FORCING WITH A COHERENT SOUSLIN TREE

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Abstract. We investigate the effect of forcing with a coherent Souslin trees on two standard set-theoretic dichotomies and several topological chain conditions.

1. Introduction

Forcing axioms are strong Baire category principles with many applications due mostly to the fact that they or many of their consequences are accessible to the general mathematicians with no particular expertise in forcing. This could perhaps be most easily seem by browsing through the collection of consequences of Martin's axiom found in [8]. During the completion of the work on [8] and shortly afterwards several extensions of Martin's axiom appeared culminating to the provably strongest one introduced in [7]. This forty year old research area of set theory reveals an interesting fact that these strong Baire category principles tend to offer classification results in cases where their alternatives such as the CH or the diamond principle $\diamondsuit$ offer only chaos. There was, however, one important metrization problem in topology posed by M. Katětov more than sixty years ago (see [10]) where neither forcing axioms nor their alternatives CH and $\diamondsuit$ give a complete answer. It turned out that for complete solution of Katětov's problem one needs to relativize the standard forcing axiom to the existence of a single coherent Souslin tree S and that, in fact, the further forcing extension of S gives the answer to this problem (see [11]). The present paper is a continuation of this line of work. We introduce a relativization $PFA(S)$ of the standard Proper Forcing Axiom to proper forcing notions that preserve the coherent Souslin tree S and show that under $PFA(S)$ the Souslin tree S forces some important set theoretic dichotomies true under PFA as well as several topological consequences. For example, we show that while $PFA(S)$ does not imply the P-ideal dichotomy the forcing with the coherent Souslin tree S forces it. On the other hand the open graph dichotomy is both a consequence of $PFA(S)$ and it is forced by S. Moreover, we show that under $PFA(S)$, the coherent Souslin tree forces that countably tight compact spaces are sequential and that the class of subspaces of countably tight spaces identifies the chain conditions such as hereditary separability and hereditary Lindelöf property. We shall also show that under $PFA(S)$ the coherent Souslin tree forces that the class of compact T_5 spaces identifies the countable chain condition with hereditary separability and hereditary Lindelöf property. What makes the forcing extension of S so special is that it combines many of the consequences of the two contradictory set theoretic axioms, the weak-diamond principle $2^{\aleph_0} < 2^{\aleph_1}$, and the PFA. It is therefore natural to expect that this new area of research on forcing axiom would be as full of applications as the classical theory and more importantly would find applications that are out of reach of the classical theory.

2. Preliminaris

We use standard notation and definitions. For example, we use $\subseteq^*$ to denote the inclusion modulo a finite set and $<^*$ to denote the eventual dominance order on ω^ω. This leads us to the corresponding cardinal characteristics of the continuum that we shall also use. The cardinal $\mathfrak{p}$ is the minimal cardinality of a family $\mathcal{F}$ of infinite subsets of ω such that $\bigcap \mathcal{F}_0$ is infinite for all finite $\mathcal{F}_0 \subseteq \mathcal{F}$ but such that there is no infinite $b \subseteq \omega$ with the property that $b \subseteq^* a$ for all $a \in \mathcal{F}$. The cardinal $\mathfrak{b}$ is the minimal cardinality of a subset of ω^ω unbounded in the ordering of eventual dominance.

An *ideal* on an index set A is a collection $\mathcal{I}$ of subsets of A such that $a \subseteq b \in \mathcal{I}$ implies $a \in \mathcal{I}$ and such that $a \cup b \in \mathcal{I}$ for all $a, b \in \mathcal{I}$. A typical example of an ideal is the ideal FIN_A of finite subsets of A. For our purpose here this is the minimal ideal on A that we shall consider and so we shall implicitly assume that $\mathrm{FIN}_A \subseteq \mathcal{I}$ for every ideal $\mathcal{I}$ on A that we consider. Another important ideal for us is the ideal $[A]^{\leq \aleph_0}$ of countable subsets of A. More precisely, we shall frequently consider ideals $\mathcal{I}$ consisting only of countable subsets of A, i.e., ideals $\mathcal{I}$ that are included in $[A]^{\leq \aleph_0}$. We shall use the ideal FIN_A to define the notion of orthogonality for subsets of A. Thus we shall say that X and Y are two *orthogonal* subsets of A and write $X \perp Y$ whenever $X \cap Y$ is a finite subset of A. For a family $\mathcal{F}$ of subsets of A, let

$$\mathcal{F}^\perp = \{X \subseteq A : (\forall Y \in \mathcal{F}) |X \cap Y| \in \mathrm{FIN}_A\}.$$

We shall say that a subset X of A is *orthogonal* to a family $\mathcal{F}$ of subsets of A, if X has a finite intersection with every set from $\mathcal{F}$, or in other words, X belongs to $\mathcal{F}^\perp$.

An ideal $\mathcal{I}$ on A is a *P-ideal* if for every sequence a_n $(n < \omega)$ of elements of $\mathcal{I}$ there is $b \in \mathcal{I}$ such that $a_n \subseteq^* b$ for all $n < \omega$. Clearly, this notion has its full meaning for $\mathcal{I} \subseteq [A]^{\leq\aleph_0}$ since clearly $[A]^{\leq\aleph_0}$ is the maximal P-ideal of countable subsets of A. Note that FIN_A is also a P-ideal for trivial reasons and as said above it is the minimal such ideal on A that we shall consider. The following useful fact is supplying us with an array of P-ideals on A.

2.1 Lemma. *For every family $\mathcal{F}$ of subsets of A such that $|\mathcal{F}| < \mathfrak{b}$, the ideal $\mathcal{F}^{\perp} \cap [A]^{\leq\aleph_0}$ is a P-ideal.*

This leads us to the following natural dichotomy about ideals of countable subsets introduced in [26].

P-ideal dichotomy, PID: For every ideal $\mathcal{I}$ of countable subsets of some set A, either

(1) there is an uncountable set $B \subseteq A$ such that $[B]^{\aleph_0} \subseteq \mathcal{I}$, or else
(2) the set A can be decomposed into countably many subsets orthogonal to $\mathcal{I}$.

We follow the standard topological terminology and notation as seen in textbooks like [5]. For example, we shall use the standard notion of a *free sequence* due to A.V. Arhangel'skii. Recall that a transfinite sequence x_ξ $(\xi < \beta)$ of elements of some topological space is said to be a *free sequence in* X whenever $\overline{\{x_\xi : \xi < \alpha\}} \cap \overline{\{x_\xi : \alpha \leq \xi < \beta\}} = \emptyset$ for all $\alpha < \beta$. Recall also that a topological space has *countable tightness* or is *countably tight* if for every $A \subseteq X$ and $x \in X$, if $x \in \overline{A}$ then there is a countable $A_0 \subseteq A$ such that $x \in \overline{A_0}$. We shall use the standard fact that a compact space K has countable tightness if and only if every free sequence of elements of K must be of countable length (see [5]). Recall that a *sequentially closed* subset of a topological space X is any subset F of X with the property that $\lim_{n\to\infty} x_n \in F$ for every sequence $\{x_n : n < \omega\} \subseteq F$ that is convergent in X. A *sequential space* is any topological space in which all sequentially closed subsets are in fact closed. Clearly, every sequential space is countably tight.

3. Coherent Souslin trees

Recall that a Souslin tree is a tree of height ω_1 with no uncountable chains nor antichains. We shall be interested in such trees that are moreover *coherent*, i.e., Souslin trees representable as downward closed subsets S of the tree $I^{<\omega_1} = \bigcup_{\alpha<\omega_1} I^\alpha$ ordered by the end-extension ordering in such a way that

$$(\forall s, t \in S)\ |\{\xi \in \mathrm{dom}(s) \cap \mathrm{dom}(t) : s(\xi) \neq t(\xi)\}| < \aleph_0.$$

Clearly, the index set I can be assumed to be countable, and in fact we shall be interested only in choices like $I = \{0, 1\}$, $I = \omega$, or $I = \mathbb{Z}$.

We shall work with a coherent Souslin tree S representable in $I^{<\omega_1}$ and equal to its *homogeneous closure*

$$S^* = \{t \in I^{<\omega_1} : (\exists s \in S_{\mathrm{dom}(t)})|\{\xi \in \mathrm{dom}(t) : s(\xi) \neq t(\xi)\}| < \aleph_0.\}$$

In other words, we shall work with a fixed coherent Souslin tree S that is homogeneously closed relative to a fixed representation in $I^{<\omega_1}$ and which is therefore *strongly homogeneous* in the sense that every pair s_0 and t_0 on the same level S_{α_0} of S can be exchanged by an automorphism σ of S which *does not depend on coordinates* $\geq \alpha_0$ which in precise terms means that for every $s, t \in S$ of some length $\alpha \geq \alpha_0$

$$\sigma(s) = t \text{ iff } \sigma(s \restriction \alpha_0) = t \restriction \alpha_0 \text{ and } s \restriction [\alpha_0, \alpha) = t \restriction [\alpha_0, \alpha).$$

We recall the following two results about the existence of coherent Souslin trees.

3.1 Theorem ([4]). *The diamond principle $\diamondsuit$ implies the existence of coherent Souslin trees. So, in particular, coherent Souslin trees exists in any σ-closed forcing extension that adds a new subset to ω_1.*

3.2 Theorem ([22]). *Coherent Souslin trees exist in a single Cohen real forcing extension.*

For more on coherent trees, we refer the readers to [28].

4. A variation of PFA

Fix a coherent Souslin tree S. We relativize the standard proper forcing axiom to proper forcing that preserve the countable chain condition of S and consider the following Baire category principle:

$PFA(S)$: If $\mathcal{P}$ is a proper forcing that preserves S and if $\mathcal{D}_\alpha$ $(\alpha < \omega_1)$ is a sequence of dense-open subsets of $\mathcal{P}$ there is a filter $\mathcal{F} \subseteq \mathcal{P}$ such that $\mathcal{F} \cap \mathcal{D}_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

Quite expectedly, the consistency of this version of the Proper Forcing Axiom follows the same route as the original proof of the consistency of PFA. It is for this reason that we give only a sketch of the proof referring the reader to original sources.

