A Useful Model (Notes for 2011 Summer Topology Conference)

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Abstract

In these notes and the accompanying lectures I introduce *models of form* PFA(S)[S], a useful class of models of set theory in which various consequences of both PFA and $V = L$ hold. We list some of these consequences and topologically derive new results from them. Then we present some of the characteristic set-theoretic methods used to prove things in these models.

0 Introduction

Models we shall call "of form PFA(S)[S]" were introduced by Todorcevic in 2001 who used them to prove the consistency of *every compact hereditarily normal space satisfying the countable chain condition is hereditarily separable and hereditarily Lindelöf.* This was finally written up in [24]. These models are obtained by fixing a particular *coherent* Souslin tree S in a ground model (such trees are obtainable from $\diamondsuit$, for example), then iterating proper posets as in the consistency proof for PFA, but only those that preserve S, thus producing a model for *PFA*(S), i.e., PFA restricted to posets that preserve (the Souslinity of) S. That a countable support iteration of proper posets that preserve S preserves S is shown in [17]. Finally, one forces with S. A weaker technique, not requiring large cardinals, is to replace "proper" by "countable chain condition."

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If all models formed by forcing with S over a model of PFA(S) satisfy φ, we say "PFA(S)[S] implies φ." If a particular ground model is used, we say "φ holds in a model of form PFA(S)[S]." Which coherent S we use does not matter. The consistency of a supercompact cardinal is assumed.

Since we will be mainly dealing with locally compact spaces, for convenience we will assume all spaces are Hausdorff.

The solution by Larson and Todorcevic of Katětov's problem [16] depended on showing the remarkable fact that — using the weaker c.c.c. technique — some of the "Souslin-type" consequences [11] of MA_{ω_1}, namely that *compact, first countable, hereditarily separable spaces are hereditarily Lindelöf*, and that *first countable, hereditarily Lindelöf spaces are hereditarily separable*, are consistent with some of the "normal implies collectionwise Hausdorff" consequences of $V = L$, namely that *separable normal first countable spaces are collectionwise Hausdorff.* Since then, the strength of both types of consequences has been increased. Larson and Tall [15] dropped the separability in the second type of consequence, by starting with a particular ground model, while Todorcevic [24] improved [16] to get from PFA(S)[S] that *compact hereditarily separable spaces are hereditarily Lindelöf.* In [22] we obtained another result in the $V = L$ column, getting that *normal spaces which are either first countable or locally compact are collectionwise Hausdorff.*

In [11] a model was constructed in which the "combinatorial" consequences of MA_{ω_1} held, but not the "Souslin-type" consequences. The current investigations of PFA(S)[S] can be viewed as complementary: we construct a model in which the Souslin-type consequences of MA_{ω_1}, indeed of PFA, hold, but not the combinatorial ones.

1 Some Easy Consequences

Balogh [1], [2] introduced a technique for proving some locally compact normal collectionwise Hausdorff (CWH) spaces to be paracompact, assuming MA_{ω_1} + Fleissner's reflection axiom, Axiom R, [5]. These hypotheses are known to be mutually consistent. The CWH hypothesis is awkward; we will consistently eliminate it from Balogh's theorems. Here is Balogh's technique:

i) Assuming additional topological hypotheses such as countable tightness, reflect via Axiom R to show that if there is a non-paracompact counterexample X, there is one with $L(X) = \aleph_1$.

ii) Write X as an ω_1-increasing union of open Lindelöf spaces with closures included in the next one. Select a point from each boundary. The selection is locally countable; by a consequence of MA_{ω_1}, it is σ-closed-discrete. By normality and CWH, expand to a σ-discrete open collection. Press down to get an uncountable discrete collection of open sets inside one of the open Lindelöf subspaces, contradiction.

The assertion that locally compact normal spaces are CWH follows from $V = L$ [27], which contradicts MA_{ω_1}. Nonetheless, we can get a model for this plus Axiom R plus the consequences of MA_{ω_1} that Balogh used.

Theorem 1. *Assuming the consistency of a supercompact cardinal, there is a model in which hold:*

a) Balogh's $\sum$ *(defined below),*

b) Locally compact normal spaces are CWH,

c) Axiom R (defined below),

d) P-ideal Dichotomy (defined below).

Although we don't need them here, it is useful to know that the following also hold:

e) compact countably tight spaces are sequential,

f) first countable hereditarily Lindelöf spaces are hereditarily separable,

g) $\mathfrak{b} = \aleph_2$, $\mathfrak{p} = \aleph_1$,

h) the Open Coloring Axiom,

i) every first countable normal space is CWH,

j) every Aronszajn tree is special.

I haven't checked the details, but probably only c) and d) require large cardinals.

2 Topological Proofs

Treating the model of Theorem 1 as a black box, let's see how we easily answer previously difficult questions.

Theorem 2. *In the model of Theorem 1,*

1) locally compact, perfectly normal spaces are paracompact,

2) (locally) compact spaces with hereditarily normal squares are metrizable,

3) compact hereditarily normal homogeneous spaces are first countable.

Just from the conjunction of $\sum$ and "locally compact normal implies CWH", useful results easily follow:

Theorem 3. *Assume* PFA(S)[S]. *Let X be locally compact, hereditarily normal, CCC. Then X is hereditarily Lindelöf.*

Proof. Taking the one-point compactification, we may assume that X is compact. It is hereditarily CWH, so has countable spread and hence countable tightness. If it were not hereditarily Lindelöf, it would have an uncountable right-separated subspace. By $\sum$, it would then have an uncountable discrete subspace, contradiction. By quoting [16] we can get *hereditarily separable* as well. As a compact, hereditarily Lindelöf space, X is first countable. But in this model there are no first countable L-spaces [16]. □

Adding "locally compact normal implies CWH" to PFA type results yields strong conclusions about homogeneity. Juhász, Nyikos, Szentmiklóssy [7] proved:

Lemma 4. *Homogeneous compacta which are homogeneous and hereditarily strongly $\aleph_1$-CWH are countably tight.*

It follows that:

Theorem 5. PFA(S)[S] *implies homogeneous hereditarily normal compacta are first countable.*

The consistency of the conclusion was first shown in a different model by de la Vega [26]. I first proved this by showing PFA(S)[S] implies open Lindelöf subspaces of compact hereditarily normal spaces have hereditarily Lindelöf closures. This conclusion was proved to imply countable tightness for such compacta in [7]. Another way of doing it is to quote a theorem in the recent preprint of Todorcevic [24]:

Theorem 6. PFA(S)[S] *implies a compact countably tight space has a point of countable character.*

This is easy to quote, but not so easy to prove. It is easy to see that this plus homogeneity yields first countability.

Lemma 7. *In a particular model (the one of [15]) of form* PFA(S)[S]*, every locally compact, hereditarily normal space which does not include a perfect pre-image of* ω_1 *is paracompact.*

We can turn this result into a characterization as follows.

Theorem 8. *There is a model of form* PFA(S)[S] *in which locally compact hereditarily normal spaces are (hereditarily) paracompact if and only if they do not include a perfect pre-image of* ω_1*.*

Proof. The backward direction follows from Lemma 7, since a space is hereditarily paracompact if every open subspace of it is paracompact, and open subspaces of locally compact spaces are locally compact. The "hereditarily" version of the other direction is because perfect pre-images of ω_1 are countably compact and not compact, and hence not paracompact. Without "hereditarily" we need:

Lemma 9 [3]**.** *In a countably tight space, perfect pre-images of* ω_1 *are closed.*

Lemma 10 [1, 2, 14]**.** *A locally compact space has a countably tight one-point compactification if and only if it does not include a perfect pre-image of* ω_1*.*

We also now have a partial characterization for locally compact spaces that are only normal:

Theorem 11. *There is a model of form PFA(S)[S] in which a locally compact normal space is paracompact and countably tight if and only if its countable subspaces have Lindelöf closures and it does not include a perfect pre-image of* ω_1*.*

The proof of Theorem 11 is quite long. It is convenient to first prove the weaker

Theorem 12. *There is a model of form* $\mathrm{PFA}(S)[S]$ *in which a locally compact normal space* X *is paracompact and countably tight if and only if the closure of every Lindelöf subspace of* X *is Lindelöf, and* X *does not include a perfect pre-image of* ω_1.

One direction is easy and is left to the reader. The other direction is harder, but much of the work has been done elsewhere. We refer to [5] for a definition of the reflection axiom *Axiom R*. However, we shall only use the following three results concerning it. We have:

Lemma 13 [14]**.** *Axiom R holds in the PFA*$(S)[S]$ *model of [15].*

Definition. $L(Y)$*, the Lindelf number of* Y*, is the least cardinal* κ *such that every open cover of* Y *has a subcover of size* $\leq \kappa$.

Lemma 14 [2]**.** *Axiom R implies that if* X *is a locally Lindelöf, regular, countably tight space such that every open* Y *with* $L(Y) \leq \aleph_1$ *has* $L(\overline{Y}) \leq \aleph_1$*, then if* X *is not paracompact, it has a clopen non-paracompact subspace* Z *with* $L(Z) \leq \aleph_1$.

Lemma 15 [2]**.** *Axiom R implies that if* X *is locally Lindelöf, regular, countably tight, and not paracompact, then* X *has an open subspace* Y *with* $L(Y) \leq \aleph_1$*, such that* Y *is not paracompact.*

We also have:

Lemma 16. *If* Y *is a subset of a locally Lindelöf space of countable tightness in which closures of Lindelöf subspaces are Lindelöf, then if* $L(Y) \leq \aleph_1$*, then* $L(\overline{Y}) \leq \aleph_1$.

