

Solutions to Practice Final 2

1. Using induction prove that

$$1^2 + 3^2 + \dots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

Solution

First we verify the base of induction. When $n = 0$ LHS = $1^2 = 1$ and RHS = $\frac{1 \cdot 1 \cdot 3}{3} = 1$.

Induction step. Assume the formula is true for $n \geq 0$ and we need to verify it for $n+1$. Then we have

$$\begin{aligned} 1^2 + 3^2 + \dots + (2n+1)^2 + (2n+3)^2 &= \frac{(n+1)(2n+1)(2n+3)}{3} + (2n+3)^2 = \\ &= \frac{(n+1)(2n+1)(2n+3) + 3(2n+3)^2}{3} = \frac{(2n+3)(2n^2 + 3n + 1 + 3(2n+3))}{3} \\ &= \frac{(2n+3)(2n^2 + 9n + 10)}{3} = \frac{(2n+3)(2n+5)(n+2)}{3} \end{aligned}$$

This completes the induction step and proves the formula for all $n \geq 0$.

2. Let a, b, c be natural numbers.

- (a) Show that the equation $ax + by = c$ has a solution if and only if $(a, b) | c$.
(b) Find all integer solutions of $6x + 15y = 9$.

Solution

- (a) Suppose $ax + by = c$ for some integer x and y . If $d|a$ and $d|b$ then obviously, $d|ax + by = c$. In particular, if $(a, b) | c$.

Conversely, suppose $(a, b) | c$ so that $c = d \cdot (a, b)$. Then $ax + by = (a, b)$ has an integer solution by a result from class. Multiplying both sides by d we get $a(xd) + b(yd) = (a, b) \cdot d = c$.

- (b) First, divide both sides by 3. we get $2x + 5y = 3$. We have $(2, 5) = 1$ and we can find integer solution of $2x + 5y = 1$ using either Euclidean algorithm or just by trying a few small numbers we get

$2 \cdot (-2) + 5 \cdot 1 = 1$. Multiplying by 3 we get $2 \cdot (-6) + 5 \cdot (3) = 3$ so $x_0 = -6, y_0 = 3$ is a solution of $2x + 5y = 3$.

It's easy to see that $x = -6 - 5k, y = 3 + 2k$ is a solution of $2x + 5y = 3$ for any k . We claim that any integer solution of $2x + 5y = 3$ has this form.

Suppose $2x + 5y = 3$. we also have $2 \cdot (-6) + 5 \cdot (3) = 3$. Subtracting these equations we get $2(-6 - x) + 5(3 - y) = 0$ or $2(-6 - x) = 5(y - 3)$. This implies that $2|(y - 3)$ so that $y - 3 = 2k$ or $y = 3 + 2k$. This gives $2(-6 - x) = 5(y - 3) = 6k, -6 - x = 3k, x = -6 - 3k$.

Thus the general solution is $x = -6 - 5k, y = 3 + 2k$ where k is any integer.

3. Find the last digit of the sum

$$2(1 + 3 + 3^2 + 3^3 + \dots + 3^{309})$$

Solution

First, we compute

$$2(1 + 3 + 3^2 + 3^3 + \dots + 3^{309}) = 2 \cdot \frac{3^{310} - 1}{3 - 1} = 3^{310} - 1.$$

We have $\phi(10) = \phi(2 \cdot 5) = 1 \cdot 4 = 4$. By Euler's theorem this implies that $3^4 \equiv 1 \pmod{10}$. Of course, this can also be seen directly as $3^4 = 81$.

Therefore $3^{4k} \equiv 1 \pmod{10}$. We have $310 = 308 + 2$ and $4|308$. Therefore $3^{310} \equiv 3^2 \pmod{10}$. This means that the last digit of 3^{310} is 9 and hence the last digit of $3^{310} - 1$ is 8.