4.1 Theorem. *It there is a supercompact cardinal κ there is a proper forcing $\mathcal{P}$ that preserves the Souslin tree S and forces $PFA(S)$ and $\check{\kappa} = \omega_2$.*

Proof. The poset $\mathcal{P}$ is the limit of a countable support iteration

$$\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \kappa \rangle$$

of proper partial orderings that preserve S. The choice of $\dot{\mathcal{Q}}_\alpha$ is done using Laver's sequence at κ (see [12]) and Shelah's proper forcing iteration lemma (see [17]) in the same way as in the original Baumgartner proof of the consistency of PFA (see, for example, [3]). The fact that S remains Souslin in the final extension follows from a result of Miyamoto [13] which says that Souslin trees are preserves by countable support iterations of proper forcings. □

Similarly to PFA the relativized version of the axiom also has considerable large cardinal strength that could be inferred on the basis of the following fact.[1]

4.2 Theorem. *$PFA(S)$ implies that $\square(\theta)$ fails for every ordinal θ of cofinality $> \omega_1$.*

Proof. This follows easily from Theorem 6.1 below and the result of [26] which says that PID implies that $\square(\theta)$ fails for every ordinal θ of cofinality $> \omega_1$. However, working along the lines of the corresponding result for PFA, we give a more direct argument which shows that the stronger conclusion of Theorem 4 of [20] holds for $PFA(S)$ as well. We start with a coherent sequence C_α $(\alpha \in E)$ indexed by a subset E of some ordinal θ of cofinality $> \omega_1$ such that

$$\{\alpha < \theta : \mathrm{cf}(\alpha) = \omega_1\} \subseteq E.$$

We shall show that this sequence must be trivial in the sense that there is a closed and unbounded subset C of θ such that for every limit point α of C, belongs to E and $C_\alpha = C \cap \alpha$ for every such point α. Assume by way to a contradiction that C_α $(\alpha \in E)$ is a nontrivial coherent sequence. For $\alpha, \beta \in E$ set $\alpha \prec \beta$ if α and β belong to E and if α is a limit point of C_β. Clearly, this is a tree ordering on E. Note that the non-triviality of C_α $(\alpha \in C \cap E)$ is equivalent to the nonexistence of a chain of $(E, \prec)$ that is unbounded in θ as a set of ordinals, or equivalently to the nonexistence of a closed unbounded subset D of θ that is a chain under $\prec$. Let $\mathcal{P}$ be the σ-closed poset of countable closed subsets of θ ordered by extension. Clearly $\mathcal{P}$ preserves the Souslin tree S as well as the fact that C_α $(\alpha \in E)$ is nontrivial. For example, this follows from Lemma 4.3 of [30] since $(E, \prec)$ can't contain a subtree isomorphic to the complete binary tree $2^{<\omega_1}$. Work in the forcing extension of $\mathcal{P}$ and let C be the generic closed unbounded

[1]Recall that for an ordinal θ the principle $\square(\theta)$ says that there is a sequence C_α $(\alpha < \theta)$ such that C_α is a closed and unbounded subset of α for all $\alpha < \theta$ which is on one hand *coherent*, i.e., $C_\alpha = C_\beta \cap \alpha$ whenever α is a limit point of C_β and on the other hand *nontrivial* in the sense that there is no closed and unbounded subset C of θ such that $C_\alpha = C \cap \alpha$ for all limit points α of C.

subset of θ of order-type ω_1. We now consider the restriction $(C, \prec)$ of the tree $(E, \prec)$ to the closed and unbounded set C. We know that the non-triviality of C_α $(\alpha \in C \cap E)$ is equivalent to the nonexistence of an uncountable chain in the tree $(C, \prec)$ and this in turn is equivalent to the nonexistence of a closed unbounded subset D of C that is a chain under $\prec$. It follows that forcing with the coherent Souslin tree S does not add uncountable branches to $(C, \prec)$ and, therefore, preserves the countable chain condition of the poset $\dot{\mathcal{Q}}$ of strictly increasing maps from $(C, \prec)$ into $\mathbb{Q}$. From this we infer that $\dot{\mathcal{Q}}$ preserves the Souslin tree S. Going to the ground model, we infer that the iteration $\mathcal{P} * \dot{\mathcal{Q}}$ preserves S, so $PFA(S)$ applies to it. Applying $PFA(S)$ to the iteration, we get a set C of order type ω_1 closed in its supremum δ and a strictly increasing map from the tree $(C, \prec)$ into $\mathbb{Q}$. But this is a contradiction since the intersection of the set of all limit points of C_δ with C is clearly an uncountable chain of this tree.

□

Many, but of course not all, consequences of PFA are also consequences of $PFA(S)$. For example, $PFA(S)$ does not imply the Souslin Hypothesis or the P-ideal dichotomy. On the other hand, $PFA(S)$ implies $MA_{\aleph_1}$ for the class of posets with strong countable chain conditions such as, for example, the class of σ-centered posets. It is for this reason that $PFA(S)$ implies inequalities like $\mathfrak{p} > \omega_1$. It turns out that, similarly with PFA, the relativized version is giving us also some equalities between standard characteristics of the continuum.

4.3 Theorem. *$PFA(S)$ implies $\mathfrak{p} = \mathfrak{b} = \mathfrak{c} = \omega_2$.*

Proof. By Theorem 5.2 below and the result of [23], we know that $PFA(S)$ implies $\mathfrak{b} = \omega_2$. So it remains to show that $PFA(S)$ implies $\mathfrak{c} = \mathfrak{b}$. This follows from the corresponding proof for PFA given in [2] which uses again an iteration $\mathcal{P} * \dot{\mathcal{Q}}$ where $\mathcal{P}$ is a σ-closed collapse of ω_2 and where $\dot{\mathcal{Q}}$ is a ccc poset of finite subsets of some fixed unbounded chain A in $(\omega^\omega, <^*)$ which realize particular oscillations. Since the oscillation theory and therefore the ccc property of $\dot{\mathcal{Q}}$ depends only on the unboundedness of A in $(\omega^\omega, <^*)$, the poset $\dot{\mathcal{Q}}$, and therefore the iteration $\mathcal{P} * \dot{\mathcal{Q}}$ preserve the coherent Souslin tree S. Applying $PFA(S)$ to this kind of iterations, we prove as in [2] that every real is coded by an ordinal $< \mathfrak{b}$ in a one-to-one fashion. □

4.4 Corollary. *Assume $PFA(S)$. The coherent Souslin tree S forces $\mathfrak{p} = \omega_1$ and $\mathfrak{b} = \mathfrak{c} = \omega_2$.*

As in the classical theory of forcing axioms (see, for example [27]), we can consider the bounded forms of $PFA(S)$. This typically lowers the consistency strengths and so could be of interest to readers concerned with the consistency strength of a particular statement forced by S under the assumption of $PFA(S)$. As indicated, for example, in [20] and [21], the optimal consistency strength can frequently be achieved by replacing the $\in$-chains of countable elementary submodels as side conditions in the constructions below by the corresponding matrices of isomorphic countable elementary submodels. This is based on the fact that this way we obtain proper poset satisfying a strong $\aleph_2$-chain condition which on the basis of Chapter IX of [17] can be iterated preserving cardinals. Thus, in particular, we claim that this minor modification of the side conditions gives that no statements considered below, with the exception of the P-ideal dichotomy, require large cardinals for their consistency. Finally, we would like to mention that one may also consider the relativized form $MM(S)$ of the Martin's Maximum (consistent on the basis of [14]) and obtain interesting application beyond reach of the $PFA(S)$. We leave this to the interested reader.

5. Graph Dichotomies

Consider an *open graph* $\mathcal{G} = (X, E)$ on a separable metric space X, i.e., a graph whose vertex set is the separable metric space and the edge set E is viewed as an open symmetric subset of $X^2 \setminus \{(x,x) : x \in X\}$. Suppose $\mathcal{G}$ is not countably chromatic and consider the corresponding co-ideal

$$\mathcal{H}(\mathcal{G}) = \{Y \subseteq X : \mathrm{Chr}(\mathcal{G} \restriction Y) > \aleph_0\}.$$

Note that $\mathcal{H}(\mathcal{G})$ is a countably complete co-ideal in the sense that of $Y = \bigcup_{n<\omega} Y_n$ is a countable decomposition of a set Y from $\mathcal{H}(\mathcal{G})$ then there must be an $n < \omega$ such that Y_n also belongs to $\mathcal{H}(\mathcal{G})$. There is a natural way to take a finite Fubini power $\mathcal{H}^n(\mathcal{G})$ of $\mathcal{H}(\mathcal{G})$, a co-ideal on X^n defined recursively by $\mathcal{H}^1(\mathcal{G}) = \mathcal{H}^n(\mathcal{G})$ and for $n > 1$,

$$\mathcal{H}^n(\mathcal{G}) = \{W \subseteq X^n : \{\bar{x} \in X^{n-1} : W_{\bar{x}} \in \mathcal{H}(\mathcal{G})\} \in \mathcal{H}^{n-1}(\mathcal{G})\},$$

where for $W \subseteq X^n$ and $\bar{x} \in X^{n-1}$,

$$W_{\bar{x}} = \{y \in X : \bar{x}^\frown y \in W\}.$$

It is easily seen that each $\mathcal{H}^n(\mathcal{G})$ is also a countably complete co-ideal on X^n. The following fact captures an essential property of open graphs on separable matric spaces.

5.1 Lemma. *Suppose $\mathcal{G} = (X, E)$ on a separable metric space (X, d) and that n is a positive integer. Let $W \in \mathcal{H}^n(\mathcal{G})$ and let*

$$\partial W = \{\bar{x} \in W : (\forall \varepsilon > 0)(\exists \bar{y} \in W)(\forall i < n)[(x_i, y_i) \in E \wedge d(x_i, y_i) < \varepsilon]\}$$

Then $W \setminus \partial W \notin \mathcal{H}^n(\mathcal{G})$.

Proof. The proof is by induction on n. The case $n = 1$ is clear. Suppose $n > 1$ and that the conclusion of the lemma holds for $n - 1$ but it fails for some $W \in \mathcal{H}^n(\mathcal{G})$. Since (X, d) has a countable base $\mathcal{B}$ and since the co-ideal $\mathcal{H}^n(\mathcal{G})$ is countably complete there is a subset V of W such that $V \in \mathcal{H}^n(\mathcal{G})$ and such that V contains no two n-tuples $\bar{x} = (x_0, ..., x_{n-1})$ and $\bar{y} = (y_0, ..., y_{n-1})$ with the property that $(x_i, y_i) \in E$ for all $i < n$. Fix $\bar{w} \in X^{n-1}$ with the property that $V_{\bar{w}} \in \mathcal{H}(\mathcal{G})$. Then from the initial inductive case and the openness of the edge relation E we can find two disjoint members $A_{\bar{w}}, B_{\bar{w}}$ of the basis $\mathcal{B}$ which intersect $V_{\bar{w}}$ and have the property that $A_{\bar{w}} \times B_{\bar{w}} \subseteq E$. Since the set of $\bar{w} \in X^n$ for which $V_{\bar{w}} \in \mathcal{H}(\mathcal{G})$ belongs to the countably complete co-ideal $\mathcal{H}^{n-1}(\mathcal{G})\}$ there exist $U \in \mathcal{H}^{n-1}(\mathcal{G})\}$ and $A, B \in \mathcal{B}$ such that for all $\bar{w} \in U$ the set $V_{\bar{w}}$ belongs to $\mathcal{H}(\mathcal{G})$ and $A_{\bar{w}} = A$ and $B_{\bar{w}} = B$. By the inductive hypothesis we can find $\bar{u}, \bar{v}$ in U such that $(u_i, v_i) \in E$ for all $i < n-1$. Pick $x \in A$ and $y \in B$. Then the two n-tuples $\bar{u}^\frown x$ and $\bar{v}^\frown y$ belong to V but contradict its property that it contains no two n-tuples $\bar{x} = (x_0, ..., x_{n-1})$ and $\bar{y} = (y_0, ..., y_{n-1})$ such that $(x_i, y_i) \in E$ for all $i < n$. This finishes the proof. □

We are ready now to prove the following result.

