EMBEDDING FUNCTION SPACES INTO ℓ_∞/c_0

STEVO TODORCEVIC

ABSTRACT. We show that if the space ℓ_∞/c_0 contains an isometric copy of every function space over a first countable compactum or every function space over a Corson copactum of weight not exceeding the continuum then every subset of $\mathbb{R}^2$ belongs to the σ-field generated by sets of the form $A_1 \times A_2$. We prove a similar result about isomorphic rather than isometric embeddings into ℓ_∞/c_0 in terms of the σ-field of subsets of $\mathbb{R}^k$ generated by sets of the form $A_1 \times \cdots \times A_k$ for other positive integers k.

1. Introduction

Fix an infinite index set S. Let $\mathcal{R}(S^2)$ be the σ-field of subsets of S^2 generated by 'rectangles', i.e., the sets of the form $A \times B$ for $A, B \subseteq S$. It is a classical problem of set theory[1] to determine for which index sets S do we have that $\mathcal{R}(S^2)$ includes *all* subsets of S^2. Clearly, this depends only on the cardinality of the set S. From a classical result of Rothberger [7] it follows easily that this is the case for every set S of cardinality at most $\aleph_1$, though this result appears explicitly only in the work of Kunen [4] and Rao [6] from the late 1960's. The purpose of this note is to connect this problem with a classical problem from the geometry of Banach space which ask to which extent is the space ℓ_∞/c_0 universal in the class of Banach spaces of density at most continuum. Recall, for example, the classical result of Parovichenko [5] stating that the function space $C(K)$ over an arbitrary compactum of weight at most $\aleph_1$ is isometric to a subspace of ℓ_∞/c_0. So it is natural to ask if some structural conditions on K rather than the restriction on weight would guarantee the isometric or isomorphic embedding of $C(K)$ into ℓ_∞/c_0. We show that if ℓ_∞/c_0 contains the isometric copy of every function space of the form $C(K)$ for K either a first countable compactum or a Corson compactum of weight at most continuum then the σ-field of subsets of $\mathbb{R}^2$ generated by rectangles contains all subsets of $\mathbb{R}^2$. We prove a similar result about the isomorphic rather than isometric universality of the space ℓ_∞/c_0 in terms of the σ-field of subsets $\mathbb{R}^k$ generated by sets of the form $A_1 \times \cdots \times A_k$ for other

[1]Due originally to S.M. Ulam (Problème No. 74, Fund. Math. 30 (1938), 365).

positive integers k. As it is well known the universality of ℓ_∞/c_0 cannot be demonstrated on the basis of the usual axioms of set theory. So, as expected, all the properties of the σ-field of subsets of $\mathbb{R}^2$ or, more generally, of $\mathbb{R}^k$ that we mention here cannot be demonstrated on the basis of the usual axioms of set theory as well. In fact, as first shown by Kunen [4], all the consequences about the σ-fields generated by rectangles that we obtain here fail when more that continuum many Cohen reals are added to a model of set theory.

We use standard set-theoretic terminology as well as the standard terminology about Banach spaces which the reader can find in many textbooks devoted to these areas. The elements of quotient space ℓ_∞/c_0 will be treated in terms of their representatives in ℓ_∞. This has some advantages since ℓ_∞ carries the separable metric topology induces from the power $\mathbb{R}^{\mathbb{N}}$. For example, if we let $\| \cdot \|$ denote the norm of ℓ_∞/c_0 then for a fixed real number M the condition like $\| x + y \| > M$ gives us an $F_{\sigma\delta}$-subset of $\ell_\infty \times \ell_\infty$ equipped with the separable metric topology. Now note that for every index set S and every separable metric topology τ on S every Borel subset of $(S,\tau)^k$ belongs to the σ-field generated by sets of the form $A_1 \times \cdots \times A_k$.