Proof. Left to the reader. □

To finish the proof of Theorem 12 it therefore suffices to prove:

Theorem 17. $\mathrm{PFA}(S)[S]$ *implies that if* X *is a locally compact normal space with* $L(X) \leq \aleph_1$*, closures of Lindelöf subspaces of* X *are Lindelöf, and* X *includes no perfect pre-image of* ω_1*, then* X *is paracompact.*

Crucial ingredients in proving this are $\aleph_1$**-CWH** and:

Lemma 18 [4]**.** PFA$(S)[S]$ *implies*

$\sum$**:** *if X is compact and countably tight, and $Z \subseteq X$ is such that $|Z| \leq \aleph_1$ and there exists a collection $\mathcal{V}$ of open sets, $|\mathcal{V}| \leq \aleph_1$, and a collection $\mathcal{U} = \{U_V : V \in \mathcal{V}\}$ of open sets, such that $Z \subseteq \bigcup \mathcal{V}$, and for each $V \in \mathcal{V}$, there is a $U_V \in \mathcal{U}$ such that $V \subseteq \overline{V} \subseteq U_V$, and $|U_V \cap Z| \leq \aleph_0$, then Z is σ-closed discrete in $\bigcup \mathcal{V}$.*

The conclusion of Lemma 18 had previously been shown under MA_{ω_1} by Balogh [1]. The weaker conclusion asserting that Z is σ-discrete, if it's locally countable, was established by Todorcevic. A modification of his proof yields the stronger result [4]. It follows that:

Corollary 19. PFA$(S)[S]$ *implies that if X is locally compact, includes no perfect pre-image of ω_1, and $L(X) \leq \aleph_1$, and $Y \subseteq X$, $|Y| = \aleph_1$, is such that each point in X has a neighbourhood meeting at most countably many points of Y, then Y is σ-closed-discrete.*

We now need some results of Nyikos:

Definition. *A space X is of* ***Type I*** *if $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where each X_α is open, $\alpha < \beta$ implies $\overline{X}_\alpha \subseteq X_\beta$, and each $\overline{X}_\alpha$ is Lindelöf. $\{X_\alpha : \alpha < \omega_1\}$ is* ***canonical*** *if for limit α, $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$.*

Lemma 20 [20]**.** *If X is locally compact, $L(X) \leq \aleph_1$, and every Lindelöf subset of X has Lindelöf closure, then X is of Type I, with a canonical sequence.*

Lemma 21 [19]**.** *If X is of Type I, then X is paracompact if and only if $\{\alpha : \overline{X}_\alpha - X_\alpha \neq 0\}$ is non-stationary.*

Proof of Theorem 17. If X is paracompact, this is straightforward. Suppose X were not paracompact. X is of Type I so we may pick a canonical sequence and we may pick a stationary $S \subseteq \omega_1$ and $x_\alpha \in \overline{X}_\alpha - X_\alpha$, for each $\alpha \in S$. By Corollary 19, $\{x_\alpha : \alpha \in S\}$ is σ-closed-discrete, so there is a stationary set of limit ordinals $S' \subseteq S$ such that $\{x_\alpha : \alpha \in S'\}$ is closed discrete. Let $\{U_\alpha : \alpha \in S'\}$ be a discrete collection of open sets expanding it. Pressing down yields an uncountable closed discrete subspace of some X_α, contradiction. $\square$

Note that Lemma 7 follows from Theorem 12, for consider the closure of a Lindelöf subspace Y of a locally compact, hereditarily normal space which does not include a perfect pre-image of ω_1. The following argument in Nyikos [20] will establish that $\overline{Y}$ is Lindelöf. First consider the special case when Y is open. Let B be a right-separated subspace of the boundary of Y. We claim B is countable, whence the boundary is hereditarily Lindelöf, so $\overline{Y}$ is Lindelöf. Since the one-point compactification of $\overline{Y}$ is countably tight, by Lemma 18, if B is uncountable, it has a discrete subspace D of size $\aleph_1$. D is closed discrete in $Z = \overline{Y} - (\overline{D} - D)$, so in Z there is a discrete open expansion $\{U_d : d \in D\}$ of D, because $\overline{Y}$ is hereditarily strongly $\aleph_1$-collectionwise Hausdorff. Since $Y \subseteq Z$, $\{U_d \cap Y : d \in D\}$ is a discrete collection of non-empty subsets of Y, contradicting Y's Lindelöfness.

Now consider an arbitrary Lindelöf Y. Since X is locally compact, Y can be covered by countably many open Lindelöf sets. The closure of their union is Lindelöf and includes $\overline{Y}$.

A consequence of Corollary 19 is that we can improve Theorem 17 for spaces with Lindelöf number $\leq \aleph_1$ to get:

Theorem 22. PFA$(S)[S]$ *implies that if X is a locally compact normal space with $L(X) \leq \aleph_1$, and X includes no perfect pre-image of ω_1, then X is paracompact.*

Proof. As before, it suffices to consider the case of an open Lindelöf Y. If the closure of Y were not Lindelöf, since it has Lindelöf number $\leq \aleph_1$ there would be a locally countable subspace Z of size $\aleph_1$ included in $\overline{Y} - Y$. That subspace would then be σ-closed-discrete by Corollary 19. As in the proof of Lemma 7 from Theorem 12, we obtain a contradiction by getting an uncountable closed discrete subspace of Y. Since we have σ-*closed*-discrete, we only need normality rather than hereditary normality. □

In retrospect, Theorem 22 is perhaps not so surprising: a phenomenon first evident in [1] is that "normal plus $L \leq \aleph_1$" can often substitute for "hereditarily normal" in this area of investigation.

An immediate corollary of Theorem 22 is:

Corollary 23. PFA$(S)[S]$ *implies every locally compact normal space of size $\leq \aleph_1$ with a G_δ-diagonal is metrizable.*

3 Applications of P-ideal Dichotomy

In order to prove Theorem 11, we introduce some known ideas about ideals.

Definition. *A collection $\mathcal{I}$ of countable subsets of a set X is a* ***P-ideal*** *if each subset of a member of $\mathcal{I}$ is in $\mathcal{I}$, finite unions of members of $\mathcal{I}$ are in $\mathcal{I}$, and whenever $\{I_n : n \in \omega\} \subseteq \mathcal{I}$, there is a $J \in \mathcal{I}$ such that $I_n - J$ is finite for all n.*

P *(short for* ***P-ideal Dichotomy****): For every P-ideal $\mathcal{I}$ on a set X, either*

i) there is an uncountable $A \subseteq X$ such that $[A]^{\leq\omega} \subseteq \mathcal{I}$

or ii) $X = \bigcup_{n<\omega} B_n$ such that for each n, $B_n \cap I$ is finite, for all $I \in \mathcal{I}$.

Todorcevic's proof that $PFA(S)[S]$ implies **P** appears in [12] and [24]. In [3], Eisworth and Nyikos proved the following remarkable result:

Lemma 24. PID *implies that if X is a locally compact space, then either*

a) X is the union of countably many ω-bounded subspaces,

or b) X does not have countable extent,

or c) X has a separable closed subspace which is not Lindelöf.

Recall a space is **$\boldsymbol{\omega}$-bounded** if every countable subspace has compact closure. ω-bounded spaces are obviously countably compact.

From [6] we have:

Lemma 25. *An ω-bounded space is either compact or includes a perfect pre-image of ω_1.*

We can now prove Theorem 11.

The forward direction follows from Theorem 12. To prove the other direction, it suffices to show that if Y is a Lindelöf subspace of our space X, then $\overline{Y}$ is Lindelöf. Applying Lemma 24, we see that by Lemma 25, $\overline{Y}$ will be σ-compact if we can exclude alternatives b) and c). c) is excluded by hypothesis, so it suffices to show that $\overline{Y}$ has countable extent. But that is easily established, since $\overline{Y}$ is locally compact normal and hence $\aleph_1$**-CWH**. A closed discrete subspace of size $\aleph_1$ in $\overline{Y}$ could thus be fattened to a *discrete* collection of open sets. Their traces in Y would contradict its Lindelöfness. □.

Corollary 26. *There is a model of form* PFA(S)[S] *in which a locally compact space is metrizable if and only if it is normal, has a G_δ-diagonal, and every separable closed subspace is Lindelöf.*

Proof. Theorem 12 applies, since spaces with G_δ-diagonals do not include perfect pre-images of ω_1. □

This characterization does not hold in ZFC; the tree topology on a special Aronszajn tree is a locally compact Moore space, and hence has a G_δ-diagonal. Under MA_{ω_1}, it is (hereditarily) normal. See e.g. the survey article [23]. Every separable subspace of an ω_1-tree is bounded in height, and so is countable.

4 The model, coherent Souslin trees

MA_{ω_1} (Martin's Axiom for meeting $\aleph_1$ dense sets) is proved consistent by a length ω_2 iteration of countable chain condition partial orders. A key observation is that one only has to consider partial orders of size $\leq \aleph_1$; one adds a generic filter for $\aleph_1$ dense subsets of each such partial order at an initial stage of the iteration. For PFA, this reduction to partial orders of size $\aleph_1$ is not available; instead, one iterates supercompact many times and does a reflection argument to prove that suffices. A crucial technical lemma is that *a countable support iteration of proper partial orders is proper*. A variation of the consistency proof for PFA will yield the model of Theorem 1. A weaker version of PFA is produced by starting with a particularly nice Souslin tree S (a *coherent* (defined below) one), and iterating partial orders that keep S Souslin. Not surprisingly, this produces a model for PFA(S), PFA restricted to partial orders that preserve S. Of course one has to show that a countable support iteration of posets that preserve S preserves S. This was accomplished by Miyamoto [17]. Having gone to the trouble of preserving S, one then kills it by forcing with it! We have created *a model of form* PFA(S)[S]. We can create different such models by starting with particular ground models.