5.2 Theorem. *Assume $PFA(S)$. Then every open graph on a separable metric space is countably chromatic unless it contains an uncountable complete subgraph.*

Proof. Since continuous pre-image of an open graph is an open graph, we may concentrate on open graphs whose vertex sets are sets of reals. So, let $\mathcal{G} = (X, E)$ be a graph where X is a set of reals and where E is an open symmetric irreflexive relation on $\mathbb{R}$. We assume that the graph (X, E) is not countably chromatic and work towards showing that it contains an uncountable complete subgraph. As explained in Chapter 8 of [23] (see Theorem 8.0 and the remark on p.85) there are two natural ways to force this dichotomy and both of them work for our purpose here. It is perhaps easier to see that the forcing from the proof of Theorem 8.0 of [23] preserves any Souslin tree T. It has the form $\mathcal{P} * \dot{\mathcal{Q}}$, where $\mathcal{P}$ is a σ-closed poset which, as is well known, preserves the Souslin tree T, and which introduces an uncountable subset $\dot{Y}$ of X such that the poset $\dot{\mathcal{Q}}$ of finite cliques of the induced subgraph

$(\dot{Y}, \check{E})$ satisfies the countable chain condition. Working in the forcing extension of $\mathcal{P}$ note that of for some positive integer n we have a *separated family*[2] $\mathcal{F} \subseteq \dot{\mathcal{Q}} \cap [\dot{Y}]^n$ such that $s \cup t \notin \dot{\mathcal{Q}}$ for all $s \neq t \in \mathcal{F}$ then the same is true of its closure $\overline{\mathcal{F}}$ in $[Y]^n$. Since T adds no new reals, this shows that $\dot{\mathcal{Q}}$ remains ccc in the further forcing extension of T, or equivalently, that T remains Souslin in the forcing extension of $\mathcal{P} * \dot{\mathcal{Q}}$, as required. We now present a detailed proof that the forcing notion from p.85 of [23] also preserves Souslin trees as it is this version of the proof that is more relevant to the questions we raise in the concluding section of this paper.

Let $\mathcal{I}$ be the σ-ideal of subsets Y of subsets of X for which the induced subgraph (Y, E) is countably chromatic. Removing a relatively open subset of X, we may assume that no nonempty relatively open subset of X belongs to $\mathcal{I}$.

Let $\mathcal{P}$ be the collection of all mappings $p : \mathcal{N}_p \to X$, where $\mathcal{N}_p$ is a finite non-empty $\in$-chain of countable elementary submodels of $(H_{\mathfrak{c}^+}, \in)$ containing all the above objects such that

(1) $(\forall N \in \mathcal{N}_p)(\forall Z \in \mathcal{I} \cap N)\ p(N) \notin Z$.
(2) $(\forall M \in N \in \mathcal{N}_p)[p(M) \in N$ and $(p(M), p(N)) \in E]$.

We order $\mathcal{P}$ by the inclusion.

5.2.1 Claim. *$\mathcal{P}$ is proper and it preserves S.*

Proof. Consider an arbitrary condition $p_0 \in \mathcal{P}$. Choose a countable elementary submodel M of $(H_{(2^{\mathfrak{c}})^+}, \in)$ containing all the above objects. Thus in particular p_0 belongs to M. Let $M_0 = M \cap H_{\mathfrak{c}^+}$ and let

$$q_0 = p_0 \cup \{(M_0, q(M_0))\},$$

where $q_0(M_0)$ is any element of $X \setminus \bigcup(\mathcal{I} \cap M_0)$ with the property that $(p_0(N), q_0(M_0)) \in E$ for all $N \in \mathcal{N}_{p_0}$. To see that $q_0(M_0)$ can indeed be chosen, pick an open interval I with rational end points containing $p_0(\bar{N})$, where $\bar{N}$ is the last model of $\mathcal{N}_{p_0}$, such that $(p_0(N), x) \in E$ for all $N \in \mathcal{N}_{p_0} \cap \bar{N}$ and $x \in I$. Let $Y = X \cap I$. Then $Y \in \bar{N}$ and $p_0(\bar{N}) \in Y$ and therefore $Y \notin \mathcal{I}$. Let $\mathcal{U}$ be the collection of all open intervals J such that $J \cap Y \in \mathcal{I}$ and let $\bar{Y} = Y \setminus \bigcup \mathcal{U}$. Then again $\bar{Y} \in \bar{N}$ and $p_0(\bar{N}) \in \bar{Y}$ and therefore $\bar{Y} \notin \mathcal{I}$ and in fact no nonempty relatively open subset of $\bar{Y}$ can belong to $\mathcal{I}$. Let $\bar{Z}$ be the collection of all $z \in \bar{Y}$ for which there are no $y \in \bar{Y}$ different from z such that $(y, z) \in E$. Clearly $\bar{Z} \in \bar{N}$ and $\bar{Z}$ is a discrete subgraph of (X, E). So, in particular $p_0(\bar{N}) \notin \bar{Z}$.

[2]We say that $\mathcal{F} \subseteq \dot{\mathcal{Q}} \cap [\dot{Y}]^n$ is *separated* if there is a sequence $I_1, I_2, ..., I_n$ of pairwise disjoint intervals of $\mathbb{R}$ with rational end-points such that $I_i \times I_j \subseteq E$ whenever $i \neq j$ and such that every element q of $\mathcal{F}$ intersects every interval I_i in exactly one point.

so we can find $y \in \bar{Y} \setminus \{p_0(\bar{N})\}$ such that $(p_0(\bar{N}), y) \in E$. Since E is open, there is an open interval $J \subseteq I$ with rational intervals containing y such that $(p_0(\bar{N}), x) \in E$ for all $x \in X \cap J$. By the property of $\bar{Y}$, we know that the intersection $\bar{Y} \cap J$ does not belong to $\mathcal{I}$, so finally we can pick our $q_0(M_0)$ in $(\bar{Y} \cap J) \setminus \bigcup(\mathcal{I} \cap M_0)$ which will now have the required property that $(p_0(N), q_0(M_0)) \in E$ for all $N \in \mathcal{N}_{p_0}$.

We show that q_0 is an $(M, \mathcal{P})$-generic condition. So pick a dense-open subset $\mathcal{D}$ of $\mathcal{P}$ belonging to M and an extension q of q_0. We need to find $\bar{q} \in M \cap \mathcal{D}$ compatible with q. We may assume $q \in \mathcal{D}$. Let $n = |\mathcal{N}_q \setminus M|$ and let $M_0, ..., M_{n-1}$ be the increasing enumeration of $\mathcal{N}_q \setminus M$. Let W be the collection of all n-tuples $(x_0, ..., x_{n-1})$ of elements of X for which we can find $r \in \mathcal{D}$ end-extending $q \cap M$ such that $|\mathcal{N}_r \setminus M| = n$ and such that if $N_0, ..., N_{n-1}$ is the increasing enumeration of $\mathcal{N}_r \setminus M$ then $x_i = r(N_i)$ for all $i < n$. Clearly $W \in M$ and $(q(M_0), ..., q(M_{n-1}) \in W$. From the condition (1) from the definition of our poser $\mathcal{P}$ we can easily conclude that $W \in \mathcal{H}^n(\mathcal{G})$. In fact, $V \in \mathcal{H}^n(\mathcal{G})$ for every $V \subseteq X^n$ such that $V \in M_0$ and $(q(M_0), ..., q(M_{n-1}) \in V$. So, by Lemma 5.1, the fact that $W \cap N_0$ is dense in W, and the fact that E is an open relation, we conclude that for every $\varepsilon > 0$ there exist $x \in W \cap N_0$ such that

$$(\forall i < n)[(q(M_i, x_i) \in E \wedge d(q(M_i), x_i) < \varepsilon)].$$

It follows that the union $\{q(M_0, ..., q(M_{n-1})\} \cup \{x_0, ..., x_{n-1}\}$ is a complete subgraph of $\mathcal{G}$. So if we pick $r \in \mathcal{D} \cap M_0$ end-extending $q \cap M$ such that $|\mathcal{N}_r \setminus M| = n$ and such that if $N_0, ..., N_{n-1}$ is the increasing enumeration of $\mathcal{N}_r \setminus M$ then $x_i = r(N_i)$ for all $i < n$. It follows that $q \cup r$ is a common extension of q and r. This finishes the proof that $\mathcal{P}$ is proper.

Let $\delta = M \cap \omega_1$. We shall show that p_0 does not force that S in not Souslin by showing that for any choice of $s_0 \in S_\delta$, the pair (q_0, s_0) is a $\mathcal{P} \times S$-generic condition. So pick a dense open subset $\mathcal{D}$ of $\mathcal{P} \times S$ belonging to M and pick an arbitrary extension (q, s) of (q_0, s_0). We need to find $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$ compatible to (q, s). We may assume that (q, s) belongs to $\mathcal{D}$ and that s is not member of the last model of $\mathcal{N}_q$. Go to the generic extension V[G] of the universe obtained by adding a generic branch G through S containing the node s. Note that since E is open the closure $\overline{Y}$ of every every $Y \subset X$ with the property $Y^2 \cap E = \emptyset$ also has this property. It follows that the σ-ideal $\mathcal{I}$ as defined in $V[G]$ is generated by its V-version. For $i < n$ let $M_i[G]$ be the set of all G-interpretations of S-names belonging to M_i. Then $M_0[G], ..., M_{n-1}[G]$ is an $\in$-chain of countable elementary submodels of $H_{\mathfrak{c}^+}[G]$. Working still in $V[G]$, let W be the collection of all n-tuples $(x_0, ..., x_{n-1})$ of elements of X for which we can find $(r, t) \in \mathcal{D}$

with r end-extending $q \cap M$ and t belonging to the generic branch $[G]$ such that $|\mathcal{N}_r \setminus M| = n$ and such that if $N_0, ..., N_{n-1}$ is the increasing enumeration of $\mathcal{N}_r \setminus M$ then $x_i = r(N_i)$ for all $i < n$. Clearly $W \in M_0[G]$ and $(q(M_0), ..., q(M_{n-1}) \in \mathcal{F})$. Applying the fact from the proof of properness of $\mathcal{P}$ above to the sequence $M_0[G], ..., M_{n-1}[G]$ of models and the n-tuple $(q(M_0), ..., q(M_{n-1})$, which has the desired property that $q(M_i) \notin \mathcal{I} \cap M_i[G]$ for all $i < n$, we find $(x_0, ..., x_{n-1}) \in M_0 \cap W$ such that $(q(M_i), x_j) \in E$ for all $i, j < n$. Pick a pair $(\bar{q}, \bar{s}) \in M[G] \cap \mathcal{D}$ witnessing the membership of $(x_0, ..., x_{n-1})$ in $\mathcal{F}$. Then $\bar{s} <_S s$ as both nodes belong to the generic branch G, so indeed $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$ is the desired compatible copy of (q, s). This finishes the proof of Claim. □