2. First Countable Compacta

Fix a set $E \subseteq [I]^2$ of unordered pairs of the unit interval $I = [0,1]$. For an integer $k \geq 2$, set

$$E^{[k]} = \{\{a_1, ..., a_k\} \in [I]^k : \{a_i, a_j\} \in E \text{ for all } i < j\}.$$

Clearly $E^{[2]} = E$. For every integer $k \geq 2$, we define a topology on $X_{E^{[k]}} = I^k \cup E^{[k]}$ by letting all points of $E^{[k]}$ isolated while a neighborhood of an $(a_1, ..., a_k) \in I^k$ is given by

$$\{(a_1, b_2, ..., b_k) : |b_i - a_i| < \varepsilon\} \cup \{\{a_1, b_2, ..., b_k\} \in E^{[k]} : |b_i - a_i| < \varepsilon\}$$

for some $\varepsilon > 0$. Then $X_{E^{[k]}} = I^k \cup E^{[k]}$ is a locally compact first countable space since for each $a \in I$, the set

$$D_a = (\{a\} \times I^{k-1}) \cup \{\{a_1, ..., a_k\} \in E^{[k]} : a_i = a \text{ for some } i\}$$

is an open compact first countable space being essentially homeomorphic to closed subspace of the Alexandroff duplicate of the power I^{k-1} with some isolated points removed. It turns out that $X_{E^{[k]}}$ has a first countable compactification $K_{E^{[k]}} = X_{E^{[k]}} \cup [I]^{\leq k}$ by adding to the family of neighborhoods of $X_{E^{[k]}}$ the following family of sets

$$\{V \cup (f^{-1}(V) \setminus C) : V \text{ open in } [I]^{\leq k}, C \text{ compact in} X_{E^{[k]}}\},$$

where $f : X_{E^{[k]}} \to [I]^{\leq k}$ is the continuous map defined by $f(a_1, ..., a_k) = \{a_1, ..., a_k\}$ and $f(\{a_1, ..., a_k\}) = \{a_1, ..., a_k\}$ and where we take $[I]^{\leq k}$ with the compact metric topology induced from the hyperspace $\exp(I)$. Recall that $\exp(I)$ is the collection of closed subsets of I with the topology given by the sets of the form $\langle F\rangle = \{H \in \exp(I) : H \subseteq F\}$ and $[F] = \{H \in \exp(I) : H \cap F \neq \emptyset\}$ for $F \in \exp(I)$ as subbasis for closed sets . The map f is finite-to-one and is continuous because

$$f^{-1}(\langle F\rangle \cap [I]^{\leq k}) = F^k \cup ([F]^k \cap E^{[k]})$$

and

$$f^{-1}([F] \cap [I]^{\leq k}) = \{\vec{a} \in I^k : a_i \in F \text{ for some } i\} \cup \{s \in E^{[k]} : s \cap F \neq \emptyset\}$$

are closed subsets of $X_{E^{[k]}}$ for every closed subset F of I. The case $k = 2$ of this construction is due to M. Bell [1] where the reader can also find a proof of a topological result due to Uljanov [10] that lies behind the idea that this kind of locally compact space might admit first countable compactifications.

2.1 Lemma. *Suppose that ℓ_∞/c_0 contains a subspace with Banach-Mazur distance to $C(K_{E^{[k]}})$ smaller than k. Then $E^{[k]}$ and $(E^c)^{[k]}$ can be separated by a set belonging to the σ-field of subsets of I^k generated by 'k-cubes' i.e., products of the form $A_1 \times \cdots A_k$.*[2]