4. Let S be infinite and $A \subset S$ be finite. Prove that $|S| = |S \setminus A|$.

Solution

Let $A = \{s_1, \dots, s_n\}$. Since S is infinite the set $S \setminus A$ is non empty. Pick any $s_{n+1} \in S \setminus A = S \setminus \{s_1, \dots, s_n\}$. Next, since $S \setminus \{s_1, \dots, s_{n+1}\} \neq \emptyset$ we can choose $s_{n+2} \in S \setminus \{s_1, \dots, s_{n+1}\}$. Proceeding by induction we can construct $s_{m+1} \in S \setminus \{s_1, \dots, s_m\}$ for any $m \geq n$.

Now define $f: S \rightarrow S \setminus A$ by the formula $f(s_i) = s_{i+n}$ for any i and $f(x) = x$ if $x \in S \setminus \{s_1, s_2, \dots\}$. By construction, f is 1-1 and onto.

5. Let $S = [0, 1]$ and $T = [0, 2)$. Let $f: S \rightarrow T$ be given by $f(x) = x$ and $g: T \rightarrow S$ be given by $g(x) = x/2$.

(a) Find S_S, S_T, S_∞ ;

(b) give an explicit formula for a 1-1 and onto map $h: S \rightarrow T$ coming from f and g using the proof of the Schroeder-Berstein theorem.

Solution

(a) Note that $1 \notin g(T)$ and therefore $1 \in S_S$. Next, we see that $1/2 \in S_S$ also. Indeed, $1/2 = g(1)$ and $1 = f(1)$. So 1 is the last ancestor of $1/2$ and hence $1/2 \in S_S$. proceeding by induction we see that $\frac{1}{2^n} \in S_S$ for any $n \geq 0$.

Next observe that $(1/2, 1) \subset S_T$. Indeed, if $1/2 < x < 1$ then $x = g(2x)$ and $1 < 2x < 2$ so that $2x \notin f(S)$.

Proceeding by induction we claim that $(\frac{1}{2^{n+1}}, \frac{1}{2^n}) \in S_T$ for any $n \geq 0$. We just verified the base of induction.

Induction step. Suppose we know the statement of $n \geq 0$ and we need to prove it for $n + 1$. Let $\frac{1}{2^{n+2}} < x < \frac{1}{2^{n+1}}$ then $x = g(2x)$ and $\frac{1}{2^{n+1}} < 2x < \frac{1}{2^n}$. Also, $2x = f(2x)$. By induction assumption, $2x \in S_T$ and the last ancestor of x is the last ancestor of $2x$ so $x \in S_T$ also.

This concludes the induction step.

It's obvious that $0 \in S_\infty$. Therefore $S_\infty = \{0\}$, $S_S = \{1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots\}$ and $S_T = \{x \in [0, 1] \text{ such that } x \neq 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$.

- (b) By the proof of the Schroeder-Bernstein Theorem the following map $h: S \rightarrow T$ is 1-1 and onto.

$$h(x) = \begin{cases} x & \text{if } x = 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \\ 2x & \text{if } x \neq 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \end{cases}$$

6. Let $n = 2p$ where p is an odd prime. Find the remainder when $\phi(n)!$ is divided by n . Here $\phi(n)$ is the Euler function of n .

Solution

We have $\phi(n) = \phi(2p) = (2-1)(p-1) = p-1$. By Wilson's theorem $\phi(n)! = (p-1)! \equiv -1 \pmod{p} \equiv p-1 \pmod{p}$. This means that $p \mid (p-1)! - (p-1)$. Since p is odd $p-1$ is even and therefore $2 \mid (p-1)! - (p-1)$ also. Since $(2, p) = 1$ this implies that $2p \mid (p-1)! - (p-1)$ or, equivalently $(p-1)! \equiv p-1 \pmod{2p}$.

Answer: $p-1$.

7. Prove that $q_1\sqrt{3} + q_2\sqrt{5} \neq q'_1\sqrt{3} + q'_2\sqrt{5}$ for any rational q_1, q_2, q'_1, q'_2 unless $q_1 = q'_1, q_2 = q'_2$.