For $\alpha < \omega_1$, let

$$\mathcal{D}_\alpha = \{p \in \mathcal{P} : (\exists N \in \mathcal{N}_p)\ \alpha \in N\}.$$

The argument from the proof of Claim 5.2.1 showing how to choose $q_0(M_0)$ also shows that $\mathcal{D}_\alpha$ is a dense-open subset of $\mathcal{P}$ for all $\alpha < \omega_1$. Applying $PFA(S)$, to $\mathcal{P}$ and this sequence of dense-open sets, we get a filter $\mathcal{F}$ such that the corresponding mapping $q = \bigcup \mathcal{F}$ has uncountable $\in$-chain $\mathcal{N} = \bigcup_{p \in \mathcal{F}} \mathcal{N}_p$ as a domain. Its range is, therefore, an uncountable subset of X spanning a complete subgraph of $\mathcal{G} = (X, E)$. This completes the proof. □

5.3 Corollary. *Assume $PFA(S)$. Then the coherent Souslin tree S forces that every open graph on a separable metric space is countably chromatic unless it contains an uncountable complete subgraph.*

Proof. Let $(\dot{X}, \dot{E})$ be an S-name for an graph where X is a set of reals and where $\dot{E}$ is an open symmetric irreflexive relation on $\mathbb{R}$. Being a countably describable object, we may assume that the open binary relation is in the ground model, i.e., that we may treat only S-names for open graphs of the form $(\dot{X}, \check{E})$, where E is a given open symmetric irreflexive relation on $\mathbb{R}$. Assume that S forces that the chromatic number of $(\dot{X}, \check{E})$ is not countable. Let

$$Y = \{x \in \mathbb{R} : (\exists s \in S)\ s \Vdash x \in \dot{X}\}.$$

For each $x \in Y$ fix an $s_x \in S$ such that $s_x \Vdash x \in \dot{X}$. Since, in particular, the open graph (Y, E) is not countably chromatic, by Theorem 5.2, we know that there is uncountable $Z \subseteq Y$ such that $Z^2 \setminus \triangle \subseteq E$. Let $\dot{Z}_G$ be the S-name for the collection of all $z \in Z$ for which s_z belongs to the generic branch. Then there is an $s \in S$ forcing that $\dot{Z}_G$ is an uncountable subset of $\dot{X}$ that spans a complete subgraph of $(\dot{X}, \dot{E})$. This completes the proof. □

We finish this section by mentioning another graph axiom forces by the coherent Souslin tree S established in [11] and needed below in Section 10.

5.4 Theorem. *Assume $MA_{\aleph_1}(S)$. The coherent Souslin tree S forces that for every graph (ω_1, E) on the vertex-set ω_1, contains either an uncountable complete subgraph, or a pair of uncountable subsets A and B of ω_1 such that there are no uncountable $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $(\alpha, \beta) \in E$ whenever $\alpha \in A_0$, $\beta \in B_0$, and $\alpha < \beta$.*

6. P-ideal dichotomy

The purpose of the section is to prove the following result.

6.1 Theorem. *Assume $PFA(S)$. Then S forces the P-ideal dichotomy.*

Proof. Let $\dot{\mathcal{I}}$ be an S-name for a P-ideal on some ordinal $\check{\nu}$. Assume that S forces that the second alternative of PID for $\dot{\mathcal{I}}$ fails. We fix a large enough regular cardinal θ so that the structure $(H_\theta, \in)$ contains all the relevant objects. We equip H_θ with a well ordering $<_\theta$ and with no explicit mention we consider only elementary submodels of the expanded structure $(H_\theta, \in, <_\theta)$. We shall also frequently define objects that are minimal relative to this well-ordering and satisfying some conditions. We shall keep these conventions even for the structure $(H_\theta[G], \in, <_\theta)$ after we add a generic branch G through S. For example, we let $\dot{\xi}_N$ be the S-name of the minimal ordinal $\xi < \check{\nu}$ such that $\xi \notin Y$ for every $Y \in \dot{\mathcal{I}}^\perp \cap N[G]$. Moreover for every countable elementary submodel N of H_θ we define an S-name $\dot{b}_N$ for an element of $\dot{\mathcal{I}}$ with the property that $b \subseteq N \cap \nu$ and $a \subseteq^* b$ for all $a \in N[G] \cap \dot{\mathcal{I}}$ as follows. Fix an $s \in S_{N\cap\omega_1}$. Using the fact that s forces $\dot{\mathcal{I}}$ to be a P-ideal, we can find a maximal antichain A_s of extensions of s and for each $t \in A_s$ a set $b_t \subseteq N \cap \nu$ such that t forces b_t to be an element of $\dot{\mathcal{I}}$ which almost contains all sets $a \in N$ forced by some $v <_S s$ to be members of $\dot{\mathcal{I}}$. Interpolating the pre-gap, we now can find a set b almost included in all the b_t ($t \in A_s$) but still almost including all sets $a \in N$ forced by some $v <_S s$ to be members of $\dot{\mathcal{I}}$. Let b_s be the $<_\theta$-minimal such set. Finally, let $\dot{b}_N$ be the S-name for an element of $\dot{\mathcal{I}}$ that is equal to b_s for the $s \in S_{N\cap\omega_1}$ belonging to the generic branch. Let $\mathcal{P}$ be the collection of all mappings $p : \mathcal{N}_p \to S$, where

(1) $\mathcal{N}_p$ is a finite $\in$-chain of countable elementary submodels of H_θ,
(2) $M \in N$ in $\mathcal{N}_p$ implies $p(M) \in N$,
(3) $p(M)$ decides the values of $\dot{b}_M$ and $\dot{\xi}_M$.

We let $p \leq q$ if,

(4) $\mathcal{N}_p \supseteq \mathcal{N}_q$ and $q = p \restriction \mathcal{N}_q$,
(5) $(\forall M \in \mathcal{N}_q)(\forall N \in (\mathcal{N}_p \cap M) \setminus \mathcal{N}_q)[p(N) <_S q(M) \Rightarrow q(M) \Vdash \dot{\xi}_N \in \dot{b}_M]$.

6.1.1 Claim. *$\mathcal{P}$ is proper and it preserves S.*

Proof. Since the proof of the properness of $\mathcal{P}$ will follow from the proof of the preservation of S we concentrate on preserving S. Choose a countable elementary submodel M of $H_{(2^\theta)^+}$ containing all the relevant objects and pick a condition $p_0 \in \mathcal{P} \cap M$. Let $M_0 = M \cap H_\theta$ and let

$$q_0 = p_0 \cup \{(M_0, q_0(M_0))\},$$

where $q_0(M_0)$ is any node of $S \setminus N$ deciding the values of $\dot{b}_0 = \dot{b}_{M_0}$ and $\dot{\xi}_0 = \dot{\xi}_{M_0}$ and where $q_1(M_0)$ is any ordinal $< M_0 \cap \omega_1$ that is larger than all ordinals of the form $N \cap \omega_1$ for $N \in \mathcal{N}_{p_0}$. Let $\delta = M \cap \omega_1$. Note that p_0 will not be able to force that S fails to be a Souslin tree once we show that for every choice of $s_0 \in S_\delta$ the pair (q_0, s_0) is an $(M, \mathcal{P} \times S)$-generic condition. To this end, consider a dense-open subset $\mathcal{D}$ of $\mathcal{P} \times S$ such that $\mathcal{D} \in M$ and consider an arbitrary extension (q, s) of (q_0, s_0). We need to find $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$ compatible with (q, s). We may and we shall assume that (q, s) belongs to $\mathcal{D}$. We may also assume that the height of s is bigger than the height of any node of the range of q and in fact that there is a countable elementary submodel M' of H_θ containing q such that $s \notin M'$ and such that s decides $\dot{\xi}_{M'}$. Choose $\gamma_0 < \delta$ such that

$$(\forall N, N' \in \mathcal{N}_q \setminus M) q(N) \restriction [\gamma_0, \delta) = q(N') \restriction [\gamma_0, \delta), \text{ and so}$$

$$(\forall N, N' \in \mathcal{N}_q \setminus M)[q(N) \restriction \delta \neq q(N') \restriction \delta \Rightarrow q(N) \restriction \gamma_0 \neq q(N') \restriction \gamma_0].$$

Let $N_0, ..., N_{n-1}$ be the increasing enumeration of those $N \in \mathcal{N}_q \setminus M$ with the property that $q(N)$ and s agree on ordinals in common domain that are $\geq \gamma_0$. For $i < n$ fix an automorphism $\sigma_i \in M$ of S which maps $q(N_i) \restriction \gamma_0$ to $s \restriction \gamma_0$, and therefore maps $q(N_i)$ below s. Let $\mathcal{F}$ be the collection of all sequences $\{(t_i, \xi_i) : i < n\}$ of elements of $S \times \nu$ for which we can find $(\bar{q}, \bar{s}) \in \mathcal{D}$ end extending $(p_0, s \restriction \gamma_0)$, and in fact isomorphic with (q, s) over p_0, γ_0, and σ_i $(i < n)$ such that $t_i = \sigma_i(\bar{q}(\bar{N}_i))$ and $\xi_i = \xi_{\bar{M}_i}$ for all $i < n$, where $\bar{N}_0, ..., \bar{N}_{n-1}$ is the increasing enumeration of those $\bar{N} \in \mathcal{N}_{\bar{q}} \setminus \mathcal{N}_{p_0}$ for which $\bar{q}(\bar{N})$ and $\bar{s}$ agree on ordinals in common domain that are $\geq \gamma_0$. For $\upsilon = \{(t_i, \xi_i) : i < n\}$ in $\mathcal{F}$ let $f(\upsilon)$ be the minimal node $\bar{s}$ extending t_{n-1} and appearing as the second coordinate of some pair $(\bar{q}, \bar{s}) \in \mathcal{D}$ witnessing $\upsilon \in \mathcal{F}$. Clearly $f, \mathcal{F} \in M \cap H_\theta = M_0$ and $\{(\sigma_0(q(N_0)), \xi_{N_0}), ..., (\sigma_{n-1}(q(N_{n-1})), \xi_{N_{n-1}})\} \in \mathcal{F}$. Note that we have arranged that $\{(\sigma_0(q(N_0)), \xi_{N_0}), ..., (\sigma_{n-1}(q(N_{n-1})), \xi_{N_{n-1}})\}$ is separated by the elementary submodels $N_0 \in N_1 \in ... \in N_{n-1}$. For a sequence $\upsilon = \{(t_i, \xi_i) : i < n - 1\}$, let $\dot{X}_\upsilon$ be the S-name for the collection of all ordinals $\xi < \nu$ for which we can find a node $t \in S$ such that $\upsilon^\frown(t,\xi) \in \mathcal{F}$ and such that $\sigma_{n-1}^{-1}(f(\upsilon^\frown(t,\xi)))$ belonging to the generic branch. Let $\partial\mathcal{F}$ be the collection of sequences $\upsilon = \{(t_i,\xi_i) : i < n-1\}$ for which there is $u \in S$ extending t_{n-1} such that $\sigma_{n-1}^{-1}(u) \Vdash \dot{X}_\upsilon \notin \dot{\mathcal{I}}^\perp$. Extend the mapping f from $\mathcal{F}$ to $\partial\mathcal{F}$ that picks the minimal such a node u. Clearly $\partial\mathcal{F}$ and the extended function still belong to M_0 and therefore to all the models of the chain $N_0 \in N_1 \in ... \in N_{n-1}$. So a simple elementarity argument over N_{n-1} shows that