Proof. Fix an isomorphic linear embedding $T : C(K_{E^{[k]}}) \to \ell_\infty/c_0$ such that $k >\| T \|\| T^{-1} \|$. For $a \in I$, let $1_a : K_{E^{[k]}} \to \{0,1\}$ be the characteristic function of the compact open subset D_a and let $x_a = T(1_a)$. Fix a k-sequence $a_1 < ... < a_k$ of elements of I. Then $\{a_1, ..., a_k\} \in E^{[k]}$ implies that $\{a_1, ..., a_k\} \in \bigcap_{i=1}^k D_{a_i}$, or in other words $\| 1_{a_1} + ... + 1_{a_k} \|_\infty \geq k$. It follows that

$$\| x_{a_1} + ... + x_{a_k} \| \geq k/ \| T^{-1} \| > \| T \| .$$

On the other hand, $\{a_1, ..., a_k\} \in (E^c)^{[k]}$ implies that $D_{a_i} \cap D_{a_j} = \emptyset$ for $i \neq j$ and so $\| 1_{a_1} + ... + 1_{a_k} \|_\infty = 1$. Then $\| x_{a_1} + ... + x_{a_k} \| \leq \| T \|$. Hence, we can separate $E^{[k]}$ from $(E^c)^{[k]}$ by a member of the σ-field of subsets of I^k generated by k-cubes. □

For a set $E \subseteq [I]^2$ of unordered pairs of elements of I, let K_E be the one-point compactification of the topological sum $\bigoplus_{k\geq 2} K_{E^{[k]}}$.

2.2 Corollary. *If $C(K_E)$ is isomorphic to a subspace of ℓ_∞/c_0 then for all but finitely many k the sets $E^{[k]}$ and $(E^c)^{[k]}$ are separated by a set belonging to the σ-field of subsets of I^k generated by k-cubes.*

[2] Here E^c denotes the complement of E, i.e, the set $[I]^2 \setminus E$, and we identify $E^{[k]}$ and $(E^c)^{[k]}$ with subsets of I^k obtained identifying a k-element subset of I with its increasing enumeration according to the usual ordering of I.

Note that since $E^{[2]} = E$, the condition that E and its complement E^c can be separated by a a set belonging to the σ-field generated by rectangles simply means that E itself belongs to that σ-field. So the following two results summarize what we have proved so far.

2.3 Theorem. *Suppose that ℓ_∞/c_0 contains an isometric copy of every function space $C(K)$ over a first countable compactum K. Then every subset of $\mathbb{R}^2$ belongs to the σ-field generated by rectangles.*

2.4 Theorem. *Suppose that ℓ_∞/c_0 contains an isomorphic copy of every function space $C(K)$ over every first countable compactum K. Then for every binary relation E on I and for all but finitely many positive integers k, the sets $E^{[k]}$ and $(E^c)^{[k]}$ are separated by a member of the σ-field of subsets of I^k generated by k-cubes.*

We finish this section with the following corollary of Theorem 2.3.

2.5 Theorem. *Suppose ℓ_∞/c_0 contains an isometric copy of every function space $C(K)$ over a first countable compactum K. Then $2^\theta = \mathfrak{c}$ for every infinite cardinal $\theta < \mathfrak{c}$ such that $\mathrm{cf}(2^\theta) \leq \mathfrak{c}$.*

Proof. It suffices to use the the existence of a cardinal $\theta < \mathfrak{c}$ such that $2^\theta > \mathfrak{c} \geq \mathrm{cf}(2^\theta)$ to produce a subset of $\mathbb{R}^2$ not in the σ-field generated by rectangles. Let $\lambda = \mathrm{cf}(2^\theta)$. Choose a subset C of $\mathbb{R}$ partitioned as $C = \bigcup_{\xi<\lambda} C_\xi$ into λ pairwise disjoint subsets C_ξ all of size θ. Choose an increasing decomposition $\mathcal{P}(C) = \bigcup_{\xi<\lambda} \mathcal{F}_\xi$ of the power set of C into families $\mathcal{F}_\xi$ of subsets of C such that $|\mathcal{F}_\xi| < 2^\theta$. Then for every $\xi < \theta$ the σ-field of subsets of $\mathbb{R}^2$ generated by rectangles of the form $A \times B$ with $A, B \in \mathcal{F}_\xi$ has cardinality $< 2^\theta$, so we can find $E_\xi \subseteq C_\xi^2$ not belonging to this σ-field. Let $E = \bigcup_{\xi<\lambda} E_\xi$. Then E is a subset of $\mathbb{R}^2$ not belonging to the σ-field generated by rectangles. □