All we ever use about coherence is that such trees satisfy a strong homogeneity property: the cones above any two elements on the same level are isomorphic. This ensures that any two generic branches yield essentially the same model. When trying to prove something about a model of form PFA(S)[S], one assumes PFA(S) and tries to see what happens when one forces with S. This entails that the usual proper forcing arguments become more difficult, because one has to prove something about, say, an S-name for a topological space after forcing with S, rather than just proving something about a space. Coherence restores some measure of control in that, roughly speaking, it ensures that different generic interpretations of that S-name will be homeomorphic.

It is not immediately clear why one would want to work much harder so as to establish that consequences of PFA still hold in models of form PFA$(S)[S]$. The reason of course is that some important propositions that contradict PFA hold in models of form PFA$(S)[S]$. From our point of view, the most important such propositions are that *normal spaces which are either first countable or locally compact are CWH.* Probably there is a common generalization to point-countable type, but I haven't proved that. Notice that since first countable normal spaces are CWH, there are no Q-sets, so $\mathfrak{p} = \aleph_1$. A useful heuristic is that, just as MA(σ-centred) captures the "combinatorial" consequences of MA but not the "Souslin-type" consequences, model of form PFA$(S)[S]$ capture the Souslin-type consequences of MA_{ω_1}, but not the combinatorial ones.

Another useful heuristic is that forcing with a Souslin tree is somewhat like countably closed forcing to add a subset of ω_1. Although one only has ω-distributivity rather than countable closure, that plus the countable chain condition is sometimes enough to push a similar argument through.

5 My Plan

The well-prepared participant for the remainder of the workshop would be someone familiar with Todorcevic's proof, using *forcing with elementary submodels as side conditions*, that *PFA implies there are no S-spaces*, as in e.g. his book [25]. Unfortunately I suspect that only a few participants in the workshop are "well-prepared". Even with well-prepared participants, it would take at least twice the time I have to go through the details of *one* of the proofs I would like to present. What to do? My goal is to give enough of the intuition behind the proofs and enough of the technical machinery so that the determined reader will be able to work through the write-ups of these proofs without undue pain.

In addition to the technical set-theoretic machinery, these proofs use some rather elementary topology. Suppose A is a locally countable subset of $\aleph_1$ points in a compact spaces X, say $A = \{a_\alpha : \alpha < \omega_1\}$. Let $\{U_\alpha : \alpha < \omega_1\}, \{V_\alpha : \alpha < \omega_1\}$ be open such that $a_\alpha \in U_\alpha \subseteq \overline{U_\alpha} \subseteq V_\alpha$, and $|V_\alpha \cap A| \leq \aleph_0$. By compactness, A has a complete accumulation point z. If X has some countability property, e.g. Fréchet, there will be a sequence $\langle a_{\alpha_n} : n < \omega \rangle$ from A converging to z. Then for any finite $B \subseteq A$, almost all a_n's will miss $\bigcup\{U_\beta : \beta \in B\}$, for if infinitely many were in that union, z would be in some $\overline{U_\beta} \subseteq V_\beta$, contradiction.

Another variation of the same idea is that a finite union of compact neighbourhoods must miss almost all of the members of a countable discrete collection of sets.

The march towards the consistency of *there are no S-spaces* started with Kunen's observation that the way to kill an S-space was to force an uncountable right-separated subspace to have an uncountable discrete subspace. So one has $\alpha \in U_\alpha$, and $\alpha < \beta$ implies $\beta \notin \overline{U_\alpha}$. As forcing conditions one uses finite subsets T of ω_1 for which the associated U_α's witness the discreteness of T. The main difficulty is to show that the forcing doesn't collapse ω_1. Kunen [10] showed this if finite powers of X were hereditarily separable; Szentmiklóssy [21] if X compact, and finally Todorcevic [25] in general. Balogh [1] observed that for X compact countably tight, Szentmiklóssy's proof could be improved so as to get a locally countable subset σ-discrete, hence obtaining $\sum$. One way to do this is to force an $f : X \to \omega$ such that each $f^{-1}(\{n\})$ will be discrete, and each x will be in one of them. Kunen, Szentmiklóssy, and Balogh used countable chain condition forcing; Todorcevic used proper forcing.

The PFA(S)[S] proofs are mainly more difficult versions of the PFA proofs; one has to show that a "natural" partial order $\mathcal{P}$ is both proper and preserves S, and that forcing with S then creates the desired object from the generic filter forced by $\mathcal{P}$. For example, $\mathcal{P}$ might create a tree of discrete subspaces indexed and ordered by the Souslin tree, so that forcing a generic branch through the Souslin tree will make uncountably many of these discrete subspaces cohere, so that there will be an uncountable discrete subspace.

How about the proof that *there is a model of* PFA(S)[S] *in which every locally compact normal space is CWH*? How does that fit in with the general scenario, given that the conclusion contradicts PFA? The answer is interesting:

1. By starting with a particular ground model, we need only prove that

PFA(S)[S] *implies locally compact normal spaces are* $\aleph_1$*-CWH* (which also contradicts PFA). (We actually prove $\aleph_1$-CWH on a club $C \subseteq \omega_1$. One can then quote a theorem of Taylor to get full $\aleph_1$-CWH, or else complicate the proof to eliminate the club. For the first alternative, see [22]; for the second, see [4].)

2. Both PFA and PFA(S)[S] imply each closed discrete subspace of size $\aleph_1$ in a locally compact normal space can be expanded to a discrete collection of compact sets of countable character.

3. After forcing with a Souslin tree, normal spaces are collectionwise normal with respect to collections of $\aleph_1$ sets of countable character.

Thus the essential core of the proof is a (new) PFA consequence, added to that is a Souslin tree forcing version of a countably closed forcing argument. We shall see that 2) shares several features with the argument for making locally countable subspaces σ-discrete.

I won't give the proof for 3); it is the same as the argument for getting *normal first countable implies* $\aleph_1$*-CWH*, which was presented by Paul Larson at a couple of topology conferences around 2004, and appears in our recent paper in Fundamenta [15]. The idea for that proof is a mashup of Fleissner's $V = L$ proof and my countably closed forcing proof for getting *normal + character* $\leq \aleph_1$ *implies* $\aleph_1$*-CWH.* We inductively define a name for a partition of the closed discrete subset in the extension, such that any assignment of open sets to members of the discrete set that witnesses normality for that partition is actually a separation. (We are sliding over some details involving a club.)

2) begins to look similar to the idea for getting Balogh's $\sum$ when we throw in a couple more easy topological ideas. The first uses an old idea of Steve Watson [27]. I'll leave the easy proof to you:

Lemma 27. *Let* $D = \{x_\alpha : \alpha < \omega_1\}$ *be a closed discrete subspace of a locally compact normal space. Then* $\{x_\alpha : \alpha < \omega_1\}$ *has a right-separated expansion by compact* G_δ*'s.*

The following lemma is also not difficult, and is also left to you:

Lemma 28. *If* D *as above has such a right-separated expansion which is also* σ*-relatively discrete, then* D *has a discrete expansion by compact* G_δ*'s.*

Thus, as with $\sum$, we shall force a right-separated collection to be σ-discrete.

6 Some Proofs

PFA(S)[S] proofs are technically rather difficult. I could spend the whole workshop trying to prove one theorem – in fact, it took me around five hours to go through the proof of *locally compact normal implies CWH* for one reasonably bright set-theorist. My intention is therefore to go through a relatively simple proof of something not very interesting – which nonetheless contains many of the characteristic PFA(S)[S] features – with the idea that if you understand this proof, you will be able to follow the write-ups of more difficult proofs.

Theorem 29. PFA(S)[S] *implies that if* $Y = \{y_\alpha\}_{\alpha<\omega_1}$ *is a locally countable subspace of a compact Fréchet space, then* Y *includes an uncountable discrete subspace.*

We'll actually get that there is a club $C \subseteq \omega_1$ such that $\{y_\alpha : \alpha \in C\}$ is σ-discrete, which suffices. This theorem tells us that PFA(S)[S] implies there are no first-countable compact S-spaces. There are significant additional steps needed in order to weaken "Fréchet" to "sequential" and then to "countably tight", hence getting rid of all compact S-spaces. One can also improve the conclusion to get Y to be σ-discrete, and indeed to get Balogh's $\sum$.

Proofs that require looking carefully at the forcing with a coherent Souslin tree are technically difficult – at least for me – but usually there are simple ideas that make the machinery work in a particular case. One of the ideas that pops up in several crucial instances is that if a sequence from a subset Y of a space X converges to an $x \in X - Y$, then almost all of the points in the sequence are outside of Y. I say "almost all" rather than "all but finitely many" because this principle is true for the more general notion of convergence along an ultrafilter $\mathcal{U}$ on ω. We say that $x_n \to_{\mathcal{U}} x$ if for each open V containing x, $\{n : x_n \in V\} \in \mathcal{U}$. Thinking of a set in the ultrafilter as of measure 1, we say "almost all" elements of the sequence are in V.

Every PFA(S)[S] proof needs the following:

Lemma 30. $\mathcal{P}$ *is proper and preserves* S *if for all sufficiently large regular* θ *and for a closed unbounded family* $\mathcal{C}$ *(in* $[H_\theta]^{\aleph_0}$*) of countable elementary submodels* M *of* H_θ *with* $\mathcal{P}, S \in M$*, letting* $\delta = M \cap \omega_1$*, for every* $p \in \mathcal{P} \cap M$*, there is a* $q \leq p$ *such that for all* $s \in S$ *of height* δ*,* $\langle q, s\rangle$ *is* $(\mathcal{P}\times S, M)$*-generic.*

Proof. This is due to Miyamoto [17]. Since the lemma is not quite stated there in this form, and the proof is short, we give it here. First of all, for any $\langle q, s\rangle \in \mathcal{P} \times S$, if $\langle q, s\rangle$ is $(\mathcal{P} \times S, M)$-generic, then q is $(\mathcal{P}, M)$-generic, so $\mathcal{P}$ is proper. Suppose $\mathcal{P}$ forces $\dot{A}$ to be a maximal antichain of S. Let $A' = \{\langle r, s\rangle \in \mathcal{P} \times S : r \Vdash s \in \dot{A}\}$. Let $p \in \mathcal{P}$. Take θ regular and sufficiently large, and let $M \in \mathcal{C}$ be a countable elementary submodel of H_θ containing p, A', $\mathcal{P}$, and S. A' is predense in $\mathcal{P} \times S$, and by assumption, there is a $q \le p$ such that for all s of height δ, $\langle q, s\rangle$ is $(\mathcal{P} \times S, M)$-generic. Thus $A' \cap M$ is predense below $\langle q, s\rangle$ for all s of height δ. Therefore $q \Vdash$ "for all s of height δ, there is a $t \in \dot{A}$ such that s extends t." But then $q \Vdash$ "$\dot{A} \subseteq S \mid \delta$." $\square$

The overall strategy for using Miyamoto's lemma is the same as in the proof that PFA implies there are no S-spaces, and many other proofs as well: "copy" the "growth" of a condition into an elementary submodel by a finite induction, using elementarity at each step.