$$\{(\sigma_0(q(N_0)),\xi_{N_0}),...,(\sigma_{n-2}(q(N_{n-2})),\xi_{N_{n-2}})\} \in \partial\mathcal{F}.$$

Similarly for a sequence υ of length $n-2$ we let $\dot{X}_\upsilon$ be the S-name for the collection of all ordinals $\xi < \nu$ for which we can find a node $t \in S$ such that such that $\upsilon^\frown(t,\xi) \in \partial\mathcal{F}$ and such that $\sigma_{n-1}^{-1}(f(a^\frown(t,\xi)))$ belonging to the generic branch. Let $\partial^2\mathcal{F}$ be the collection of all sequences a for which there is $u \in S$ extending t_{n-2} such that $\sigma_{n-2}^{-1}(u) \Vdash \dot{X}_\upsilon \notin \dot{\mathcal{I}}^\perp$. We extend f to $\partial^2\mathcal{F}$ by letting $f(\upsilon)$ be the minimal such node u. We still have that $\partial^2\mathcal{F}$ and the extended f belongs to M_0 and therefore to all submodels $N_0 \in N_1 \in ... \in N_{n-1}$. So a simple elementarity argument over N_{n-2} shows that

$$\{(\sigma_0(q(N_0)),\xi_{N_0}),...,(\sigma_{n-3}(q(N_{n-3})),\xi_{N_{n-3}})\} \in \partial^2\mathcal{F}.$$

Proceeding in this way we arrive at $\partial^{n-1}\mathcal{F} \in M_0$ and the extended mapping $f \in M_0$ such that $\{(\sigma_0(q(N_0)),\xi_{N_0})\} \in \partial^{n-1}\mathcal{F}$. Let $\dot{X}_\emptyset$ be the S-name for the collection of all $\xi < \nu$ for which we can find a node t extending $s \restriction \gamma_0$ such that $\{(t,\xi)\}$ and such that $\sigma_0^{-1}(f(\{(t,\xi)\}))$ belong to the generic branch. Using the elementarity M_0 and the choice of the name $\dot{\xi}_{N_0}$ one easily shows that there must be a node $u_0 \in M_0$ such that $s \restriction \gamma_0 \leq u_0 <_S s$ such that $\sigma_0^{-1}(u_0) \Vdash \dot{X}_\emptyset \notin \dot{\mathcal{I}}^\perp$. Since $q(N_0) \restriction \delta$ is an S-generic branch there must be an infinite subset a_0 of ν in M_0 and $v_0 \geq \sigma_0^{-1}(u)$ belonging to this generic branch (i.e., $v_0 <_S q(N_0) \restriction \delta$) such that $v_0 \Vdash a \subseteq \dot{X}_\emptyset$ and $a_0 \in \dot{\mathcal{I}}$. Note that for $N \in \mathcal{N}_q \setminus M$ the node $q(N)$ decides the value of $\dot{b}_N$ which we simply denote by b_N, and so we have the following

$$(\forall N \in \mathcal{N}_q \setminus M)[q(N) \restriction \delta = q(N_0) \restriction \delta \Rightarrow a \subseteq^* b_N].$$

So there is $\xi_0 \in a_0$ such that $\xi_0 \in b_N$ for all $N \in \mathcal{N}_q \setminus M$ with property $q(N) \restriction \delta = q(N_0) \restriction \delta$. It follows that $v_0 \Vdash \xi_0 \in \dot{X}_\emptyset$, so we can find a node u_1 in the generic branch $q(N_0) \restriction \delta$ and node $t_0 \in S$ such that $(t_0,\xi_0) \in \partial^{n-1}\mathcal{F}$ such that u_1 forces that $\sigma_0^{-1}(f(t_0,\xi_0))$ belongs to the generic branch. So in particular $u_1 \geq_S f(t_0,\xi_0)$, and therefore,

$$u_1 \Vdash \dot{X}_{\{(t_0,\xi_0)\}} \notin \dot{\mathcal{I}}^\perp.$$

So, we can continue picking $v_1 \geq_S u_1$ and infinite $a_1 \subseteq \nu$ in M_0 such that v_1 forces a_1 belongs to the ideal $\dot{\mathcal{I}}$ and is included in $\dot{X}_{\{(t_0,\xi_0)\}}$. So we can pick $\xi_1 \in a_1$ belonging to all b_N for $N \in \mathcal{N}_q \setminus M$ with the property $q(N) \restriction \delta = q(N_1) \restriction \delta$, and finally find $t_1 < s \restriction \delta$ such that $\{(t_0,\xi_0),(t_1,\xi_1)\} \in \partial^{n-2}\mathcal{F}$ and find u_2 in the branch $q(N_1) \restriction \delta$ such that u_2 forces that $\sigma_1^{-1}(f(\{(t_0,\xi_0),(t_1,\xi_1)\})$ belongs to the generic branch and so, in particular, $u_1 \geq_S f(\{(t_0,\xi_0),(t_1,\xi_1)\})$ and therefore,

$$u_1 \Vdash \dot{X}_{\{(t_0,\xi_0),(t_1,\xi_1)\}} \notin \dot{\mathcal{I}}^{\perp}.$$

Proceeding in this way we will arrive at a sequence

$$\{(t_i,\xi_i) : i < n\} \in \mathcal{F} \cap M$$

so that if $(\bar{q},\bar{s}) \in \mathcal{D} \in M$ is the corresponding witness then $(q,s))$ and $(\bar{q},\bar{s})$ are compatible in $\mathcal{P} \times S$. This finishes the proof of Claim.

□

For $\alpha < \omega_1$ let

$$\mathcal{D}_\alpha = \{p \in \mathcal{P} : (\exists N \in \mathcal{N}_p)\alpha \in N\}.$$

Then $\mathcal{D}_\alpha$ is dense-open in $\mathcal{P}$ for all $\alpha < \omega_1$. Let $\mathcal{G}$ be a filter of $\mathcal{P}$ intersecting all $\mathcal{D}_\alpha$'s. Then $\mathcal{N} = \bigcup_{p\in\mathcal{G}} \mathcal{N}_p$ is an uncountable $\in$-chain of countable elementary submodels of H_θ and $g = \bigcup \mathcal{G}$ is a function from $\mathcal{N}$ into S with the property that for every $N \in \mathcal{N}$,

$$g(N) \Vdash \{\xi_M : M \in \mathcal{N} \cap N\} \subseteq^* b_N.$$

Let $\dot{X}$ be the S-name for the set of all ξ_N where $N \in \mathcal{N}$ and where $g(N)$ belongs to the generic branch of S. Since the range of g is an uncountable subset of S there is $s \in S$ which forces that $\dot{X}$ is an uncountable subset of ν with the property that $[\dot{X}]^{\aleph_0} \subseteq \dot{\mathcal{I}}$. It follows that s forces the first alternative of PID. This finishes the proof.

□

We finish this section by a natural variation of the above proof that in turn gives a natural variation of the P-ideal dichotomy (first analyzed in [9]) true after forcing by S.

6.2 Theorem. *Assume $PFA(S)$. The coherent Souslin tree S forces that for every P-ideal $\mathcal{I}$ of countable subsets of some regular uncountable cardinal θ, either there is a stationary set $E \subseteq \{\delta < \theta : \mathrm{cf}(\delta) = \omega\}$ orthogonal to $\mathcal{I}$ or there is a set $C \subseteq \theta$ of order-type ω_1 closed in its supremum such that $[C]^{\aleph_0} \subseteq \mathcal{I}$.*

Proof. Fix an S-name $\dot{\mathcal{I}}$ for a P-ideal of countable subsets of θ. We assume that S forces the negation of the first alternative and we shall produce an S-name $\dot{C}$ for a subset of θ of order type ω_1 that is closed and unbounded in its supremum with the property that some $s \in S$ forces $[\dot{C}]^{\aleph_0} \subseteq \dot{\mathcal{I}}$.

For a countable elementary submodel N of $(H_{(2^\theta)^+}, \in, <_{(2^\theta)^+})$. Note that this time, since S forces that no stationary subset of the set

$$\{\delta < \theta : \mathrm{cf}(\delta) = \omega\}$$

is orthogonal to the ideal $\dot{\mathcal{I}}$, it follows that S-forces that the ordinal $\xi_N = \sup(N \cap \theta)$ has the property that

$$\Vdash_S (\forall X \in \dot{\mathcal{I}}^\perp \cap N[G])\ \check{\xi}_N \notin X.$$

As before, we fix a S-name $\dot{b}_N$ for $b \in \dot{\mathcal{I}}$ with the property that $b \subseteq N \cap \theta$ and $a \subseteq^* b$ for all $a \in \dot{\mathcal{I}} \cap N[G]$ as follows. Fix $s \in S_{\delta_N}$, fix a maximal antichain A_s of successors of s and for every $t \in A_s$ a set $b_t \subseteq N \cap \theta$ such that

$$t \Vdash (\forall a \in \dot{\mathcal{I}} \cap N[G])\ a \subseteq^* \check{b}_t.$$

Since A_s is countable, we can interpolate the pre-gap formed by

$$\{a \subseteq N \cap \omega_1 : (\exists v <_S s) v \Vdash \check{a} \in \dot{\mathcal{I}}\}$$

and $\{b_t : t \in A_s\}$ and find a set $b \subseteq N \cap \theta$ such that

$$s \Vdash (\forall a \in \dot{\mathcal{I}} \cap N[G])\ a \subseteq^* \check{b}.$$

We let b_s be the $<_{(2^\theta)^+}$-minimal such set. Finally let $\dot{b}_N$ be the S-name for a subset of $N \cap \theta$ that is equal to b_s for the $s \in S_{\delta_N}$ belonging to the generic branch.

