Practice Final 2

1. Using induction prove that

$$1^2 + 3^2 + \ldots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

2. Let a, b, c be natural numbers.

   (a) Show that the equation $ax + by = c$ has a solution if and only if $(a,b)|c$.

   (b) Find all integer solutions of $6x + 15y = 9$.

3. Find the last digit of the sum

$$2(1 + 3 + 3^2 + 3^3 + \ldots + 3^{309})$$

4. Let S be infinite and $A \subset S$ be finite. Prove that $|S| = |S \backslash A|$.

5. Let $S = [0,1]$ and $T = [0,2)$. Let $f\colon S \to T$ be given by $f(x) = x$ and $g\colon T \to S$ be given by $g(x) = x/2$.

   (a) Find S_S, S_T, S_∞;

   (b) give an explicit formula for a 1-1 and onto map $h\colon S \to T$ coming from f and g using the proof of the Schroeder-Berenstein theorem.

6. Let $n = 2p$ where p is an odd prime. Find the remainder when $\phi(n)!$ is divided by n. Here $\phi(n)$ is the Euler function of n.

7. Prove that $q_1\sqrt{3} + q_2\sqrt{5} \neq q_1'\sqrt{3} + q_2'\sqrt{5}$ for any rational q_1, q_2, q_1', q_2' unless $q_1 = q_1', q_2 = q_2'$.

8. Let a be a root of $x^5 - 6x^3 + 2x^2 + 5x - 1 = 0$. Construct a polynomial with integer coefficients which has a^2 as a root.

   Hint: separate even and odd powers.

9. Find all complex roots of $x^6 + 7x^3 - 8 = 0$.

   Reminder: Real numbers are also complex numbers.

10. Represent $\sin(5\theta)$ as a polynomial in $\sin(\theta)$.

11. Is $\frac{\sqrt[6]{5} - \sqrt{5}}{1 + 2\sqrt{7}}$ constructible? Justify your answer.

12. For each of the following answer "true" or "false". Justify your answer.

    a) If $\frac{x}{y}$ is constructible then both x and y are constructible.

b) If x is constructible then $\frac{1}{x}$ is constructible.

c) There is an angle θ such that $\cos\theta$ is constructible but $\sin\theta$ is not constructible.

d) $\sqrt[3]{\frac{10}{27}}$ is constructible.

13. Prove that the equation

$$(1 + x^{19})^3 + (1 + x^{19})^2 - 3 = 0$$

has no constructible solutions.