Todorcevic's proof [24] that $\mathrm{PFA}(S)[S]$ implies there are no compact S-spaces depends on showing that such spaces are sequential. This allows him to reduce an uncountable amount of information down to a countable amount, which Souslin tree forcing can handle. Our proof that PFA$(S)[S]$ implies locally compact normal spaces are $\aleph_1$-CWH is along the same lines: we in effect use the fact that any countably infinite subset of an uncountable closed discrete subspace in a locally compact normal space has a discrete expansion by compact G_δ's which converges to the point at infinity in the one-point compactification of the space.

Large portions of PFA$(S)[S]$ proofs are independent of the particular problem we are working on, but instead involve general properties of Souslin trees, in particular, coherent ones. To emphasize this and to render the technology more accessible we have organized much of the proofs of our baby version of $\sum$ and of the CWH theorem as a sequence of lemmas and notation having nothing to do with topology.

Lemma 31. *Let S be a Souslin tree and N a countable elementary submodel of some H_θ containing S. Suppose $A \subseteq S$, $A \in N$, $t \in A - N$. Suppose there is an $s \in S \cap N$, s below t. Then there is a $u \in [s, t) \cap N$ such that A is dense above u.*

Proof. If s itself is not the desired u, let

$$E = \{x \in S : s \text{ is below } x, \text{ the cone above } x \text{ does not contain a member of } A, \text{and } x \text{ is minimal among elements of } [s, x] \text{ with that property}\}.$$

Then $E \in N$ and E is an antichain of S, so E is countable. Therefore $E \subseteq N$. Let $\eta = \sup\{ht(x) : x \in E\}$. Then $\eta \in N$. Let u be the predecessor of t on the $(\eta + 1)$th level of S. Then $u \in N$ and $u \in [s, t)$. If A were not dense above u, there would be a y above u such that the cone above y would not include a member of A. The height of the least such y would be greater than η, contradiction. □

Definition. *An **m-chain with possible repetitions** is an m-tuple $\langle a_1, \ldots, a_m\rangle$, each $a_i \in S$, such that a_{i+1} extends a_i. We admit the possibility that $a_{i+1} = a_i$.*

Definition. *Let $\mathcal{A}$ be a family of chains with possible repetitions of a Souslin tree S. $\mathcal{A}$ is **dense above** $s \in S$ if for each s' extending s, there is an $A \in \mathcal{A}$ such that $\min A$ extends s'. We shall use "s' above s" and "s' extends s" synonymously, and admit the possibility that $s' = s$.*

Corollary 32. *Let S be a Souslin tree and N a countable elementary submodel of some H_θ containing S. Suppose $\mathcal{A}$ is a family of chains with possible repetitions of S, $\mathcal{A} \in N$, and suppose there is an $A_0 \in \mathcal{A}$, $\min A_0 \notin N$. Suppose $s \in S \cap N$, s below $t = \min A_0$. Then there is a $u \in S \cap N$, $u \in [s, t)$, such that $\mathcal{A}$ is dense above u.*

Proof. Let $\mathcal{A}^* = \{\min A : A \in \mathcal{A}\}$. Apply Lemma 31. □

Before proceeding further, let us say what "coherent" means, since we will be using it. We quote from [16]; also see the references listed there, as well as [9].

Definition. *A **coherent** tree is a downward closed subtree S of ${}^{<\omega_1}\omega$ with the property that $\{\vec{\xi} \in \operatorname{dom} s \cap \operatorname{dom} t : s(\vec{\xi}) \neq t(\vec{\xi})\}$ is finite for all $s, t \in S$. A **coherent Souslin tree** is a Souslin tree given by a coherent family of functions in ${}^{<\omega_1}\omega$ closed under finite modifications.*

As noted in [16], for S a coherent (König calls these *uniformly coherent*) Souslin tree, and s, t on the same (ηth) level of S, there is a canonical isomorphism σ^S_{st} between the cones above (we think of our trees as growing upwards) s and t, defined by letting $\sigma^S_{st}(s')(\alpha)$ be $t(\alpha)$ if $\alpha < \eta$ and $s'(\alpha)$ otherwise, for each s' extending s. These isomorphisms are such that $\sigma^S_{su} = \sigma^S_{tu} \circ \sigma^S_{st}$ and $\sigma^S_{st} = (\sigma^S_{ts})^{-1}$. See [13] for a construction of a coherent Souslin tree from $\diamondsuit$.

Intuitively, what coherence does for us is it deals with the following problem: in trying to go from a PFA proof to a $\mathrm{PFA}(S)[S]$ proof, we have much less control over what the $\mathcal{P}$-generic S-name becomes when we force with S, than we would have over simply an object — rather than a name — we construct with PFA. A coherent Souslin tree, however, has — up to automorphism — only one generic branch. Therefore the possible interpretations of a name will be "isomorphic," i.e. although there are many possible objects to deal with, they are all essentially the same. We do not yet, however, have a clear understanding of under which circumstances this intuition leads to a $\mathrm{PFA}(S)[S]$ proof from a PFA proof.

It is somewhat easier to force an uncountable discrete subspace than it is to make the latter σ-discrete; however, once one has figured out the right notation, it is not too much more difficult to do the σ-discrete version. Moreover, it is a useful technique I have not found written anywhere other than in my recent as yet unpublished papers, although it is due to Todorcevic. So we will do σ-proofs.

In dealing with proofs of properness, one first wants to fix a regular κ such that everything one wants to talk about, e.g. $\mathcal{P}$, is in H_κ. Then one looks at an even bigger cardinal θ, say $\theta = (2^\kappa)^+$.

I want to go as far as I can toward proving that a partial order $\mathcal{P}$ is proper and preserves S without actually telling you what the partial order is. The elements of $\mathcal{P}$ will consist of ordered pairs, the first coordinate of which will be a finite partial function (or a k-tuple of such functions) taking values in V, where $V \models \mathrm{PFA}(S)$. (Unlike the situation with proper posets, I do not know how to make the machinery work with countable partial functions). The second coordinate will be a finite $\in$-chain of countable elementary submodels of H_κ. (If your knowledge of elementary submodels is minimal, I recommend chapter 24 of [8]). Let $p = \langle f_p, \mathcal{N}_p \rangle \in \mathcal{P}$. E.g. if we want to generically partition a set of size $\aleph_1$ into countably many pieces, f_p (or its first component) will be a finite partial function from ω_1 into ω. We think of $f_p^{-1}(\{n\})$ as a "level" and require that $\mathcal{N}_p$ *separates* each level in the sense that if $f_p(s) = f_p(s') = n$, then if $s \neq s'$, there is an $N \in \mathcal{N}_p$ such that $s \in N$ and $s' \notin N$.

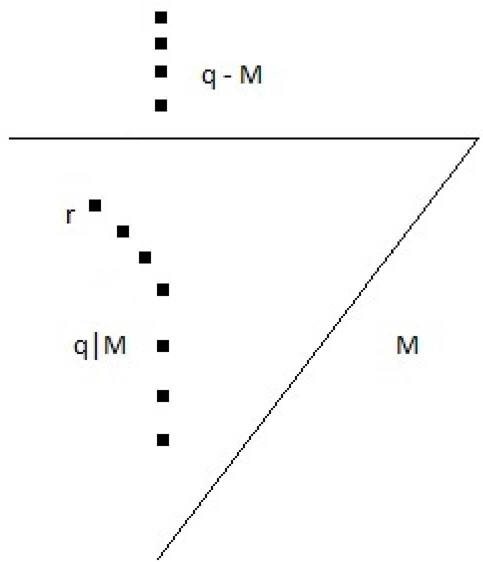
Let M be a countable elementary submodel of H_θ containing everything relevant. (There will be a club of such M's.) Let $\delta = M \cap \omega_1$. Let $p \in \mathcal{P} \cap M$. Let $p^M = \langle f_p, \mathfrak{N}_p \cup \{M \cap H_\kappa\}\rangle$. Then, by a standard argument, $p^M \in \mathcal{P}$.

The advantage of the method of proper forcing with elementary submodels as side conditions is that you know what the generic condition should be, namely p^M. In the PFA$(S)[S]$ situation, there is a natural variation on this.

Let t_M be an arbitrary node at the δth level of S. We will show $\langle p^M, t_M\rangle$ is generic. Let $\mathcal{D} \in M$ be a given dense open subset of $\mathcal{P} \times S$ and let $\langle q, t\rangle$ be a given extension of $\langle p^M, t_M\rangle$. We need to show $\langle q, t\rangle$ is compatible with some member of $\mathcal{D} \cap M$. Extending $\langle q, t\rangle$, we may assume that $\langle q, t\rangle \in \mathcal{D}$. Moreover, by extending further (since $\mathcal{D}$ is open), we may assume that t is not in the largest model of $\mathfrak{N}_q$, and that this model contains all the members of $\operatorname{dom} f_q$. (Here and elsewhere, we omit proofs that are simple if one works and plays well with elementary submodels. If not, the proofs are in [22].)