By a *finite continuous $\in$-chain* of countable elementary submodels of $H_{(2^\theta)^+}$ we mean a finite partial function $\{(\alpha, N_\alpha) : \alpha \in D\}$, where D is a finite subset of ω_1 and there N_α's are countable elementary submodels of $H_{(2^\theta)^+}$ such that for some *continuous* $\in$-chain $(M_\alpha : \alpha < \omega_1)$, we have that $N_\alpha = M_\alpha$ for all $\alpha \in D$. We shall identify $\{(\alpha, N_\alpha) : \alpha \in D\}$ with its range $\mathcal{N} = \{N_\alpha : \alpha \in D\}$ whenever there is no danger of confusion. So, similarly as before we let $\mathcal{P}$ be the collection of all mappings $p : \mathcal{N}_p \to S$, where

(1) $\mathcal{N}_p$ is a finite *continuous* $\in$-chain of countable elementary submodels of $H_{(2^\theta)^+}$,
(2) $M \in \mathcal{N}_p$ implies $p(M) \in S_{M \cap \omega_1}$,

We let $p \leq q$ if,

(3) $\mathcal{N}_p$ extends $\mathcal{N}_q$ as a function and p extends q as a function,
(4) $(\forall M \in \mathcal{N}_q)(\forall N \in (\mathcal{N}_p \cap M) \setminus \mathcal{N}_q)[q(M) \Vdash \check{\xi}_N \in \dot{b}_M]$.

6.2.1 Claim. *$\mathcal{P}$ is proper and it preserves S.*

Proof. In order to show that a given condition $p_0 \in \mathcal{P}$ does not force that S is not a Souslin tree, we take an arbitrary countable elementary submodel M of $(H_{(2^{(2^\theta)})^+}, \in)$, we form its extension

$$q_0 : \mathcal{N}_{q_0} \to S,$$

by letting

$$\mathcal{N}_{q_0} = \mathcal{N}_{p_0} \cup \{(\delta_0, M_0)\} \text{ and } q_0(M_0) \in S_\delta$$

for $M_0 = M \cap H_{\mathfrak{c}^+}$ and $\delta_0 = M \cap \omega_1$, and then we show that (q_0, s_0) is an $(M, \mathcal{P} \times S)$-generic condition for every choice $s_0 \in S_\delta$. To this end choose a dense-open subset $\mathcal{D}$ of $\mathcal{P}$ such that $\mathcal{D} \in M$ and choose an arbitrary extension (q, s) of (q_0, s_0). We need to find $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$ compatible to (q, s). Clearly, we may assume $(q, s) \in \mathcal{D}$.

Let $M_0, ..., M_{n-1}$ be the increasing enumeration of $\mathcal{N}_q \setminus M$. Choose $\gamma_0 < \delta_0$ higher than $N \cap \omega_1$ for all $N \in \mathcal{N}_q \cap M$ such that

$$(\forall N, N' \in \mathcal{N}_q \setminus M)[q(N) \restriction \delta_0 \neq q(N') \restriction \delta_0 \Rightarrow q(N) \restriction \gamma_0 \neq q(N') \restriction \gamma_0].$$

For each $i < n$ choose an automorphism σ_i of S such that $\sigma_i \in M$ and such that for every u of length α extending $s \restriction \gamma_0$,

$$\sigma_i(u) = (q(N_i) \restriction \gamma_0)^\frown(u \restriction [\gamma_0, \alpha)).$$

We may assume that $\sigma_i = \sigma_j$ whenever $q(N_i) \restriction \gamma_0 = q(N_j) \restriction \gamma_0$, or equivalently, whenever $q(N_i) \restriction \delta_0 = q(N_j) \restriction \delta_0$.

Let $\mathcal{F}$ be the family of all sequences $\upsilon = \{(t_i, \xi_i) : i < n\}$ for which we can find a $(\bar{q}, \bar{s}) \in \mathcal{D}$ end-extending $(q \cap M, s \restriction \gamma_0)$ and isomorphic to (q, s) over $(q \cap M, s \restriction \gamma_0)$ such that if $\bar{N}_0, ..., \bar{N}_{n-1}$ is the increasing enumeration of $\mathcal{N}_{\bar{q}} \setminus (\mathcal{N}_q \cap M)$ then for all $i < n$,

$$t_i = \bar{s} \restriction (\bar{N}_i \cap \omega_1) \text{ and } \xi_i = \sup(\bar{N}_i \cap \theta).$$

Then $\mathcal{F} \in M \cap H_{(2^\theta)^+} = M_0 \in ... \in M_{n-1}$ and

$$\{(s \restriction (M_0 \cap \omega_1), \xi_{M_0}), ..., (s \restriction (M_{n-1} \cap \omega_1), \xi_{M_{n-1}})\} \in \mathcal{F}.$$

Clearly, we can also choose a function $f \in M$ from $\mathcal{F}$ into S such that $\bar{s} = f(\upsilon)$ is the second coordinate of a witness $(\bar{q}, \bar{s}) \in \mathcal{D}$ for the membership of $\upsilon = \{(t_i, \xi_i) : i < n\}$ in $\mathcal{F}$.

For $\upsilon = \{(t_i, \xi_i) : i < n-1\}$ of length $n-1$ let $\dot{X}_\upsilon$ be the S-name for the set of all $\xi < \theta$ for which we can find $t \in S$ such that $\upsilon^\frown(t, \xi) \in \mathcal{F}$ and such that $\sigma_{n-1}(f(\upsilon^\frown(t, \xi)))$ belongs to the generic branch. Let $\partial\mathcal{F}$ be the collection of all such $\upsilon = \{(t_i, \xi_i) : i < n-1\}$ for which we can find u extending t_{n-2} and forcing that $\dot{X}_\upsilon$ is a stationary subset of θ. Note that since for

$$\upsilon = \{(s \restriction (M_0 \cap \omega_1), \xi_{M_0}), ..., (s \restriction (M_{n-2} \cap \omega_1), \xi_{M_{n-2}})\}$$

the S-name $\dot{X}_v$ belongs to the model N_{n-1} and since $\sigma_{n-1}(s)$ forces that $\xi_{M_{n-1}}$ belongs to $\dot{X}_v$ there must be a $u \in S \cap M$ below s forcing $\dot{X}_v$ is a stationary subset of θ. It follows that

$$\{(s \upharpoonright (M_0 \cap \omega_1), \xi_{M_0}), ..., (s \upharpoonright (M_{n-2} \cap \omega_1), \xi_{M_{n-2}})\} \in \partial\mathcal{F}.$$

Note that $\partial\mathcal{F} \in M$ and that we can extend f to $\mathcal{F} \cup \partial\mathcal{F}$ that picks the node $u \in S$ forcing $\dot{X}_v$ is stationary such that $f \in M$. Continuing this way n-times we arrive at a function $f : \bigcup_{i<n} \partial^i\mathcal{F} \to S$ such that $f \in M$ and at the conclusion that the empty sequence $\emptyset$ belongs to $\partial^n\mathcal{F}$ and that the node $f(\emptyset)$ belongs to the (M, S)-generic branch $s \upharpoonright \delta_0$ while its image $\sigma_0(f(\emptyset))$ forces that $\dot{X}_\emptyset$ is a stationary subset of θ. We now work in M and use these data as follows. Since $\sigma_0(f(\emptyset))$ forces that $\dot{X}_\emptyset$ is not orthogonal to $\dot{\mathcal{I}}$, we can find $u_0 \geq f(\emptyset)$ belonging to the generic branch $s \upharpoonright \delta_0$ and an infinite set $a_0 \subseteq \theta$ in M such that

$$\sigma_0(u_0) \Vdash \check{a}_0 \subseteq \dot{X}_\emptyset \text{ and } \check{a}_0 \in \dot{\mathcal{I}}.$$

Let $\mathcal{M}_0$ be the collection of all models $N \in \mathcal{N}_q \setminus M$ such that $q(N) \upharpoonright \delta_0 = q(M_0) \upharpoonright \delta_0$. Then for $N \in \mathcal{M}_0$, the node $q(N)$ extends $\sigma_0(u_0)$ and so it forces that a_0 is almost included in $\dot{b}_N$. Since for $N \in \mathcal{M}_0$, the node $q(N)$ also decides $\dot{b}_N$, we can find a single ordinal $\xi_0 \in a_0$ such that

$$(\forall N \in \mathcal{M}_0)\ q(N) \Vdash \check{\xi}_0 \in \dot{b}_N.$$

By the definition of $\dot{X}_0$, we can assume that for some $t_0 \in S \cap M$, we have that $\{(t_0, \xi_0)\} \in \partial^{n-1}\mathcal{F}$ and that $\sigma_0(u_0)$ forces $\sigma_0(f(\{(t_0, \xi_0)\}))$ in in the generic branch. This means that $u_0 \geq_S f(\{(t_0, \xi_0)\})$, and so in particular $\sigma_1(u_0)$ forces that $\dot{X}_{\{(t_0,\xi_0)\}}$ is stationary. Hence, we can continue and find $u_1 \geq u_0$ on the branch $s \upharpoonright \delta_0$ and an infinite set $a_1 \subseteq \theta$ in M such that

$$\sigma_1(u_1) \Vdash \check{a}_1 \subseteq \dot{X}_{\{(t_0,\xi_0)\}} \text{ and } \check{a}_1 \in \dot{\mathcal{I}}.$$

So, as above, we can find a single ordinal $\xi_1 \in a_0$, such that

$$(\forall N \in \mathcal{M}_0)\ q(N) \Vdash \check{\xi}_0 \in \dot{b}_N,$$

where $\mathcal{M}_1$ is the collection of all models $N \in \mathcal{N}_q \setminus M$ such that $q(N) \upharpoonright \delta_0 = q(M_1) \upharpoonright \delta_0$. It is clear that proceeding in this way, we arrive at

$$\{(t_0, \xi_0), ..., (t_{n-1}, \xi_{n-1})\} \in \mathcal{F} \cap M$$

such that id $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$ is the witnessing condition, then (q, s) and $(\bar{q}, \bar{s})$ are compatible in $\mathcal{P} \times S$. This finishes the proof of the Claim.

