Practice Final 1

1. (a) What is $\phi(20^{100})$ where ϕ is Euler's ϕ-function?

   (b) Find an integer x such that $140x \equiv 133 \pmod{301}$.
   Hint: $\gcd(140, 301) = 7$.

2. (a) Prove, by mathematical induction, that $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ for every natural number n.

   (b) Prove that for p an odd prime (that is, p is a prime that is not equal to 2), $1^p + 2^p + 3^p + ... + (p-1)^p \equiv 0 \pmod{p}$.

3. Prove that for any odd integer a, a and a^{4n+1} have the same last digit for every natural number n.

4. Recall that a "perfect square" is a number of the form n^2 where n is a natural number. Show that 9120342526523 is not the sum of two perfect squares. Hint: Consider values modulo 4.

5. (a) Are there rational numbers a and b such that $\sqrt{3} = a + b\sqrt{2}$? Justify your answer.

   (b) Prove that $\frac{\sqrt{5}}{\sqrt{2}+\sqrt{11}}$ is irrational.

6. (a) What is the cardinality of the set of roots of polynomials with constructible coefficients? Justify your answer.

   (b) Let $\mathbb{N}$ denote the set of all natural numbers. What is the cardinality of the set of all functions from $\mathbb{N}$ to $\{1, 3, 5\}$? Justify your answer.
   Hint: You can use the fact that $|P(\mathbb{N})| = |\mathbb{R}|$.

7. Let θ be an angle between 0 and 90 degrees. Suppose that $\cos\theta = \frac{3}{4}$. Prove that $\frac{\theta}{3}$ is not a constructible angle.

8. For each of the following numbers, state whether or not it is constructible and justify your answer.

   (a) $\cos\theta$ where the angle $\frac{\theta}{3}$ is constructible

   (b) $\sqrt{7 + \sqrt{5}}$

   (c) $(0.029)^{1/3}$

   (d) $\tan 22.5°$

9. Find all complex solutions of the equation $z^6 + z^3 + 1 = 0$.

10. Let $p = 3, q = 11$ and $e = 7$. Let $N = 3 \cdot 11 = 33$. The receiver broadcasts the numbers $N = 33, e = 7$. The sender sends a secret message M to the receiver using RSA encryption. What is sent is the number $R = 6$.

    Decode to find the original message M.

11. Construct a polynomial with integer coefficients which has $\sqrt{2} + \sqrt{5}$ as a root.