Let $q_M = q \,|\, M$. What this means will depend on the partial order $\mathcal{P}$. E.g. if f_q is a single partial function, $q \,|\, M = \langle f_q \cap M, \mathcal{N}_q \cap M\rangle$. Since finite subsets of M are members of M, $q_M \in M$. If the partial order is at all reasonable, q_M will be in M and q will extend it. $f_q \cap M$ will be another finite partial function like f_q; to see that $\mathcal{N}_q \cap M$ separates the levels of $\mathcal{N}_q \cap M$, consider an $N \in \mathfrak{N}_q$ such that $ht(s) \in N$ and $ht(s') \notin N$. Since $ht(s') \in M \cap H_\kappa \in \mathfrak{N}_q$, N must be a member of $M \cap H_\kappa$, so $N \in \mathfrak{N}_{q_M}$.

Here is the picture, temporarily ignoring the t's:



$\langle q, t\rangle \in \mathcal{D}$; we want to get an $\langle r, t_r\rangle \in M$, t_r below t, such that $\langle r, t_r\rangle$ "is just like $\langle q - M, t\rangle$" and in particular, r and $q|M$ have a common extension $R \in M$ such that $\langle R, t_r\rangle$ is in $\mathcal{D}$. By elementarity this is not difficult to do; it is also not difficult to take r sufficiently high up in M so that it is compatible with $q|M$. The hard part is to find such an r which is also compatible with $q - M$. This is accomplished by a finite induction, at each stage of which one decides one of the finitely many elements of r. Toward making that decision, one constructs $\aleph_1$ many possible candidates for that element. Then, finally using not just general machinery, but the particulars of the partial order and the assumptions of whatever result one is trying to prove, one argues that there are $\aleph_0$ of the possible candidates such that almost all of them are suitable – they don't conflict with $q - M$. This is the place where one uses e.g. the easy topological facts about sequences I mentioned earlier. This process is repeated finitely many times, winding up with an r such that $\langle$ the common extension of r and $q - M, t_r\rangle$ is in $\mathcal{D} \cap M$ and is compatible with $\langle q, t\rangle$.

The rather weak assumptions we have made about $\mathcal{P}$ now enable us to develop some machinery.

We may assume that the maximal model of $\mathfrak{N}_{q_M}$ contains all the members of dom f_{q_M} (= dom $f_q \cap M$), else we could have extended $\mathfrak{N}_q$ to ensure this. For let N^* be that maximal model. Since $N^* \in M \cap H_\kappa$, it is not the maximal model of $\mathfrak{N}_q$, so $N^* \in N'$, where N' is the minimal model of $\mathfrak{N}_q$ which is not in M. Then $N' = M \cap H_\kappa$. Then we can adjoin to $\mathfrak{N}_q$ a countable elementary submodel of H_κ in M containing N^* and $\operatorname{dom} f_q \cap M$.

Let δ_M be the intersection of ω_1 with the maximal model of $\mathfrak{N}_{q_M}$. By taking the maximal model large enough, we may ensure that the projection of $(\operatorname{dom} f_q \cup \{t\}) - M$ on the δth level of S has the same size as its projection on the δ_Mth level. To see this, note that there is a $\delta^* < \delta$ such that the projection of $(\operatorname{dom} f_q \cup \{t\}) - M$ on the δ^*th level has the same size as its projection on the δth level, since δ is a limit ordinal and S is a normal tree. Then add to $\mathfrak{N}_q$ a countable elementary submodel N of $H_\kappa, N \in M$, with δ^* and the maximal model of $\mathfrak{N}_{q_M}$ as members.

Let $\{u_1, \ldots, u_n\}$, $\{v_1, \ldots, v_n\}$ respectively enumerate these projections on the δ_Mth and δth levels, such that $u_i = v_i \,|\, \delta_M$, $i \leq n$, and such that $u_1 = t \,|\, \delta_M$ and $v_1 = t \,|\, \delta$. For $1 \leq i, j \leq n$, let σ_{ij} be the canonical isomorphism which moves u_i to u_j. Note $\sigma_{ij}^{-1} = \sigma_{ji}$, and σ_{ii} is the identity isomorphism.

Let N_{q_M} be the maximal model of $\mathcal{N}_{q_M}$. For any $\langle r, t_r \rangle \in \mathcal{P} \times S$ that is $\leq \langle q_M, u_1 \rangle$, define:

$$F_r = \{x \in (\operatorname{dom} f_r \cup \{t_r\}) - N_{q_M} : \quad x \,|\, \delta_M = \text{ some } u_{i_x} \text{ and some } \sigma_{1 i_x}(t) \text{ extends } x\}.$$

Then, considering t as t_q, claim:

$$F_q = \{x \in (\operatorname{dom} f_q \cup \{t\}) - M : x \,|\, \delta = \text{ some } v_{i_x} \text{ and } \sigma_{1 i_x}(t) \text{ extends } x\}.$$

Clearly F_q includes the right-hand side. On the other hand, if $x \in \operatorname{dom} f_q - N_{q_M}$, then $x \notin M$, so $ht(x) \geq \delta$. No two v_i's project onto the same u_j, so if $x \,|\, \delta_M = u_{i_x}$, then $x \,|\, \delta = v_{i_x}$. $\square$

We claim that if v_i and v_j are projections of elements of F_q, then $\sigma_{ij}(v_i) = v_j$. To see this, first note that if $x \in F_q$ extends v_i, then $\sigma_{i1}(v_i) \leq \sigma_{i1}(x) \leq t$. Hence $\sigma_{i1}(v_i) = v_1$, since both are of height δ below t. It follows that $\sigma_{ij}(v_i) = \sigma_{1j} \circ \sigma_{i1}(v_i) = \sigma_{1j}(v_1) = v_j$. We then have that for such v_i and v_j, $v_i \mid [\delta_M, \delta) = v_j \mid [\delta_M, \delta)$.

For $x \in F_q$, let $\hat{x} = \sigma_{1 i_x}^{-1}(x)$. For an m-tuple $\vec{x} = \langle x_1, \ldots, x_m \rangle$, if $\hat{x}_j$ is defined for $1 \leq j \leq m$, let $\hat{\vec{x}} = \langle \hat{x}_1, \ldots, \hat{x}_m \rangle$. In particular, let $\hat{F}_q$ be the chain with possible repetitions of length $|F_q| : \langle \hat{x} : x \in F_q \rangle$. Similarly define F_q^l and $\hat{F}_q^l$ for $l \in L = \{l : \operatorname{dom}_l f_q \neq 0\}$. We can make analogous definitions of $\hat{F}_r$ etc. for an arbitrary $\langle r, t_r\rangle$ extending $\langle q_M, u_1\rangle$. Let $c = \text{length } \hat{F}_q = |F_q|$ and $c_l = \text{length } \hat{F}^l_q = |F^l_q|$. We use sequence notation and chains with possible repetitions to avoid losing information when we pass from F_q to $\hat{F}_q$.

The intent of the next three paragraphs is to define *in M* the set of all conditions in $\mathcal{D}$ that "look just like $\langle q, t\rangle$".

Let $\mathcal{D}_0 = \{\langle r, t_r\rangle \in \mathcal{D} : \langle r, t_r\rangle \leq \langle q_M, u_1\rangle$ and

i) q_M is an *initial part* of r, i.e. for each l, $\operatorname{dom}_l f_{q_M}$ is an initial segment of $\operatorname{dom}_l f_r$, and $\mathfrak{N}_{q_M}$ is an initial segment of $\mathfrak{N}_r$,

ii) the height of each node in $F_r - F_q$ is $> \delta_M$,

iii) $L_r = L$, each $|F^l_r| = c_l$, $|F_r| = c$,

iv) $f_r(\text{the } j\text{th element of } F_r) = f_q(\text{the } j\text{th element of } F_q)$,

v) the height of t_r is greater than the height of any of the nodes in dom $f_r\}$.

The above requirements will ensure that the natural correspondence between r and q induces a natural correspondence of F_r and $\hat{F}_r$ to F_q and $\hat{F}_q$ respectively.

Notice that the u_i's and hence the σ_{ij}'s are in M, and so $\mathcal{D}_0 \in M$ by definability. Clauses iii) and iv) do not violate definability, since c and the c_l's are just natural numbers and so are in M. Similarly, the range of f_q is just a finite subset of ω, so we could rewrite iv) using specific natural numbers.

$$\mathcal{F} = \left\{ F \in S^c : F = \hat{F}_r \text{ for some } \langle r, t_r\rangle \in \mathcal{D}_0 \right\},$$

and

$$\mathcal{F}_l = \left\{ F \in S^{c_l} : F = \hat{F}^l_r \text{ for some } \langle r, t_r\rangle \in \mathcal{D}_0 \right\}$$

are also in M and in H_κ as well.

Since $M \cap H_\kappa \in N$ for each $N \in \mathfrak{N}_q - M$, it follows that $\mathcal{F}$ and $\mathcal{F}_l \in N$, for all such N. Note that $\hat{F}_q \in \mathcal{F}$ and $\hat{F}^l_q \in \mathcal{F}_l$, since $\langle q, t\rangle \in \mathcal{D}_0$. Note also that the terms of $\hat{F}^l_q$ are separated by models of $\mathfrak{N}_q$. To see this, recall t is not in the largest model of $\mathfrak{N}_q$, which does contain all the members of dom f_q. If $\hat{x}, \hat{x}'$ are terms of $\hat{F}^l_q$, then there is an $N \in \mathfrak{N}_q$ such that $x \in N$ and $x' \notin N$. Then $\hat{x} \in N$, and $\hat{x}' \notin N$, else $x' \in N$. $N \notin M$, so the σ_{ij}'s $\in N$.