□

The variation is made in order to ensure that if $\mathcal{G}$ is a sufficiently generic filter of $\mathcal{P}$ then $\mathcal{N} = \bigcup_{p\in\mathcal{G}} \mathcal{N}_p$ is a *continuous* $\in$-chain of length ω_1 of countable elementary submodels of H_θ. Let $g = \bigcup \mathcal{G} : \mathcal{N} \to S$ and let $C = \{\xi_N : N \in \mathcal{N}\}$. Then C is a subset of θ of order type ω_1 that is a closed subset of its supremum $\sup(C)$ and that has the property that if s is any node of S above which the range of g is dense then

$$s \Vdash [\dot{C}]^{\aleph_0} \subseteq \dot{\mathcal{I}}.$$

This finishes the proof.

□

6.3 Corollary. *Assume PFA(S). Then the coherent Souslin tree S forces that for every P-ideal $\mathcal{I}$ of countable subsets of ω_1, either there is a closed and unbounded set $C \subseteq \omega_1$ such that $[C]^{\aleph_0} \subseteq \mathcal{I}$ or there is a stationary subset $D \subseteq \omega_1$ orthogonal to $\mathcal{I}$.*

6.4 Corollary. *Assume PFA(S). Then the coherent Souslin tree S forces that for every $\aleph_1$-generated ideal $\mathcal{J}$ of of subsets of ω_1, either there is closed and unbounded set $C \subseteq \omega_1$ such that $[C]^{\aleph_0} \subseteq \dot{\mathcal{J}}^\perp$, or there is a stationary subset $D \subseteq \omega_1$ such that $[D]^{\aleph_0} \subseteq \dot{\mathcal{J}}$.*

Proof. This follows by applying the previous result to the ideal

$$\dot{\mathcal{I}} = \{A \in [\omega_1]^{\leq\aleph_0} : A \perp \dot{\mathcal{J}}\}.$$

Note that by our assumption that S forces that $\dot{\mathcal{J}}$ is $\aleph_1$-generated and the fact that $\mathfrak{b} = \omega_2$ after forcing by S, we can conclude that $\dot{\mathcal{I}}$ is indeed a P-ideal in the forcing extension of S. □

6.5 Corollary. *Assume PFA(S). The coherent Souslin tree S forces that every coherent sequence C_α ($\alpha \in E$) indexed by a subset E of some regular cardinal $\theta > \omega_1$ such that $\{\alpha < \theta : \mathrm{cf}(\alpha) = \omega_1\} \subseteq E$ must be trivial.*

Proof. For $\alpha < \beta < \theta$ put $\alpha \prec \beta$ whenever α is a limit point of C_β and consequently C_α and C_β are both defined, i.e., $\alpha, \beta \in E$. Then $\prec$ is a tree ordering on θ. Let $\mathcal{I}$ be the ideal of countable subsets A of θ with the property that every infinite subset of A is unbounded in the tree $(\theta, \prec)$. Then it is easily seen that $\mathcal{I}$ is a P-ideal on θ with no $C \subseteq \theta$ of order type ω_1 that is a closed subset of $\delta = \sup(C)$ such that $[C]^{\aleph_0} \subseteq \mathcal{I}$. By Theorem 6.2 there is stationary $E \subseteq \{\delta < \theta : \mathrm{cf}(\delta) = \omega\}$ orthogonal to $\mathcal{I}$ and this easily gives us that the C_α ($\alpha \in E$) must be trivial, i.e., that there is a closed and unbounded set $D \subseteq \theta$ such that $C_\beta = D \cap \beta$ for every limit point β of D. □

Note that this gives another although quite closely related proof of Theorem 4.2 above. For more on this variation on the P-ideal dichotomy the reader is referred to [9].

7. PID and the S-space problem

Recall that the P-ideal dichotomy was first discovered in its equivalent dual form as the combinatorial essence lying behind the solution of the S-space problem ([19],[21]; see also [29]). In this section we review this connection and make a variation that will be relevant to our further analysis of forcing by S.

Suppose K is a compact space and that X is a subspace of K that is not Lindelöf. We would like to conclude that either,

(a) X contains an uncountable discrete subspace, or
(b) K is not countably tight.

Going to a subspace of X, we may assume that $|X| = \aleph_1$ and that there is a sequence U_x $(x \in X)$ of open subsets of K such that

$$(\forall x \in X)\ x \in U_x \text{ and } |U_x \cap X| \leq \aleph_0.$$

Moreover, we choose another sequence V_x $(x \in X)$ of open subsets of K such that

$$(\forall x \in X)\ x \in V_x \subseteq \overline{V_x} \subseteq U_x.$$

Let $\mathcal{I} = \{A \in [X]^{\leq\aleph_0} : (\forall x \in X)\ |A \cap V_x| < \aleph_0\}$. Note that if $\mathfrak{b} = \omega_2$, the ideal $\mathcal{I}$ is a P-ideal, so let us examine the two alternatives of PID. If there is an uncountable $Y \subseteq X$ such that $[Y]^{\aleph_0} \subseteq \mathcal{I}$, then the Y would be the desired uncountable discrete subspace of X. The other alternative would give us a countable decomposition of X into subsets that are orthogonal to $\mathcal{I}$, so let us examine one such subset Z of X.

7.1 Lemma. *If $\mathfrak{b} = \omega_2$, the sequential closure $\overline{Z}^{\omega_1}$ in K of any subset Z of X that is orthogonal to $\mathcal{I}$ must be included in $\bigcup_{x\in X} \overline{V_x}$.*

Proof. We shall prove by induction on $\alpha < \omega_1$ that any subset Z of X with the property that $\overline{Z}^{\alpha} \nsubseteq \bigcup_{x\in X} \overline{V_x}$ contains an infinite subset belonging to $\mathcal{I}$. The case $\alpha = 1$ is clear as any sequence $A = \{z_n : n < \omega\}$ converging to a point in $K \setminus \bigcup_{x\in X} \overline{V_x}$ has finite intersection with all V_x's and so it belongs to $\mathcal{I}$. Suppose the conclusion is true for some α and that we have a point $y \in \overline{Z}^{\alpha+1}$ not belonging to $\bigcup_{x\in X} \overline{V_x}$. Pick a sequence $\{y_n : n < \omega\} \subseteq \overline{Z}^{\alpha}$ converging to x. For $n < \omega$, let $X_n = \{x \in X : y_n \notin \overline{V_x}\}$. By the inductive hypothesis, for each $n < \omega$, there is an infinite set $A_n = \{z_n^k : k < \omega\} \subseteq Z$ such that $A_n \cap V_x$ is finite for all $x \in X_n$. We may assume A_n's are pairwise disjoint. Pick an $x \in X$. Then there is $m_x < \omega$ such that $x \in X_n$ for all $n \geq m_x$.

Pick $f_x \in \omega^\omega$ such that $z_n^k \notin V_x$ for all $n \geq m_x$ and all $k \geq f_x(n)$. Since $\mathfrak{b} > |X|$ there is $g \in \omega^\omega$ such that $f_x <^* g$ for all $x \in X$. Let $A = \{z_n^{g(n)} : n < \omega\}$. Then $A \cap V_x$ is finite for all $x \in X$ and so $A \in \mathcal{I}$. This finishes the proof. □

Before we proceed further, let us state what has been proven so far in this section.

7.2 Theorem. *Assume PID. The following statements are equivalent:*

(1) *Hereditarily separable subspaces of sequential compacta are hereditarily Lindelöf.*
(2) $\mathfrak{b} = \omega_2$.

Proof. The implication from (1) to (2) follows from the result of Chapter 1 of [23]. □

Recall, that the second alternative of the P-ideal dichotomy applied to the ideal $\mathcal{I} = \{A \in [X]^{\leq\aleph_0} : (\forall x \in X)\ |A \cap V_x| < \aleph_0\}$ above has much stronger implications on the space X if we assume $\mathfrak{p} = \omega_2$ rather than $\mathfrak{b} = \omega_2$, the conclusion that X can be covered by countably many subsets that are *free sequences* in K. So, in particular PID and $\mathfrak{p} = \omega_2$ imply that hereditarily separable regular spaces are hereditarily Lindelöf. Unfortunately, this implication cannot be directly used here in our analysis of the forcing relation of the Souslin tree S, since S forces the equality $\mathfrak{p} = \omega_1$. The above analysis, however, does suggest a way out. To describe this we need the following two natural variations of the sequential closure and the cardinal $\mathfrak{p}$.

7.3 Definition. Fix a non-principal ultrafilter $\mathcal{U}$ on ω. A subset Z of some topological space X is *$\mathcal{U}$-sequentially closed* if for every sequence $\{z_n : n < \omega\} \subseteq Z$, if $\lim_{n\to\mathcal{U}} z_n$ exists in X then it must belong to Z. A space X is *$\mathcal{U}$-sequential* if $\mathcal{U}$-sequentially closed subsets of X are in fact closed.

Clearly, sequential spaces are $\mathcal{U}$-sequential for every non-principal ultrafilter $\mathcal{U}$ on ω and $\mathcal{U}$-sequential spaces are countably tight.

For a non-principal ultrafilter $\mathcal{U}$ on ω, let

$$\mathfrak{p}_\mathcal{U} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{U} \text{ and } (\forall b \in [\omega]^\omega)(\exists a \in \mathcal{A})\ b \not\subseteq^* a\}.$$

Clearly, $\mathfrak{p}_\mathcal{U} \geq \mathfrak{p}$ for every non-principal ultrafilter $\mathcal{U}$ on ω, and in fact

$$\mathfrak{p} = \min\{\mathfrak{p}_\mathcal{U} : \mathcal{U} \in \beta\omega \setminus \omega\}.$$

Out interest in these notions here is based on the following two facts.

7.4 Lemma. *Assume $PFA(S)$. Then for every non-principal ultrafilter $\mathcal{U}$ on ω, the Souslin tree S foces that $\mathfrak{p}_{\check{\mathcal{U}}} = \omega_2$.*

Proof. Fix a non-principal ultrafilter $\mathcal{U}$ on ω and an S-name $\dot{\mathcal{X}}$ for a subset of $\check{\mathcal{U}}$ of cardinality $\aleph_1$. By the countable chain condition of S or the fact that S has cardinality $\aleph_1$ we can find a subset $\mathcal{Y}$ of $\mathcal{U}$ of cardinality $\aleph_1$ such that every node of S forces the inclusion $\dot{\mathcal{X}} \subseteq \check{\mathcal{Y}}$. We have already mentioned that $PFA(S)$ implies that $\mathfrak{p} = \omega_2$ so we can find infinite $B \subseteq \omega$ such that $B \setminus A$ is finite for all $A \in \mathcal{Y}$. It follows that every node of S forces that $\check{B}$ is almost included in every element of $\dot{\mathcal{X}}$. This finishes the proof. □

7.5 Theorem. *Assume $PFA(S)$ and let $\mathcal{U}$ be any non-principal ultrafilter on ω. Then the Souslin tree S forces that if $\dot{X}$ is a non-Lindelöf subspace of some compact space $\dot{K}$, then either*

(1) *$\dot{X}$ contains an uncountable discrete subspace, or*
(2) *The $\check{\mathcal{U}}$-sequential closure in K of some subset of X is a $\check{\mathcal{U}}$-sequentially closed non-closed subset of $\dot{K}$.*

8. Countably tight compacta are $\mathcal{U}$-sequential

Fix a non-principal ultrafilter $\mathcal{U}$ on ω. A *well-founded $\mathcal{U}$-tree* is a collection T of finite subsets of ω such that

(1) $\emptyset \in T$ and $t \in T$ implies that all initial segments of t are in T,
(2) if $t \cup \{m\} \in T$ for some $m > \max(t)$ then
$$\{n < \omega : n > \max(t) \text{ and } t \cup \{n\} \in T\} \in \mathcal{U},$$
(3) there is no infinite subset M of ω such that all finite initial segments of M belong to T.