Solution

Suppose $q_1\sqrt{3} + q_2\sqrt{5} = q'_1\sqrt{3} + q'_2\sqrt{5}$. Then $(q_1 - q'_1)\sqrt{3} + (q_2 - q'_2)\sqrt{5} = 0$. Let $a = q_1 - q'_1, b = q_2 - q'_2$ are rational and $a\sqrt{3} + b\sqrt{5} = 0$. We want to show that $a = b = 0$. If $a \neq 0$ this gives $\sqrt{\frac{3}{5}} = -\frac{b}{a}$ which is rational. This is a contradiction since $\sqrt{\frac{3}{5}}$ is irrational. Hence $a = 0$. Since $a\sqrt{3} + b\sqrt{5} = 0$ this implies $b\sqrt{5} = 0, b = 0$.

8. Let a be a root of $x^5 - 6x^3 + 2x^2 + 5x - 1 = 0$. Construct a polynomial with integer coefficients which has a^2 as a root.

Hint: separate even and odd powers.

Solution

We can rewrite the equation as $x^5 - 6x^3 + 5x = 1 - 2x^2$, $x(x^4 - 6x^2 + 5) = 1 - 2x^2$. Squaring both sides we get $x^2(x^4 - 6x^2 + 5)^2 = (1 - 2x^2)^2$. Clearly, $y = x^2$ satisfies $y(y^2 - 6y + 5)^2 = (1 - 2y)^2$.

9. Find all complex roots of $x^6 + 7x^3 - 8 = 0$.

Reminder: Real numbers are also complex numbers.

Solution

Let $z = x^3$. Then z satisfies $z^2 + 7z - 8 = 0$. Solving this quadratic equation we get $z = 1, z = -8$. Thus we need to solve $x^3 = 1$ and $x^3 = -8$. Solving $x^3 = 1$ gives $x = 1, x = \cos(2\pi/3) + i \sin(2\pi/3) = \frac{-1+i\sqrt{3}}{2}, x = \cos(4\pi/3) + i \sin(4\pi/3) = \frac{-1-i\sqrt{3}}{2}$

Next we write -8 as $2^3(\cos \pi + i \sin \pi)$. Thus solving $x^3 = -8$ we get $x = 2(\cos(\pi/3) + i \sin(\pi/3)) = 1 + i\sqrt{3}, x = 2(\cos(\pi/3 + 2\pi/3) + i \sin(\pi/3 + 2\pi/3)) = 2(\cos \pi + i \sin \pi) = -2, x = 2(\cos(\pi/3 + 4\pi/3) + i \sin(\pi/3 + 4\pi/3)) = 2(\cos(5\pi/3) + i \sin(5\pi/3)) = 1 - i\sqrt{3}$

10. Represent $\sin(5\theta)$ as a polynomial in $\sin(\theta)$.

Solution

We have $\cos(5\theta) + i \sin(5\theta) = (\cos \theta + i \sin \theta)^5 = (\cos \theta + i \sin \theta)^2(\cos \theta + i \sin \theta)^3$. We compute separately $(\cos \theta + i \sin \theta)^2 = (\cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta)$ and $(\cos \theta + i \sin \theta)^3 = (\cos \theta + i \sin \theta)^2(\cos \theta + i \sin \theta) = (\cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta)(\cos \theta + i \sin \theta) = (\cos^2 \theta - \sin^2 \theta) \cos \theta - 2 \sin^2 \theta \cos \theta + i(\cos^2 \theta - \sin^2 \theta) \sin \theta + 2i \sin \theta \cos^2 \theta = \cos^3 \theta - 3 \sin^2 \theta \cos \theta + i(3 \sin \theta \cos^2 \theta - \sin^3 \theta)$.

Combining these together we get $\cos(5\theta) + i \sin(5\theta) = (\cos \theta + i \sin \theta)^5 = (\cos \theta + i \sin \theta)^2(\cos \theta + i \sin \theta)^3 = (\cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta)(\cos^3 \theta - 3 \sin^2 \theta \cos \theta + i(3 \sin \theta \cos^2 \theta - \sin^3 \theta)) = (\cos^2 \theta - \sin^2 \theta)(\cos^3 \theta - 3 \sin^2 \theta \cos \theta) - 2 \sin \theta \cos \theta(3 \sin \theta \cos^2 \theta - \sin^3 \theta) + i(\cos^2 \theta - \sin^2 \theta)(3 \sin \theta \cos^2 \theta - \sin^3 \theta) + 2i \sin \theta \cos \theta(\cos^3 \theta - 3 \sin^2 \theta \cos \theta)$.