Our plan is to reflect $\langle q, t\rangle$ to an $\langle r, t_r\rangle \in \mathcal{D}_0 \cap M$ by using elementarity to systematically reflect the members of F_q down into M. Our topological hypotheses will be used to obtain such a reflection which is also compatible with $\langle q, t\rangle$. Let $\mathcal{N}'_q$ be a minimal subchain of $\mathfrak{N}_q$ containing $M \cap H_\kappa$ at its bottom and separating $\hat{F}^l_q$ for each l. Let $\mathcal{N}'_q = \{N_a\}_{a\leq m-1}$ ordered by inclusion, with $N_0 = M \cap H_\kappa$. $\hat{F}_q$ is a chain with possible repetitions; let us write it as:

$$\langle \hat{\vec{x}}_1, \ldots, \hat{\vec{x}}_{m-1}, t\rangle$$

where $\hat{\vec{x}}_a = \langle \hat{x}_{a,1}, \ldots, \hat{x}_{a,d_a}\rangle$ enumerates in increasing order $\hat{F}_q \cap (N_a - N_{a-1})$, $a \geq 1$. Thus the length of the vector $\vec{x}_a$ is equal to the size of $F_q \cap (N_a - N_{a-1})$. Since $\mathcal{F} \in N_{m-1}$,

$$\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-1}) = \big\{x \in S : \langle \hat{\vec{x}}_1, \ldots, \hat{\vec{x}}_{m-1}, x\rangle \in \mathcal{F}\big\}$$

$\in N_{m-1}$. By Lemma 31, there is a $y_m \in N_{m-1} \cap S$, $y_m \in [\max \hat{\vec{x}}_{m-1}, t)$, such that $\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-1})$ is dense above y_m. Next, consider:

$$\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2}) = \big\{\langle \vec{x}, y\rangle \in S^{d_{m-1}+1} : \langle \vec{x}, y\rangle \text{ is a chain with possible repetitions and } \mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2}, \vec{x}) \text{ is dense above } y\big\}.$$

Then $\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2}) \in N_{m-2}$ and $\langle \hat{\vec{x}}_{m-1}, y_m\rangle \in \mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2})$. As before, this time by Corollary 32, with $\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2}), \langle \hat{\vec{x}}_{m-1}, y_m\rangle, N_{m-2}$ playing the roles of $\mathcal{A}, A_0, N$ respectively, we can find a $y_{m-1} \in N_{m-2} \cap S$, $y_{m-1} \in [\max \hat{\vec{x}}_{m-2}, \min \hat{\vec{x}}_{m-1})$, such that $\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2})$ is dense above y_{m-1}. Continuing, in m steps we find a $y_1 \in N_0$, $y_1 \in [u_1, v_1)$, such that:

$$\mathcal{F}(\emptyset) = \big\{\langle \vec{x}, y\rangle \in S^{d_1+1} : \langle \vec{x}, y\rangle \text{ is a chain with possible repetitions and } \mathcal{F}(\vec{x}) \text{ is dense above } y\big\}$$

is $\in N_0$ and dense above y_1.

One of the virtues of Souslin-tree forcing is that, roughly speaking, for each countable piece of information about the final model, there is a level such that all nodes of the tree at that level decide that information. Thus, club often, all nodes of the tree decide what's going on below, e.g. whether or not the point α is in the open set $\dot{U}_\beta$ about β for α, β's below that level. It is therefore convenient to use such a club in the proofs, thereby getting a locally countable subspace is σ-discrete on a club, or a closed discrete subspace of size $\aleph_1$ is separated on a club. By working harder, one can avoid this annoyance – see [4].

Let $\dot{\mathcal{X}}_1$ be a name for:

$$\Big\{ \langle \alpha_1, \ldots, \alpha_{d_1} \rangle \in (C^\circ)^{d_1} : \text{for some } \langle \vec{z}, w \rangle \in \mathcal{F}(\emptyset), \{\vec{z}, w\} \subseteq B \text{ and} \\ \text{for each } i, 1 \le i \le d_1, ht(z_i)^- = \alpha_i \Big\}.$$

Then $\dot{\mathcal{X}}_1 \in M$. Let $\dot{\mathcal{X}}_1'$ be a name for

$$\{\vec{\xi} \in \mathcal{X}_1 : \min \vec{\xi} > \delta_M\}.$$

Claim: $y_1 \Vdash \dot{\mathcal{X}}_1' \neq 0$.

Proof. Given any y_1' extending y_1, since $\mathcal{F}(\emptyset)$ is dense above y_1, we can find a $\langle \vec{z}, w \rangle \in \mathcal{F}(\emptyset)$ with minimal element of height greater than δ_M extending y_1'. Take y_1'' above $\langle \vec{z}, w \rangle$. Then $y_1'' \Vdash \langle ht(z_1)^-, \ldots, ht(z_{d_1})^- \rangle \in \dot{\mathcal{X}}_1'$. □

There is a level of height greater than δ_M at which all extensions of y_1 at that level decide a $\dot{\vec{\xi}}$ which y_1 forces to be a member of $\mathcal{X}_1'$ to be some $\vec{\xi}$ and also decide a corresponding $\langle \vec{z}, w \rangle(\vec{\xi})$. Let μ_1 be the sup of the components of these countably many $\vec{\xi}$'s and repeat the process, extending each of the aforementioned extensions of y_1 to a level of height greater than μ_1, deciding $\vec{\xi}$ as before, but with the minimal component of such $\vec{\xi}$'s greater than μ_1. Continuing, we form a subtree of height ω of the cone above y_1, such that each element of each level of the subtree decides $\dot{\vec{\xi}} \in \dot{\mathcal{X}}_1$ and a corresponding $\langle \vec{z}, w \rangle(\vec{\xi})$, and such that the $\vec{\xi}$'s of one level all have minimal components greater than the maximal components of the $\vec{\xi}$'s of the previous level.

By elementarity, there is such a subtree in M. Therefore the sup ζ of the heights in S of the elements of the subtree is less than δ. We can thus take $\{y_{1,j} : j < \omega\}$ strictly ascending below v_1, all of height less than ζ, and associated strictly increasing $\vec{\xi}_j^1$'s and their corresponding $\langle \vec{z}_j^1, w_j^1 \rangle$'s, with $\langle \vec{z}_j^1, w_j^1 \rangle \in \mathcal{F}(\emptyset) \cap M$ and $y_{1,j} \Vdash \langle \vec{z}_j^1, w_j^1 \rangle \subseteq \dot{B}$, and $\pi_d(\vec{\xi}_j^1) = ht(\pi_d(\vec{z}_j^1))^-$, where $\pi_d(\vec{\xi}_j^1)$ ($\pi_d(\vec{z}_j^1)$) is its d-th component.

Now, finally, we want to look at some specific partial orders. The first one will show that:

Theorem 33. *PFA(S)[S] implies that if $\{x_\alpha : \alpha < \omega_1\}$ is a right-separated subspace of a compact space Z with finite products Fréchet-Urysohn, then there is a club $C \subseteq \omega_1$ such that $\{x_\alpha : \alpha \in C\}$ is σ-discrete.*

Refinement of the argument which we won't get into will enable us to

a) replace "C" by "ω_1",

b) replace "right-separated" by "locally countable",

c) replace "finite products Fréchet-Urysohn" by "sequential", and finally,

d) replace "sequential" by "countably tight".

In order to get the σ-closed-discrete version of $\sum$, we seem to need a somewhat different partial order [4].

To prove the Theorem, we have S-names $\dot{Z}$, $\dot{U}_\alpha$, $\alpha < \omega_1$, such that S forces $\dot{Z}$ is such a space and:

i) $\alpha \in \dot{U}_\alpha$, which is open,

ii) $\beta < \alpha$ implies $\alpha \notin \bigcup \left\{ \overline{\dot{U}}_\beta : \beta < \alpha \right\}$.

Let C be a closed unbounded subset of ω_1 such that for each $\delta \in C$, every node of the δth level of S decides all statements of the form $\alpha \in \overline{\dot{U}}_\beta$. To see that there is such a club, note that we may take a maximal antichain A deciding $\alpha \in \overline{\dot{U}}_\beta$. Since A is countable, we can choose $h(\alpha, \beta) < \omega_1$ above $\sup\{\mathrm{ht}(a) : a \in A\}$. Let C be closed unbounded such that $h(\alpha, \beta) < \delta$ whenever $\beta, \alpha < \delta \in C$. Let $C^\circ = \{\delta \in C : sup(C \cap \delta) < \delta\}$. For $\delta \in C^\circ$, let $\delta^- = sup(C \cap \delta)$. Note that every member of C is a δ^- for some $\delta \in C^\circ$. For $\delta \in C$, let δ^+ be the least element of C greater than δ.

Let $\mathcal{P}$ be the collection of all pairs $p = \langle f_p, \mathcal{N}_p \rangle$ where:

1) f_p is a finite partial function from $S \,|\, C^\circ$ to ω. Let $\mathrm{dom}_l\, f_p = \{s : f_p(s) = l\}$. We require that each non-empty $\mathrm{dom}_l\, f_p$ consists of nodes of different heights.

2) $\mathfrak{N}_p$ is a finite $\in$-chain of countable elementary submodels of H_κ where κ is regular and sufficiently large, containing all relevant objects, such that $\mathfrak{N}_p$ separates each $\mathrm{dom}_l\, f_p$ in the sense that if $s, s' \in \mathrm{dom}_l\, f_p$ with $s \neq s'$, then there is an $N \in \mathfrak{N}_p$ such that $s \in N$ and $s' \notin N$.

3) If $s, s' \in \mathrm{dom}_l\, f_p$ and s' strictly extends s and $\mathrm{ht}(s') = \tau$ and $\mathrm{ht}(s) = \sigma$, then $s' \Vdash \sigma^- \notin \overline{\dot{U}}_{\tau^-}$.