The above proof is an adaptation of an argument of Rothberger [8] who was using the same assumption on cardinals for a similar purpose. Let us say that a family $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$ of real functions *has countable base* if there is a sequence $(g_n) \subseteq \mathbb{R}^\mathbb{R}$ of real unctions such that for every $f \in \mathcal{F}$ there is a subsequence $(g_{n_k}) \subseteq (g_n)$ such that $f(a) = \lim_{k\to\infty} g_{n_k}(a)$ for all $a \in \mathbb{R}$. Clearly every such family has cardinality not bigger than the continuum so a natural question[3] is to ask if the converse also holds. Note that in this problem we can restrict the range of functions to the set $\{0, 1\}$ rather than $\mathbb{R}$ and so this is really a part of a more general question asking for which index sets S which families $\mathcal{F}$ of two-valued functions defined on S have countable base. In [7], Rothberger showed that every family of real functions $\mathcal{F}$ of size at most $\aleph_1$ defined on an

[3]See, Fund. Math., vol. 27 (1936) p. 293, *problème de M. Sierpiński.*

index set S which is also of size at most $\aleph_1$ has a countable base. In [9], we have extended this result to index sets S of size $\mathfrak{p}$ and families of real function defined on S of sizes not bigger than $\mathfrak{p}$.[4] The following simple fact connects the two problems.

2.6 Lemma. *If every $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ of size continuum has a countable base then every subset of $\mathbb{R}^2$ belongs to the σ-field generated by rectangles.*

Proof. Fix a subset E of $\mathbb{R}^2$. For $a \in \mathbb{R}$ define $f_a : \mathbb{R} \to 2$ by: $f_a(b) = 1$ iff $(a, b) \in E$. Then $\mathcal{F} = \{f_a : a \in \mathbb{R}\}$ is a family of real functions of size at most continuum. Let $\{g_n : \mathbb{R} \to 2 : n \in \mathbb{N}\}$ be a countable base for the family $\mathcal{F}$. Then for each $a \in \mathbb{R}$, we can fix a strictly increasing sequence $(n_k^a) \subseteq \mathbb{N}$ such that $f_a(b) = \lim_{k\to\infty} g_{n_k^a}(b)$ for all $b \in \mathbb{R}$. Now, to each $n \in \mathbb{N}$, we assign the following two sets of reals

$$A_n = \{a : n = n_k^a \text{ for some } k\} \text{ and } B_n = \{b : g_n(b) = 1\}.$$

Then $E = \bigcap_m \bigcup_{n\geq m} A_n \times B_n$ and so E belongs to the σ-field of subsets of $\mathbb{R}^2$ generated by rectangles. □

3. Corson Compacta

In this section we consider embeddings of function spaces over Corson compacta. Recall that these are the compact subspaces of the Σ-powers $\{x \in [0,1]^\Gamma : |\{\gamma \in \Gamma : x(\gamma) \neq 0\}| \leq \aleph_0\}$. Of course we will need to restrict their weights not to exceed the continuum. Let us first examine isometric embeddings into ℓ_∞/c_0.

3.1 Theorem. *Suppose ℓ_∞/c_0 contains an isometric copy of every function space $C(K)$ over a Corson compactum K of weight at most continuum. Then every subset of $\mathbb{R}^2$ belongs to the σ-field generated by rectangles.*

