Math 247S  Practice Term Test  Winter 2012

Rules: No books, no notes. You have 50 minutes to complete the test. Note: the actual test is shorter than the practice test.

(1) Let $V$ be a complex vector space with two inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$.
   Suppose $\langle v, v \rangle_1 = \langle v, v \rangle_2$ for any $v \in V$.
   Prove that $\langle u, v \rangle_1 = \langle u, v \rangle_2$ for any $u, v \in V$.

(2) For which real values of $a, b, c$ is the matrix the matrix
   \[
   A = \begin{pmatrix}
   a & b & -c \\
   -b & a & 0 \\
   ac & 0 & 1
   \end{pmatrix}
   \]
   invertible? Find the formula for $A^{-1}$ for those values of $(a, b, c)$ for which $A^{-1}$ exists.

(3) Let $V = \mathbb{R}^\infty$ i.e., $V$ is the space of infinite sequences of real numbers $a = (a_1, a_2, \ldots)$ where all but finitely many $a_i$ are zero for every $a \in V$.
   Let $f: V \to \mathbb{R}$ be given by $f(a) = \sum_{i=1}^{\infty} a_i$.
   Is it true that there exists $v \in V$ such that $f(a) = \langle a, v \rangle$ for all $a \in V$? If yes, find $v$. If not, explain why not.

(4) Mark true or false. If true, give an argument why, if false, give a counterexample.
   a) Let $V$ be a finite dimensional complex vector space with inner product and let $f: V \to \mathbb{C}$ be a linear map. Then there exists $v \in V$ such that $f(u) = \langle v, u \rangle$ for every $u \in V$.
   b) Similar matrices have equal determinants.
   c) Every orthonormal set of vectors is linearly independent;
   d) Let $V$ be a finite-dimensional vector space with inner product. Then for any $S \subset V$ we have $(S^\perp)^\perp = S$
(5) Let $W = \{(x, y, z) \in \mathbb{R}^3 \text{ such that } x + 2y - z = 0\}$.
   a) Find an orthogonal basis of $W$;
   b) Find the orthogonal projection of $(1, 1, 2)$ to $W$.

(6) Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^n$. Let $G$ be an $n \times n$ matrix with $G_{ij} = \langle v_i, v_j \rangle$. Let $P$ be the parallelepiped spanned by $v_1, \ldots, v_n$.

   Prove that $\text{vol}(P) = \sqrt{\det G}$.

   **Hint:** Look at the matrix $A$ with rows $v_1, \ldots, v_n$. 