The rationale for the "if s extends s'" clause is that we are coding the σ-discrete subspace by a generic branch, and don't care what happens off that branch. The superscript minuses are there because we only expect conditions of height α to know about things of smaller index.

It is now routine to show that this partial order does what it is intended to do. To show that the partial order is proper and preserves S, note that by compactness, $\mathcal{X}_1$, as a subspace of a finite power of Z, has a complete accumulation point x; by right-separation, x does not project to any of the x_α's. By Fréchet-Urysohn, there is a sequence $\{x_{\alpha_n}\}_{n<\omega}$ from $\mathcal{X}_1$ which converges to x. Since the projections of x are not in any of the $\overline{U}_\alpha$'s for s's of height α in the condition we are trying to get away from, for n sufficiently large, x_{α_n} will not be in them either. The endgame for this proof is almost identical to the CWH proof to which we now detour.

As mentioned earlier, in order to prove the CWH result, it will suffice to expand the points in a club $C \subseteq \omega_1$ to compact G_δ's which are σ-left-separated by the right-separating U's. We shall do this by simultaneously both approximating a countable partition of ω_1 by finite partial functions from ω_1 into ω and approximating finitely many of the desired compact G_δ's by finite decreasing sequences of compact G_δ's.

From now on, we assume PFA(S). We have an S-name $\dot{Z}$, such that S forces $\dot{Z}$ is a locally compact normal space. It is convenient to assume that $\{\alpha : \alpha < \omega_1\}$ is a closed discrete subspace of Z. We shall usually omit the "ˇ" that should be placed over elements of the ground model. Let $\dot{\mathcal{E}}$ be a name such that S forces $\dot{\mathcal{E}}$ to be the collection of non-empty compact G_δ's of $\dot{Z}$. We shall assume that for each $\alpha < \omega_1$, we have S-names $\dot{U}_\alpha$, $\dot{K}_\alpha$, $\dot{K}_{\alpha,\beta}$, $\beta < \alpha$, such that S forces:

i) $\alpha \in \dot{U}_\alpha$,

ii) $\dot{U}_\alpha$ is open, $\overline{\dot{U}}_\alpha$ is compact,

iii) $\alpha \neq \beta$ implies $\alpha \notin \overline{\dot{U}}_\beta$,

iv) $\alpha \in \dot{K}_\alpha \subseteq \dot{U}_\alpha$,

v) $\dot{K}_\alpha \in \dot{\mathcal{E}}$, $\dot{K}_{\alpha,\beta} \in \dot{\mathcal{E}}$,

vi) $\beta < \alpha$ implies $\dot{K}_\alpha \cap \dot{\overline{U}}_\beta = 0$

vii) for each α, $\{\dot{K}_{\alpha,\beta} : \beta < \alpha\} \subseteq \dot{\mathcal{E}}$ is discrete, with $\beta \in \dot{K}_{\alpha,\beta} \subseteq \dot{K}_\beta$, and if $\alpha < \gamma$, then $\dot{K}_{\gamma,\beta} \subseteq \dot{K}_{\alpha,\beta}$.

vii) is easy to accomplish: discretely separate $\{\beta : \beta < \alpha\}$, shrink the separating open sets to compact G_δ's, and then intersect with the corresponding K_β's. We then can recursively shrink the compact G_δ's to get $K_{\gamma,\beta} \subseteq K_{\alpha,\beta}$. That is, having gotten say the discrete collection $\{K'_{\gamma,\beta} : \beta < \gamma\}$, let $K_{\gamma,\beta} = K'_{\gamma,\beta} \cap \bigcap\{K_{\alpha,\beta} : \alpha < \gamma\}$.

Let C be a closed unbounded subset of ω_1 such that for each $\delta \in C$, every node of the δth level of S decides all statements of form $\dot{K}_{\gamma,\beta} \cap \dot{\overline{U}}_\alpha = 0$ for all $\beta < \gamma \leq \alpha < \delta$.

Let $\mathcal{P}$ be the collection of all triples $p = \langle f_p, \mathcal{E}_p, \mathfrak{N}_p\rangle$ where:

1) f_p is a finite partial function from $S \,|\, C^\circ$ to ω. Let $\operatorname{dom}_l f_p = \{s : f_p(s) = l\}$. We require that each non-empty $\operatorname{dom}_l f_p$ consists of nodes of different heights.

2) $\mathfrak{N}_p$ is a finite $\in$-chain of countable elementary submodels of H_κ where κ is regular and sufficiently large, containing all relevant objects, such that $\mathfrak{N}_p$ separates each $\operatorname{dom}_l f_p$ in the sense that if $s, s' \in \operatorname{dom}_l f_p$ with $s \neq s'$, then there is an $N \in \mathfrak{N}_p$ such that $s \in N$ and $s' \notin N$.

3) $\mathcal{E}_p$ is a finite partial function from $\omega \times S \,|\, C^\circ$ to ω_1 such that, letting π_2 be the projection map from $\omega \times S \,|\, C^\circ$ onto $S \,|\, C^\circ$,

   a) $\pi_2[\operatorname{dom} \mathcal{E}_p] = \operatorname{dom} f_p$

   b) $\mathcal{E}_p(n, s) \geq ht(s)$,

   c) whenever $s \in N \in \mathfrak{N}_p$, $\mathcal{E}_p(n, s) \in N$,

   d) if $s, s' \in \operatorname{dom}_l f_p$ and s' strictly extends s and $ht(s') = \tau$, then

$$s' \Vdash \bigcap\{\dot{K}_{\mathcal{E}_p(n,s),ht(s)^-} : \langle n, s\rangle \in \operatorname{dom} \mathcal{E}_p\} \cap \dot{\overline{U}}_{\tau^-} = 0$$

For $p, q \in \mathcal{P}$, we let $p \leq q$ if and only if:

4) $f_p \,|\, \operatorname{dom} f_q = f_q$,

5) $\mathcal{E}_p \,|\, \operatorname{dom} \mathcal{E}_q = \mathcal{E}_q$.

6) $\mathfrak{N}_p \supseteq \mathfrak{N}_q$.

Clause 3c) will ensure that the restriction of a condition p to $\mathfrak{N}$ will be a member of $\mathfrak{N}$.

Lemma 34. *Let $D_s = \{p \in \mathcal{P} : s \in \operatorname{dom} f_p\}$. Let $D_{s,n} = \{p \in \mathcal{P} : \langle n, s\rangle \in \operatorname{dom} \mathcal{E}_p\}$. Then for each $s \in S \mid C^\circ$, and each $n < \omega$, D_s and $D_{s,n}$ are dense.*

Proof. Given any $q \in \mathcal{P}$, if $q \notin D_s$, take $m > \max\{f_q(t) : t \in \operatorname{dom} f_q\}$. Then $\langle f_q \cup \{\langle s, m\rangle\}, (\mathcal{E}_q \cup \{\langle\langle 0, s\rangle, ht(s)\rangle\}, \mathfrak{N}_q\rangle$ is the required extension of q in D_s. Given $q \in D_s - D_{s,n}$, suppose k is least such that $\langle k, s\rangle \in \operatorname{dom} \mathcal{E}_q$. Let $q' = \langle f_q, \mathcal{E}_q \cup \{\langle\langle n, s\rangle, \mathcal{E}_q(k, s)\rangle\}, \mathfrak{N}_q\rangle$. Then q' is $\leq q$ and is a member of $D_{s,n}$. $\square$

Lemma 35. *$PFA(S)[S]$ implies that C has a σ-left-separated, right-separated expansion by compact G_δ's, and hence a discrete expansion by compact G_δ's.*

Proof. Let G be $\mathcal{P}$-generic for the D_s's and the $D_{s,n}$'s. Let $f = \bigcup\{f_p : p \in G\}$. Let $e = \bigcup\{\mathcal{E}_p : p \in G\}$. Then $e : \omega \times S \,|\, C^\circ \to \omega_1$. For $\gamma = ht(s)^-$, $s \in B \,|\, C^\circ$, where B is the generic branch of S, let $E_\gamma = \bigcap\{K_{e(n,s),\gamma} : n < \omega\}$. Then S forces that $\{E_\gamma : \gamma \in C\}^{\cdot}$ is the required right-separated, σ-left-separated expansion of C by compact G_δ's. $\square$

Since for $x \in \operatorname{dom} f_q - M$, $ht(x) \geq \delta$, such x decides whether or not $\dot{K}_{\zeta,\pi_d(\vec{\xi}^1_j)^-}$ meets $\dot{\overline{U}}_{ht(x)^-}$. Since $\overline{U}_{ht(x)^-}$ is compact and for fixed d the $\pi_d(\vec{\xi}^1_j)^-$'s are distinct, there is a $j_x \in \omega$ such that for each $d \leq d_1$, x forces:

$$(\dagger) \qquad \bigcup\left\{\dot{K}_{\zeta,\pi_d(\vec{\xi}^1_j)^-} : j \geq j_x\right\} \cap \dot{\overline{U}}_{ht(x)^-} = 0.$$

For the $\sum$ proof, we replace this by $\pi_d(\vec{\xi}^1_j)^- \notin \overline{U_h t(x)^-}$. To see this, note that x certainly forces that there is such a j_x. Then for some j_x, some extension of x forces ($\dagger$). But then x must have already forced this, since it had decided whether $\dot{K}_{\zeta,\pi_d(\vec{\xi}^1_j)^-}$ met $\dot{\overline{U}}_{ht(x)^-}$.

