MAT246 SPRING 2009 L0101
PRACTICE FINAL EXAM (FROM FALL 2008)

There are 10 problems, worth 10 points each.

(1) (a) Calculate $\phi(20^{100})$, where ϕ is the Euler ϕ–function.
(b) Find an integer x such that $140x \equiv 133 \pmod{301}$. Hint: $\gcd(140, 301) = 7$.

(2) (a) Prove by mathematical induction that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for every natural number n.
(b) Prove that for p an odd prime, $1^p + 2^p + 3^p + \cdots + (p-1)^p \equiv 0 \pmod{p}$.

(3) Prove that for any odd integer a and any natural number n, a and a^{4n+1} have the same last digit.

(4) Recall that a *perfect square* is a number of the form n^2 where n is a natural number. Show that 9120342526523 is not the sum of two perfect squares. Hint: Consider values modulo 4.

(5) (a) Are there rational numbers a and b such that $\sqrt{3} = a + b\sqrt{2}$? Justify your answer.
(b) Prove that $\frac{\sqrt{5}}{\sqrt{2}+\sqrt{11}}$ is irrational.

(6) (a) What is the cardinality of the set of roots of polynomials with constructible coefficients? Justify your answer.
(b) Let $\mathbb{N}$ denote the set of all natural numbers. What is the cardinality of the set of all functions from $\mathbb{N}$ to $\{1, 3, 5\}$? Justify your answer.

(7) (a) Let $\mathbb{R}$ denote the set of all real numbers. What is the cardinality of the set of all finite subsets of $\mathbb{R}$? Show that your answer is correct.
(b) Let S be the set of all real numbers x between 0 and 1 such that every digit in the decimal expansion of x is either 3 or 5. That is, $S = \{\, 0.a_1a_2a_3\cdots \mid a_i = 3 \text{ or } a_i = 5 \text{ for all } i \,\}$. What is the cardinality of S? Show that your answer is correct.

(8) Let θ be the unique acute angle where $\cos\theta = 3/4$. Prove that $\theta/3$ is not a constructible angle.

(9) For each of the following numbers, state whether or not it is constructible and justify your answer.
(a) $\cos\theta$, where the angle $\theta/3$ is constructible. (That is, is this constructible in general?)
(b) $\sqrt[3]{25/8}$.
(c) $\sqrt{7 + \sqrt{5}}$.
(d) $(0.029)^{1/3}$.
(e) $\tan 22.5°$.

(10) (a) Is there a circle in the plane such that for every point (x, y) in the circle, x and y are constructible numbers? Justify your answer.
(b) Does $x^3 - 3x + 3\sqrt{3}$ have a constructible root? Justify your answer.