T$ can not be written as $S \otimes U$ with $S, U \in \mathcal{T}^1(V)$.

Solution

Suppose $e_1^* \otimes e_1^* + e_2^* \otimes e_2^* = S \otimes U$ for some $S = ae_1^* + be_2^*, U = ce_1^* + de_2^*$. Then $S \otimes U = (ae_1^* + be_2^*) \otimes (ce_1^* + de_2^*) = ace_1^* \otimes e_1^* + bce_2^* \otimes e_1^* + ade_2^* \otimes e_1^* + bde_2^* \otimes e_2^* = e_1^* \otimes e_1^* + e_2^* \otimes e_2^*$. This means that $ac = 1, bc = 0, ad = 0, bd = 1$. It’s easy to see that this system has no solutions. For example, $abcd = (bc)(ad) = 0 \cdot 0 = 0$ and on the other hand, $abcd = (ac)(bd) = 1 \cdot 1 = 1$. This is a contradiction.
(7) (15 pts) Let \( \omega = e^{\sqrt{x^2+y^2}} dx + \frac{\cos(\sqrt{x^2+y^2})}{1+e^{x^2}y^2} dy \) be a 1-form on \( \mathbb{R}^2 \setminus \{0\} \).

Let \( f: \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{0\} \) be given by \( f(u,v) = (\cos(2u+v), \sin(2u+v)) \).

Compute \( f^*(d\omega) \).

Solution

First recall that \( f^*(d\omega) = df^*(\omega) \). We compute \( f^*(\omega) = e^1 d \cos(2u+v) + \frac{\cos(\pi/2)}{1+e^{\cos^2(2u+v)} \sin(2u+v)} d \sin(2u+v) = ed \sin(2u+v) \).

Therefore, \( f^*(d\omega) = df^*(\omega) = d(ed \sin(2u+v)) = e(d \circ d) \sin(2u+v) = 0 \).