Let $j_1 = \max\{j_x : x \in \operatorname{dom} f_q - M\}$. Let $\mathbf{z}_{1,d}$ be the element of height $\pi_d(\vec{\xi}^1_{j_1})^+$ below x_d, for $x_d \in F_q \cap (N_1 - N_0)$. Let $\vec{\mathbf{z}}_1 = \langle \mathbf{z}_{1,1}, \ldots, \mathbf{z}_{1,d_1}\rangle$. Let $\mathbf{w}_1 = w^1_{j_1}$. Then $\langle \hat{\vec{\mathbf{z}}}_1, \mathbf{w}_1\rangle = \langle \vec{z}^1_{j_1}, w^1_{j_1}\rangle \in \mathcal{F}(\emptyset)$ and for all $x \in \operatorname{dom} f_q - M$, x forces $\dot{K}_{\zeta,ht(\mathbf{z}_{1,d})^-} \cap \dot{\overline{U}}_{ht(x)^-} = 0$, for all $d \leq d_1$. Notice that $\langle \vec{\mathbf{z}}_1, \mathbf{w}_1\rangle \in M$.

We now need to iteratively peel off the remaining "layers" of F_q. Let $\dot{\mathcal{X}}_2$ be a name for:

$$\Big\{\langle\alpha_1,\ldots,\alpha_{d_2}\rangle \in (C^\circ)^{d_2} : \text{ for some } \langle\vec{z},w\rangle, \langle\hat{\vec{z}},w\rangle \in \mathcal{F}(\vec{\mathbf{z}}_1), \{\vec{z},w\} \subseteq B \\ \text{and for each } i, 1 \le i \le d_2, ht(z_i)^- = \alpha_i\}.$$

We now carry out the same argument as before, with an infinite strictly ascending sequence of $y_{2,j}$'s below v_1 extending y_{1,j_1} and deciding $\vec{\xi} \in \mathcal{X}_2$, where $\min\vec{\xi} > \max\vec{\xi}^1_{j_1}$. As before, we obtain a $\vec{\mathbf{z}}_2 \in M$, each $\mathbf{z}_{2,d}$ below $x_d \in F_q \cap (N_2 - N_1)$, and with each $ht(\mathbf{z}_{2,d}) > ht(\mathbf{z}_{1,d_1})$, such that for each $x \in \operatorname{dom} f_q - M$, x forces $\dot{K}_{\zeta,ht(\mathbf{z}_{2,d})^-} \cap \overline{U}_{ht(x)^-} = 0$, for all $d \le d_2$.

Continuing, after m steps we will find $\langle\hat{\vec{\mathbf{z}}}_1,\ldots,\hat{\vec{\mathbf{z}}}_m,\mathbf{w}_1\rangle \in \mathcal{F}$, each component of each $\vec{\mathbf{z}}_a$ below some v_i, and hence in M. Since $\langle\hat{\vec{\mathbf{z}}}_1,\ldots,\hat{\vec{\mathbf{z}}}_m,\mathbf{w}_1\rangle \in \mathcal{F}$, there is an $\langle r,t_r\rangle \in \mathcal{D}_0 \cap M$ such that $\hat{F}_r = \langle\hat{\vec{\mathbf{z}}}_1,\ldots,\hat{\vec{\mathbf{z}}}_m,\mathbf{w}_1\rangle$. Then $\mathbf{w}_1 = t_r$. Now $\mathbf{w}_1 = w^1_{j_1}$ is below y_{1,j_1}, since otherwise y_{1,j_1}, could not force it to be in B. Therefore it is below v_1 and so $t_r \le t$. We claim that $\langle r,t_r\rangle$ is compatible with $\langle q,t\rangle$, which will finish the proof.

Since $r \le q_M$, it follows that $f_r \cup f_q$ is a function. Let $\mathcal{E}_{r,q} = \mathcal{E}_r \cup \mathcal{E}_q \mid (\omega \times (\operatorname{dom} f_q - \operatorname{dom} f_r)) \cup \{\langle\langle n_{i,d}+1,\mathbf{z}_{i,d}\rangle,\zeta\rangle : \mathbf{z}_{i,d} \in \operatorname{dom} f_r\}$, where $n_{i,d}$ is the maximal integer such that $\langle n_{i,d},\mathbf{z}_{i,d}\rangle \in \operatorname{dom}\mathcal{E}_r$. Then $\mathcal{E}_{r,q}$ satisfies 3c) in the definition of $\mathcal{P}$.

We next note that $\mathfrak{N}_r \cup \mathfrak{N}_q$ is an $\in$-chain, for by construction, $\mathfrak{N}_r \in M$, so $\mathfrak{N}_r \cup \{M \cap H_\kappa\}$ is an $\in$-chain. Now $\mathfrak{N}_q = \mathfrak{N}_{q_M} \cup (\mathfrak{N}_q - \mathfrak{N}_{q_M})$; the elements N of $\mathfrak{N}_q - \mathfrak{N}_{q_M}$ all have $M \cap H_\kappa$ in them, for if not, such an N would be in M. $\mathfrak{N}_r \cup \mathfrak{N}_q$ is thus the $\in$-chain $\mathfrak{N}_r \cup \{M \cap H_\kappa\} \cup (\mathfrak{N}_q - \mathfrak{N}_{q_M})$.

Let $\mathcal{R} = \langle\langle f_r \cup f_q, \mathcal{E}_{r,q}, \mathfrak{N}_r \cup \mathfrak{N}_q\rangle, t\rangle$.

Since $\operatorname{dom} f_r \subseteq M$ and $r \le q_M$, each $\operatorname{dom}_l(f_r \cup f_q)$ consists of nodes of different heights. Suppose b, $c \in \operatorname{dom}_l(f_r \cup f_q)$. The only case of interest is when $b \in \operatorname{dom}_l f_r$ and $c \in \operatorname{dom}_l f_q$. If $c \in M$, then $c \in \operatorname{dom}_l f_{q_M}$ and the members of $\mathfrak{N}_r$ separate b and c since $r \le q_M$. If $c \notin M$, then an $N \in \mathfrak{N}_r$ containing b will not contain c, since $N \subseteq M$. To finish showing that the first component of $\mathcal{R}$ is a condition, suppose $s' \in \operatorname{dom}_l f_q$, $s \in \operatorname{dom}_l f_r$, and s' extends s. If $s \in \operatorname{dom}_l f_{q_M}$, this is trivial, so suppose $s \in \operatorname{dom}_l f_r - \operatorname{dom}_l f_{q_M}$. Since s' extends s and also extends some v_i, it follows that v_i extends s, which then extends u_i, since $ht(s) > \delta_M$. Then u_1 is below $\sigma_{i1}(s)$ which is below v_1 which is below t. Then s is below $\sigma_{1i}(t)$. But then $s \in F_r$. By construction then, 3d) of the definition of $\mathcal{P}$ is satisfied, so indeed $\langle f_r \cup f_q, \mathcal{E}_{r,q}, \mathfrak{N}_r \cup \mathfrak{N}_q\rangle \in$

$\mathcal{P}$ and is below both r and q. But then $\mathcal{R} \in \mathcal{P} \times S$ is below both $\langle r, t_r \rangle$ and $\langle q, t \rangle$ as required. □

The proof that *PFA*$(S)[S]$ *implies compact countably tight spaces are sequential* [24] has only recently become available. Todorcevic's original proof of the σ-discrete form of $\sum$ used this; however I found a proof of $\sum$ which used a weaker version of Moore-Mrowka, which was in Larson's notes [12] on Todorcevic's Erice lectures. To state this weaker version, I must temporarily abandon my treatment of PFA$(S)[S]$ as if it were an axiom.

Theorem 36. *Assume* PFA(S). *Let* $\mathcal{U}$ *be an ultrafilter on* ω. *Then* S *forces that if* X *is compact countably tight, and* $Y \subseteq X$ *is neither Lindelöf nor closed, then there is a sequence in* Y *that* $\mathcal{U}$*-converges to a point in* $X - Y$.

In other words, if X is countably tight, then X is $\mathcal{U}$-sequential for each ground model $\mathcal{U}$. We will see how to use this later, and give some idea of its proof.

An important notion introduced by Arhangel'skiĭ is that of a *free sequence.*

Definition. *$\langle x_\alpha : \alpha < \lambda \rangle$ is free if for each $\beta < \lambda$, $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \beta \leq \alpha < \lambda\}} = 0$.*

The canonical example of a free sequence is ω_1. Note that closed discrete sequences are free, and uncountable free sequences include uncountable discrete sequences. Arhangel'skiĭ proved that:

Lemma 37. *A compact Hausdorff space is countably tight if and only if it has no uncountable free sequences.*

The key idea in the proofs involving tightness and sequentiality is to set up a partial order such that the generic branch of S will then code an uncountable free sequence, just as in our previously presented proof sketches S codes a compact G_δ expansion or a σ-discretizing function. These proofs are easier in the respect that they do not need the σ-aspect, but harder in that in addition to the general machinery we have laid out, they also – as in the corresponding PFA proofs – involve working with filters.

Todorcevic then goes on (in his preprint [24]) to prove that if the space is $\mathcal{U}$-sequential for all such $\mathcal{U}$, then it's actually sequential. However the easier halfway result is enough to get $\sum$: we use it in the form that a non-closed set is not sequentially closed to do our usual trick of getting a sequence of candidates, almost all of which stay away from the condition we want to avoid. See [4].

To prove Theorem 36, one sets up a partial order that forces an uncountable free sequence – contradicting countable tightness. We then prove the partial order is proper and preserves S by showing that if there were a non-closed subset X which was sequentially closed, X would be countably compact, so not Lindelöf, else it would be compact and closed. We then have a sequence going from X to a point outside of X, and so can do the usual tricks.

I will mention 2 questions of interest:

1) (Todorcevic) Does PFA(S)[S] imply there are no S-spaces?

2) Is there a model of PFA(S)[S] in which every first countable perfect pre-image of ω_1 includes a copy of ω_1?

Conjecture 1. *MM(S)[S] implies every first countable perfect pre-image of ω_1 includes a copy of ω_1.*

If so, by unpublished work of mine, MM(S)[S] implies every T_5 manifold of dimension > 1 is metrizable, which would answer a 1983 question of Nyikos [18].

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