Proof. We start first with the assumption that there is a well ordering $\prec$ of $\mathbb{R}$ which as a subset of $\mathbb{R}^2$ does not belong to the σ-field generated by rectangles. Let $<$ denote the usual ordering of $\mathbb{R}$. Let K be the collection of all subsets of $\mathbb{R}$ on which the two orderings $<$ and $\prec$ agree. Then K with the topology inherited from the Cantor cube $2^{\mathbb{R}}$ when naturally identified with the power-set of $\mathbb{R}$ is a Corson compact space. We shall show that $C(K)$ is not isometric to a subspace of ℓ_∞/c_0. Suppose that there is a linear operator $T : C(K) \to \ell_\infty/c_0$ such that $\| T \| = 1$. For $a \in \mathbb{R}$, let $1_a : K \to \{0, 1\}$ be defined by

$$1_a(A) = 1 \text{ iff } a \in A.$$

[4] Recall that $\mathfrak{p}$ is the minimal cardinality of a family $\mathcal{H}$ of infinite subsets of $\mathbb{H}$ such that every finite subfamily of $\mathcal{H}$ has infinite intersection but there is no single infinite subset b of $\mathbb{N}$ such that $b \setminus a$ is finite for all $a \in \mathcal{H}$.

Clearly $1_a \in C(K)$ for all $a \in \mathbb{R}$. Let $x_a = T(1_a)$. Then $\| x_a \| = 1$ for all $a \in \mathbb{R}$. Moreover, for two different elements a and b of $\mathbb{R}$, we have that

$$\| x_a + x_b \| > 1 \text{ iff } \| 1_a + 1_b \|_\infty > 1 \text{ iff } \{a, b\} \in K.$$

Since $\{a, b\} \in K$ holds if and only if the two orderings $<$ and $\prec$ agree on $\{a, b\}$ and since clearly $<$ belongs to the σ-field generated by rectangles, we conclude that $\prec$ belongs to that field as well, a contradiction.

To prove the general result about Corson compacta, we may assume that there is a well-ordering $\prec$ of $\mathbb{R}$ which belongs to the σ-field generated by rectangles. Fix a subset E of $\mathbb{R}$ that does not belong to the σ-field of subsets of $\mathbb{R}$ generated by rectangles. We may assume that $a < b$ for all $(a, b) \in E$. Since $\prec$ is in the σ-field generated by rectangles one of the sets $E \cap \prec$ of $E \setminus \prec$ is not in the σ-field. By symmetry, we may assume that $E_0 = E \cap \prec$ is not in the σ-field. Let K_0 be the collection of all subsets A of $\mathbb{R}$ such that $(a, b) \in E_0$ for all $a < b$ in A. Then being a closed subspace of the compactum K considered above K_0 is also a Corson compact space. If $C(K_0)$ is isometric to a subspace of ℓ_∞ / c_0 then the argument above would give us a description for $a < b$ of the membership of (a, b) in E_0 in terms of the condition $\| x_a + x_b \| > 1$. So, in particular E_0 would belong to the σ-field of subsets of $\mathbb{R}^2$ generated by rectangles. □

3.2 Corollary. *Suppose ℓ_∞ / c_0 contains an isometric copy of every function space $C(K)$ over a Corson compactum K of weight at most continuum. Then $2^\theta = \mathfrak{c}$ for every infinite $\theta < \mathfrak{c}$ such that* $\mathrm{cf}(2^\theta) \leq \mathfrak{c}$.

Let us now examine similar result about isomorphic rather than isometric embedding function spaces over Corson compacta into ℓ_∞ / c_0. As before, for a binary relation E on $\mathbb{R}$ and an integer $k > 1$, we set

$$E^{[k]} = \{(a_1, ..., a_k) \in \mathbb{R}^k : a_i \neq a_j \text{ and } (a_i, a_j) \in E \text{ for all } i < j\}.$$

Let E^c denote the complementary relation, i.e., $E^c = \mathbb{R}^2 \setminus E$.