Therefore, $\sin(5\theta) = (\cos^2 \theta - \sin^2 \theta)(3 \sin \theta \cos^2 \theta - \sin^3 \theta) + 2 \sin \theta \cos \theta(\cos^3 \theta - 3 \sin^2 \theta \cos \theta) = (1 - 2 \sin^2 \theta)(3 \sin \theta(1 - \sin^2 \theta) - \sin^3 \theta) + 2 \sin \theta \cos^4 \theta - 6 \sin^3 \theta \cos^2 \theta = (1 - 2 \sin^2 \theta)(3 \sin \theta(1 - \sin^2 \theta) - \sin^3 \theta) + 2 \sin \theta(1 - \sin^2 \theta)^2 - 6 \sin^3 \theta(1 - \sin^2 \theta)$.

11. Is $\frac{\sqrt[6]{5}-\sqrt{5}}{1+2\sqrt{7}}$ constructible? Justify your answer.

Solution

$\frac{\sqrt[6]{5}-\sqrt{5}}{1+2\sqrt{7}}$ is not constructible. We argue by contradiction. Assume $\frac{\sqrt[6]{5}-\sqrt{5}}{1+2\sqrt{7}}$ is constructible. Since $\sqrt{5}$ and $\sqrt{7}$ are constructible this implies that $\sqrt[6]{5}$ is constructible and hence $(\sqrt[6]{5})^2 = \sqrt[3]{5}$ is also constructible. $\sqrt[3]{5}$ is a root of $x^3 - 5 = 0$ which is a cubic equation with integer coefficients. By a theorem from class if it has a constructible root it must have a rational root as well. Let $\frac{m}{n}$ be a rational root where $(m, n) = 1$. Then $m|5$ and $n|1$ which means that $\frac{m}{n} = \pm 1, \pm 5$. Plugging these numbers into $x^3 - 5 = 0$ we see that none of them are roots.

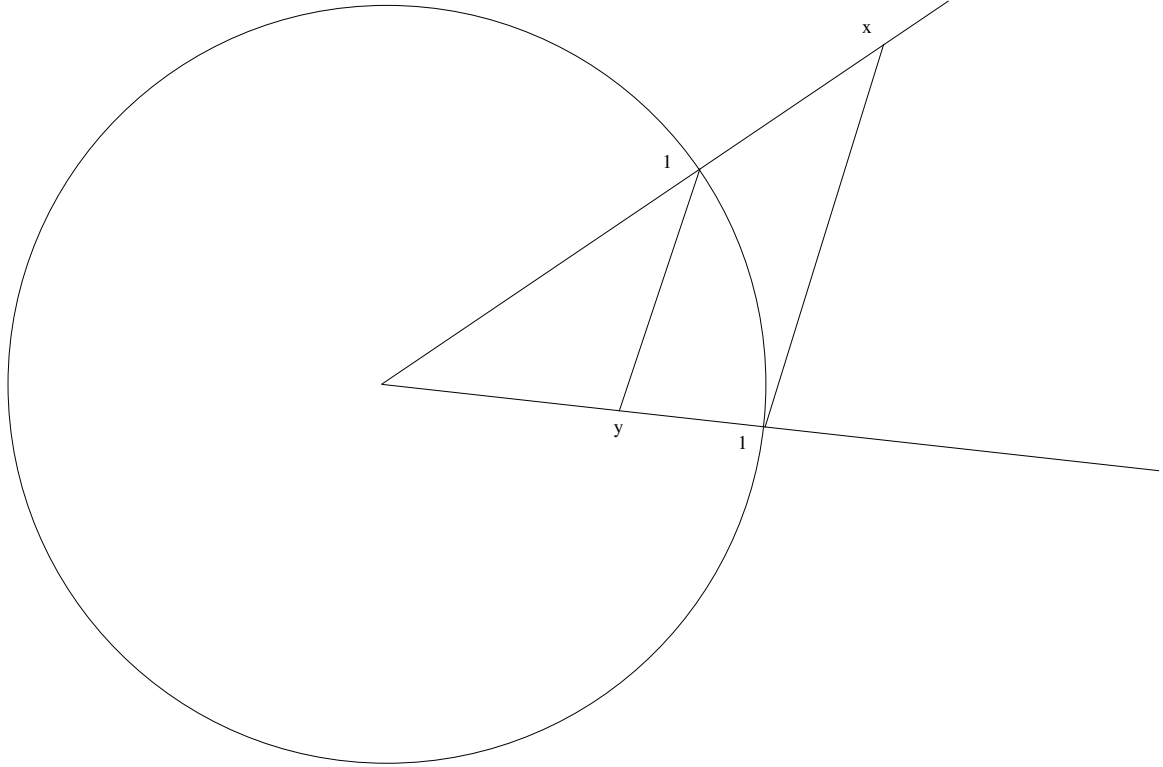
This is a contradiction and therefore $\frac{\sqrt[6]{5}-\sqrt{5}}{1+2\sqrt{7}}$ is not constructible.

