

- (1) (15 pts) Find the formula for the sum $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \dots - (2n) \cdot (2n + 1)$ and prove it by mathematical induction.

Solution

Observe that $(2n)(2n - 1) - (2n)(2n + 1) = (2n) \cdot (-2) = -4n$. Thus we need to find $-4 \cdot 1 - \dots - 4n = -4(1 + \dots + n) = -4 \frac{n(n+1)}{2} = -2n(n + 1)$.

We prove this by induction.

When $n = 1$ we have $1 \cdot 2 - 2 \cdot 3 = 2 - 6 = -4 = -2 \cdot (1) \cdot (2) = -4$.

Induction step. Suppose $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \dots - (2n) \cdot (2n + 1) = -2n(n + 1)$ then $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \dots - (2n) \cdot (2n + 1) + (2n + 1) \cdot (2n + 2) - (2n + 2) \cdot (2n + 3) = -2n(n + 1) + (2n + 1) \cdot (2n + 2) - (2n + 2) \cdot (2n + 3) = -2n(n + 1) - 2(2n + 2) = -2(n + 1)(n + 2)$.

- (2) (15 pts) Find the remainder when 6^{100} is divided by 14.

Solution

First we observe that $6 \equiv -1 \pmod{7}$. Hence $6^{100} \equiv (-1)^{100} = 1 \pmod{7}$. Thus $6^{100} \equiv 1 \pmod{7} \equiv 8 \pmod{7}$. This means that 7 divides $6^{100} - 8$. But $6^{100} - 8$ is even and 2 also divides $6^{100} - 8$. Since $(2, 7) = 1$ this means that 14 divides $6^{100} - 8$, i.e. $6^{100} \equiv 8 \pmod{14}$.

Answer: 8.

- (3) (15 pts) The triple $a = 1012, b = 45, c = 1013$ is Pythagorean, i.e. it satisfies $a^2 + b^2 = c^2$. (You don't have to check that).

Find two more Pythagorean triples with the same $b = 45$.

Solution

We rewrite $b^2 = c^2 - a^2 = (c - a)(c + a)$. Let $b^2 = mn$ where $m > n$. we put $m = c + a, n = c - a, c = \frac{m+n}{2}, a = \frac{m-n}{2}$.

In our case we have $b^2 = 45^2 = (3^2 \cdot 5)^2 = 3^4 \cdot 5^2$. Let's factor it as $(3^2 \cdot 5^2) \cdot 3^2 = 225 \cdot 9$. This gives $m = 225, n = 9$ and $c = \frac{225+9}{2} = 117, a = \frac{225-9}{2} = 108$.

Next rewrite $b^2 = 81 \cdot 25$. Put $m = 81, n = 25$. this gives $c = \frac{81+25}{2} = 53, a = \frac{81-25}{2} = 28$.

Answer: $a = 108, b = 45, c = 117$ and $a = 28, b = 45, c = 53$.

- (4) (15 pts) Find the integer $a, 0 \leq a < 37$ such that $(34!)a \equiv 1 \pmod{37}$.

Solution

Since 37 is prime, by Wilson's theorem, $36! \equiv -1 \pmod{37}$.

We rewrite $34! \cdot 35 \cdot 36 \equiv -1 \pmod{37}$. Since $36 \equiv -1 \pmod{37}$ this gives $34! \cdot 35 \equiv 1 \pmod{37}$.

Answer: $a = 35$.

- (5) (15 pts) For natural numbers a, b, c define (a, b, c) to be the greatest common divisor of a, b and c .
- (a) show that $(a, b, c) = ((a, b), c)$.
- (b) Show that the equation $ax + by + cz = (a, b, c)$ has at least one integer solution x, y, z .

Solution

- (a) If $d|a, d|b, d|c$ then $d|(a, b)$ and $d|c$.
 Conversely, if $d|(a, b)$ and $d|c$ then $d|a, d|b, d|c$. This means that a, b, c and $(a, b), c$ have the same divisors. therefore, $(a, b, c) = ((a, b), c)$.
- (b) Let $d = (a, b)$. Then by a), $(d, c) = (a, b, c)$. By Euclidean algorithm we have that $dx_1 + cz = (a, b, c)$ has an integer solution x_1, z .
 Next, applying Euclidean algorithm again we see that $ax_2 + by_2 = d$ also admits an integer solution. Plugging it into the first formula we get
 $(ax_2 + by_2)x_1 + cz = (a, b, c)$ or $a(x_1x_2) + b(y_2x_1) + cz = (a, b, c)$.
- (6) (10 pts) Let $n = pq$ where p, q are distinct odd primes. Find the remainder when $\phi(n)!$ is divided by n .

Solution

we have $\phi(n) = (p-1)(q-1)$. Since $p > 2, q > 2$ we have $p-1 \geq 2, q-1 \geq 2$. Therefore, $(p-1)(q-1) \geq 2(p-1) > p$. Therefore, $p|\phi(n)!$. Similarly, $q|\phi(n)!$. Since $(p, q) = 1$ this means $pq|\phi(n)!$, i.e. $\phi(n)! \equiv 0 \pmod{pq}$.

Answer: 0.

- (7) (15 pts) Solve the following quadratic equation

$$x^2 - (3+i)x + 2 = 0$$

Solution

By the formula for the roots of a quadratic equation we have

$$x = \frac{3+i \pm \sqrt{(3+i)^2 - 8}}{2} = \frac{3+i \pm \sqrt{6i}}{2} = \frac{3+i \pm \sqrt{6}\sqrt{i}}{2}. \text{ Thus we need to find } \sqrt{i}.$$

We have $i = 1(\cos(\pi/2) + i \sin(\pi/2))$. Therefore $\sqrt{i} = \pm(\cos(\pi/4) + i \sin(\pi/4)) = \pm(\frac{1}{\sqrt{2}} + (\frac{1}{\sqrt{2}}i))$.

$$\text{Therefore } x = \frac{3+i \pm \sqrt{6}\sqrt{i}}{2} = \frac{3+i \pm \sqrt{6}(\frac{1}{\sqrt{2}} + (\frac{1}{\sqrt{2}}i))}{2} = \frac{3+i \pm \sqrt{3}(1+i)}{2}$$