3.3 Theorem. *Suppose that ℓ_∞ / c_0 contains an isomorphic copy of every function space $C(K)$ over a Corson compactum K of weight at most continuum. Then for every binary relation E on $\mathbb{R}$ and for all but finitely many positive integers k, the sets $E^{[k]}$ and $(E^c)^{[k]}$ are separated by a member of the σ-field of subsets of $\mathbb{R}^k$ generated by k-cubes.*

Proof. As in the case of isometric embeddings, we fix a well-ordering $\prec$ of the continuum and consider first the case of the relation

$$E = \{(a, b) \in \mathbb{R}^2 : a < b \text{ and } a \prec b\}.$$

As above, let K be the Corson compactum of all subsets of $\mathbb{R}$ on which the two orderings $<$ and $\prec$ agree. Let $T : C(K) \to \ell_\infty / c_0$ be an isomorphic embedding and fix an integer $k > \| T \| \| T^{-1} \|$. As before for $a \in \mathbb{R}$ we let 1_a be the corresponding indicator function on K and $x_a = T(1_a)$ be the corresponding member of ℓ_∞/c_0.

Fix a k-sequence $a_1 < ... < a_k$ of reals. Then $(a_1, ..., a_k) \in E^{[k]}$ implies $\| 1_{a_1} + ... + 1_{a_k} \|_\infty \geq k$ and so as before

$$\| x_{a_1} + ... + x_{a_k} \| \geq k / \| T^{-1} \| > \| T \| .$$

On the other hand, $(a_1, ..., a_k) \in (E^c)^{[k]}$ implies $\| 1_{a_1} + ... + 1_{a_k} \|_\infty = 1$ and so $\| x_{a_1} + ... + x_{a_k} \| \leq \| T \|$. It follows that we can separate $E^{[k]}$ from $(E^c)^{[k]}$ by a member of the σ-field of subsets of $\mathbb{R}^k$ generated by k-cubes. To get the same conclusion about an arbitrary irreflexive binary relation E_0 on $\mathbb{R}$ we proceed as above and note that we may assume without loss of generality that it is included in the relation $E = < \cap \prec$ and consider the corresponding Corson compactum K_0 getting the desired conclusion. □

4. Embedding $C([0, \omega_2])$ into ℓ_∞/c_0

Recall that by Parovichenko's result [5], for every ordinal $\gamma < \omega_2$, the function space $C([0, \gamma])$ isometrically embeds into ℓ_∞/c_0. In hindsight, Kunen [4] was the first to show that this may not hold for the larger function space $C([0, \omega_2])$, at least if one is not willing to go beyond the usual axioms of set theory. In [2] (see also [3]), arguing along similar lines, Kunen's result has been extended to cover also the isomorphic rather than isometric embeddings of $C([0, \omega_2])$ into ℓ_∞/c_0. It turns out that working as above the embeddability of $C([0, \omega_2])$ into ℓ_∞/c_0 can also be connected with the separation properties of the σ-fields $\mathcal{R}(\omega_2^k)$.

4.1 Theorem. *If the space ℓ_∞/c_0 contains an isometric copy of the space $C([0, \omega_2])$ then the ordering of ω_2 belongs to the σ-field of subsets of $\omega_2 \times \omega_2$ generated by rectangles.*

Proof. Fix a linear operator $T : C([0, \omega_2]) \to \ell_\infty/c_0$ of norm 1. For $\alpha < \omega_2$ let $1_{[0,\alpha]}$ be the characteristic function of the interval $[0, \alpha]$ and let $y_\alpha = T(1_{[\alpha,\omega_2]})$ and $x_\alpha = y_{\alpha+1} - y_\alpha$. Then for $\alpha, \beta < \omega_2$, we have that $\alpha < \beta$ if and only if $\| x_\alpha + y_\beta \| > 0$, and so the usual ordering of ω_2 belongs to the σ-field of subsets of $\omega_2 \times \omega_2$ generated by rectangles. □

It is easily seen that having the ordering of ω_2 in the σ-field of subsets of ω_2 generated by rectangles does not imply that every other subset of $\omega_2 \times \omega_2$ is in the same σ-field. This could be shown by forcing an isometric embedding of $C([0, \omega_2])$ into ℓ_∞/c_0 while arranging the cardinal inequalities $2^\theta > \mathfrak{c} \geq \mathrm{cf}(\theta)$ for $\theta = \omega_2$. In fact, with little extra work, one can show that the same forcing extension has a function space $C(K)$ over a first countable compactum K which does not even isomorphically embed into ℓ_∞/c_0.