12. For each of the following answer "true" or "false". Justify your answer.

- a) If $\frac{x}{y}$ is constructible then both x and y are constructible.
- b) If x is constructible then $\frac{1}{x}$ is constructible.
- c) There is an angle θ such that $\cos \theta$ is constructible but $\sin \theta$ is not constructible.
- d) $\sqrt[3]{\frac{10}{27}}$ is constructible.

Solution

- a) **False.** For example, take $x = y = \pi$. Then X and y are not constructible but $x/y = 1$ is constructible.
- b) **True.** See figure below. Draw segments of lengths 1 and x on one side of an angle and a segment of length 1 on the other side. Connect x and 1 on opposite sides by a line a draw a parallel line through 1 on the same side as x . It intersect the second side of the angle at distance y . Then from similar triangles we get $\frac{x}{1} = \frac{1}{y}$ or $y = \frac{1}{x}$



- c) **False.** If $\cos \theta$ is constructible then so is $1 - \cos^2 \theta$. Hence $\sin \theta = \pm \sqrt{1 - \cos^2 \theta}$ is also constructible since a square root of a constructible number is constructible.
- d) **False.** We argue by contradiction. Suppose $x = \sqrt[3]{\frac{10}{27}}$ is constructible. It satisfies the equation $27x^3 - 10 = (3x)^3 - 10 = 0$. If x is constructible then so is $y = 3x$ which satisfies the equation $y^3 - 10 = 0$. This is a cubic equation with integer coefficients. If it has a constructible root it must also have a rational one. We can write that rational root as $\frac{a}{b}$ where $(a, b) = 1$. Then $a|10$ and $b|1$ which means that $y = \frac{a}{b} = \pm 1 \pm 2 \pm 5$ or ± 10 . By plugging these numbers into $y^3 - 10 = 0$ we see that none of them are roots. This is a contradiction and therefore $\sqrt[3]{\frac{10}{27}}$ is not constructible.

13. Prove that the equation

$$(1 + x^{19})^3 + (1 + x^{19})^2 - 3 = 0$$

has no constructible solutions.

Solution

Suppose x is a constructible root. Then $y = x^{19} + 1$ is also constructible and it satisfies $y^3 + y^2 - 3 = 0$. This is a cubic equation with integer coefficients. If it has

a constructible root it must also have a rational one. We can write that rational root as $\frac{a}{b}$ where $(a, b) = 1$. Then $a|1$ and $b|1$ which means that $y = \frac{a}{b} = \pm 1$. But neither $y = 1$ nor $y = -1$ solve $y^3 + y^2 - 1 = 0$. This is a contradiction which means that $(1 + x^{19})^3 + (1 + x^{19})^2 - 3 = 0$ has no constructible solutions.