(1) Prove by mathematical induction that \( n^3 + 5n \) is divisible by 6 for any natural \( n \).

**Solution**

We first check that the statement is true for \( n = 1 \). We have \( 1^3 + 5 = 6 \) is divisible by 6.

Suppose the statement is true for \( n \geq 1 \). Let’s show that it’s also true for \( n + 1 \).

We have \((n + 1)^3 + 5(n + 1) = n^3 + 3n^2 + 3n + 1 + 5n + 5 = (n^3 + 5n) + 3n^2 + 3n + 6\). Clearly \( 3n^2 + 3n + 6 \equiv 0 \pmod{3} \). Also, either \( n \) or \( n + 1 \) is even so that \( n(n + 1) \) is even and hence is divisible by 2. Therefore \( 3n^2 + 3n + 6 \equiv 0 \pmod{2} \). Taken together the above means that \( 3n^2 + 3n + 6 \equiv 0 \pmod{6} \). Therefore \((n + 1)^3 + 5(n + 1) = (n^3 + 5n) + 3n^2 + 3n + 6 \equiv 0 \pmod{6} \) by induction assumption.

(2) Find the remainder when \( 7^{101} \) is divided by 101.

**Solution**

Since 101 is prime, By Fermat theorem \( 7^{100} \equiv 1 \pmod{101} \) and hence \( 7^{107} \equiv 7 \pmod{101} \).

(3) Find the integer \( a \), \( 0 \leq a \leq 20 \) such that \( 13a \equiv 1 \pmod{20} \).

**Solution**

We have that \( 13 \cdot 3 = 39 \equiv -1 \pmod{20} \). Hence \( 13 \cdot (-3) \equiv 1 \pmod{20} \). Since \(-3 \equiv 17 \pmod{20} \) we have \( 13 \cdot 17 \equiv 1 \pmod{20} \).

(4) Prove that if \( m \equiv 1 \pmod{\phi(n)} \) and \((a,n) = 1 \) then \( a^m \equiv a \pmod{n} \), where \( \phi \) is Euler’s function.

**Solution**

We are given \( m \equiv 1 \pmod{\phi(n)} \), i.e \( m = k\phi(n) + 1 \) By Euler’s theorem \( a^{\phi(n)} \equiv 1 \pmod{n} \). Therefore, \( a^{k\phi(n)} \equiv 1 \pmod{n} \) and hence \( a^{k\phi(n)+1} \equiv 1 \cdot a \equiv a \pmod{n} \)

(5) Suppose \( 3^{3^{100}} \) is written in ordinary way. What are the last two digits?

**Solution**

We need to find the remainder when we divide \( 3^{3^{100}} \) by 100. Let \( n = 100 = 2^2 \cdot 5^2 \). Then \( \phi(n) = (2^2 - 2^1) \cdot (5^2 - 5^1) = 40 \). therefore, by the previous problem, \( 3^{40k+1} \equiv 3 \pmod{100} \). Next observe that \( 3^4 = 81 \equiv 1 \pmod{40} \). Therefore, \( 3^{100} = (3^4)^{25} \equiv 1 \pmod{40} \). This finally implies that \( 3^{3^{100}} \equiv 3 \pmod{100} \). This means that the last two digits of \( 3^{3^{100}} \) are 03.
(6) Prove that $\sqrt[3]{\frac{2}{7}}$ is irrational.

**Solution**

Suppose $\sqrt[3]{\frac{2}{7}} = \frac{a}{b}$ where $a, b$ are integers. we can assume that $(a, b) = 1$. Then $\frac{2}{7} = \frac{a^3}{b^3}$ and $2b^3 = 7a^3$. LHS is even which means that $a$ must be even. Hence $a = 2c$ and we have $2b^3 = 7 \cdot 8c^3$, $b^3 = 28c^3$. Now RHS is even and hence $b$ must be even. That means that both $a$ and $b$ are even which contradicts $(a, b) = 1$. 