To state a similar result for isomorphic rather than isometric embedding, for a permutation $\sigma : \{1, ..., k\} \to \{1, ..., k\}$, we let

$$<^{[\sigma]} = \{(\alpha_1, ..., \alpha_k) \in \omega_2^k : \alpha_i < \alpha_j \text{ iff } \sigma(i) < \sigma(j)\}.$$

For each positive integer k, we define also a particular permutation $\rho_k : \{1, ..., k\} \to \{1, ..., k\}$ such that $\rho_k(2j) > \rho_k(2j+2) > \rho_k(2i+1) > \rho_k(2i-1)$ for all positive integers i and j for which these numbers are smaller or equal than k.

4.2 Theorem. *Suppose that* $k > \| T \| \| T^{-1} \|$ *for some linear operator* $T : C([0, \omega_2]) \to \ell_\infty/c_0$. *Then* $<^{[\mathrm{id}_{2k-1}]}$ *and* $<^{[\rho_{2k-1}]}$ *can be separated by a set belonging to the* σ*-field of subsets of* ω_2^{2k-1} *generated by* $(2k-1)$*-products* $A_1 \times \cdots \times A_{2k-1}$.

Proof. For $\alpha < \omega_2$, we let $z_\alpha = T(1_{(\alpha,\omega_2]})$. Then $\alpha_1 < ... < \alpha_{2k-1}$ implies that $\| (z_{\alpha_1} - z_{\alpha_2}) + (z_{\alpha_3} - z_{\alpha_2}) + ... + z_{\alpha_{2k-1}} \| \leq \| T \|$. On the other hand, $(\alpha_1, ..., \alpha_{2k-1}) \in <^{[\rho_{2k-1}]}$ implies $\| (z_{\alpha_1} - z_{\alpha_2}) + (z_{\alpha_3} - z_{\alpha_2}) + ... + z_{\alpha_{2k-1}} \| \geq k / \| T^{-1} \| > \| T \|$. This gives us the conclusion of Theorem 4.2. □

References

[1] Murray G. Bell. A first countable compact space that is not an N^* image. *Topology Appl.*, 35(2-3):153–156, 1990.

[2] Christina Brech and Piotr Koszmider. On universal Banach spaces of density continuum. *preprint.*

[3] Mirna Dzamonja and Lajos Sokup. On embedding spaces into ℓ_∞/c_0 in the Cohen model. *in preparation.*

[4] Kenneth Kunen. *Inaccessibility Properties of Cardinals.* ProQuest LLC, Ann Arbor, MI, 1968. Thesis (Ph.D.)–Stanford University.

[5] I. I. Parovichenko. On a universal bicompactum of weight aleph. *Dokl. Akad. Nauk SSSR*, 150:36–39, 1963.

[6] B. V. Rao. On discrete Borel spaces and projective sets. *Bull. Amer. Math. Soc.*, 75:614–617, 1969.

[7] Fritz Rothberger. On families of real functions with a denumerable base. *Ann. of Math. (2)*, 45:397–406, 1944.

[8] Fritz Rothberger. A remark on the existence of a denumerable base for a family of functions. *Canadian J. Math.*, 4:117–119, 1952.

[9] Stevo Todorcevic. Some partitions of three-dimensional combinatorial cubes. *J. Combin. Theory Ser. A*, 68(2):410–437, 1994.

[10] V. M. Uljanov. Bicompact extensions with first axiom of countability. *Vestnik Moskov. Univ. Ser. I Mat. Meh.*, 28(2):13–18, 1973.