Let ∂T be the collection of all terminal nodes of T, the collection of nodes of T with no proper end-extensions in T. Then for every compact Hausdorff space K, any assignment $\{x_t : t \in \partial T\} \subseteq K$ extends uniquely to the global assignment $\{x_t : t \in T\}$ defined recursively on the rank of T as follows. If T has a rank 1, i.e., $T = \{\emptyset\} \cup \{\{n\} : n \in M\}$ for some $M \in \mathcal{U}$, we let $x_\emptyset = \lim_{n \to \mathcal{U}} x_{\{n\}}$. If T has rank > 1, for each $\{n\} \in T$, we apply the inductive hypothesis and extend the assignment $\{t \in \partial T : n = \min(t)\}$ to the assignment $\{t \in T : n = \min(t)\}$ and again let $x_\emptyset = \lim_{n \to \mathcal{U}} x_{\{n\}}$. We call $x_\emptyset$ *the limit of the assignment* $\{x_t : t \in \partial T\}$ and write
$$x_\emptyset = \lim_{t \in \partial T} x_t.$$
Note that this limit is in fact limit of an *ultrafilter* on the boundary ∂T of T, defined as follows. Fist of all, we define a *$\mathcal{U}$-subtree* to be any subset T_0 of T with the following properties:

(4) $\emptyset \in T_0$,

(5) if $t \in T_0$ and if $t \notin \partial T$ then

$$\{n < \omega : n > \max(t) \text{ and } t \cup \{n\} \in T_0\} \in \mathcal{U}.$$

A simple induction on the rank of T shows that the collection

$$\{\partial T_0 : T_0 \text{ a } \mathcal{U}\text{-subtree of } T\}$$

generates an ultrafilter on $\mathcal{U}_T$ on ∂T such that for every assignment $\{x_t : t \in \partial T\}$ of points in some Hausdorff space X, we have that

$$\lim_{t \in \partial T} x_t = \lim_{t \to \mathcal{U}_T} x_t,$$

whenever, of course, one of these limits exists.
The purpose of this definition is to give a more manageable representation of the $\mathcal{U}$-sequential closures of subsets of K. This is given in the following straightforward lemma.

8.1 Lemma. *Fix a subset Z of some compact space K. Let $\overline{Z}^{\mathcal{U}}$ be the $\mathcal{U}$-sequential closure of the set Z in K. Then*

$$\overline{Z}^{\mathcal{U}} = \{\lim_{t \in \partial T} x_t : T \textit{ a well-founded } \mathcal{U}\textit{-tree and } \{x_t : t \in \partial T\} \subseteq Z\}.$$

Proof. Let Z^* denotes the set on the right hand side of this equality. From the definition of $\lim_{t \in \partial T} x_t$, it is clear that $Z^* \subseteq \overline{Z}^{\mathcal{U}}$. For the other inclusion, recall that $\overline{Z}^{\mathcal{U}} = \bigcup_{\alpha < \omega_1} \overline{Z}^{\alpha}$, where $\overline{Z}^{\alpha}$ is defined recursively, by $\overline{Z}^0 = Z$,

$$\overline{Z}^{\alpha+1} = \{\lim_{n \to \mathcal{U}} x_n : \{x_n : n < \omega\} \subseteq \overline{Z}^{\alpha}\},$$

and $\overline{Z}^{\lambda} = \bigcup_{\alpha < \lambda} \overline{Z}^{\alpha}$, for a countable limit ordinal λ. So, it suffices to show by induction on $\alpha < \omega_1$ that $\overline{Z}^{\alpha} \subseteq Z^*$. For the inductive step from α to $\alpha + 1$, suppose we are give an $x = \lim_{n \to \mathcal{U}} x_n$, where $\{x_n : n < \omega\} \subseteq \overline{Z}^{\alpha}$. By the inductive hypothesis, for each $n < \omega$, we can fix a well founded $\mathcal{U}$-tree T_n such that $x_n = \lim_{t \in \partial T_n} x_t$ for some assignment $\{x_t : t \in \partial T_n\} \subseteq Z$. Let

$$T = \{\{n\} \cup t : n < \omega, t \in T_n \text{ and } \min(t) > n\}.$$

Clearly, T is a well-founded $\mathcal{U}$-tree such that

$$\partial T = \{\{n\} \cup t : n < \omega, t \in \partial T_n \text{ and } \min(t) > n\}.$$

Note also that if we define the assignment $\{y_t : t \in \partial T\} \subseteq Z$ by letting $y_{\{n\} \cup t} = x_t$, we get that $\lim_{t \in \partial T} y_t = x$. It follows that x belongs to Z^*. This finishes the proof. $\square$

We are now in a position to state and prove the main result of this section.

8.2 Theorem. *Assume $PFA(S)$. For every non-principal ultrafilter $\mathcal{U}$ on ω, the Souslin tree S forces that every countably tight compactum is $\check{\mathcal{U}}$-sequential.*

Proof. Suppose that $\dot{X}$ is an S-name for a $\check{\mathcal{U}}$-sequentially closed non-closed subset of some compact space $\dot{K}$. We shall show that S forces that $\dot{K}$ is not countably tight. Since, in particular S forces that $\dot{X}$ is not Linelöf, we can fix an S-name for an assignment

$$x \in V_x \subseteq \overline{V_x} \subseteq U_x \ (x \in \dot{X})$$

of open sets of $\dot{K}$ such that S forces that $\dot{X} \setminus \bigcup_{x\in Y} U_x \neq \emptyset$ for all countable subset Y of $\dot{X}$. We will find a proper poset $\mathcal{P}$ which preserves S and forces a sequence $\dot{x}_\xi$ $(\xi < \omega_1)$ of S-names of elements of $\dot{X}$ such that for all $\alpha < \omega_1$, S forces that

$$\{\dot{x}_\xi : \xi \leq \alpha\} \subseteq V_{\dot{x}_\alpha} \text{ and } \{x_\xi : \xi > \alpha\} \subseteq \dot{X} \setminus U_{\dot{x}_\alpha}.$$

So, in particular, S forces that $\dot{K}$ is not a countably tight compactum. We shall assume that S forces that $\dot{K}$ and $\dot{X}$ live on some ordinal $\check{\nu}$ and we shall fix a sufficiently large regular cardinal θ and consider only countable elementary submodels of the expanded structure $(H_\theta, \in, <_\theta)$, where $<_\theta$ is a fixed well-ordering of H_θ. We shall also fix an enumeration

$$N = \{a_i^N : i < \omega\}$$

for every countable elementary submodel N of $(H_\theta, \in, <_\theta)$. We also fix an S-name $\dot{\mathcal{F}}$ for a maximal filter of $\check{\mathcal{U}}$-sequentially closed subsets of $\dot{X}$ such that

$$(\forall Y \in [\dot{X}]^{\leq\aleph_0}) \ \dot{X} \setminus \bigcup_{x\in Y} U_x \in \dot{\mathcal{F}}.$$

Then for every countable elementary submodel N of $(H_\theta, \in, <_\theta)$, we can associate an S-name $\dot{x}_N$ for an element of $\dot{X}$ as follows. For each $k < \omega$, we first let $\dot{y}_k^N$ be the S name for the first element of the intersection of all subsets of $\dot{X}$ that are interpretations of S-names appearing in the finite sets $\{a_i^N : i \leq k\}$, or in other words an S-name with the property that S forces

$$\dot{y}_k^N = \min(\bigcap(\dot{\mathcal{F}} \cap \{a_i^N : i \leq k\})).$$

Thus, in particular S forces that $\{\dot{y}_k^N : k < \omega\} \subseteq \dot{X} \cap N[G]$. Finally, let

$$\dot{x}_N = \lim_{k\to\check{\mathcal{U}}} \dot{y}_k^N.$$

This, in particular, gives us the S-names $V_{\dot{x}_N}$ and $U_{\dot{x}_N}$ for open subsets of $\dot{K}$, which we simply denote as $\dot{V}_N$ and $\dot{U}_N$, respectively.

Let $\mathcal{P}$ be the collection of all mappings $p : \mathcal{N}_p \to S \times \omega_1$ where $\mathcal{N}_p$ is a finite $\in$-chain of countable elementary submodels of $(H_\theta, \in, <_\theta)$ such that for all $N \in \mathcal{N}_p$ the pair $p(N) = (p_0(N), p_1(N))$ has the following properties:

(1) $p_0(N) \in S \setminus N$ and $p_1(N) \in N \cap \omega_1$,
(2) $p_0(N)$ decides the values of $\dot{x}_N$ and $\dot{y}_k^N$'s as well as all the statements of the forms $\check{\alpha} \in \dot{V}_N$ and $\check{\alpha} \in \dot{U}_N$, for $\alpha \in N \cap \nu$.

Moreover, we require that for all $M \in N$ from $\mathcal{N}_p$, we have the following two conditions:

(3) $p_0(M) \in N$, and
(4) $p_0(M) <_S p_0(N)$ and $M \cap \omega_1 > p_1(N)$ imply $p_0(N) \Vdash \dot{x}_M \in \dot{V}_N$.

We order $\mathcal{P}$ by the inclusion.

8.2.1 Claim. *$\mathcal{P}$ is proper and it preserves S.*

Proof. Since the properness of $\mathcal{P}$ will follow from the proof of the preservation of S we again concentrate on preserving S. To this end, we pick an arbitrary p_0 in $\mathcal{P}$ and show that it does not force that S is not Souslin. Choose a countable elementary submodel M of $H_{(2^\theta)^+}$ containing all the relevant objects including our fixed condition $p^0 \in \mathcal{P}$. Let $M^0 = M \cap H_\theta$ and let

$$q^0 = p^0 \cup \{(M^0, q^0(M^0))\},$$

where $q_0^0(M^0)$ is any node of $S \setminus N$ deciding the values of $\dot{x}_N$ and $\dot{y}_k^N$'s and all statements of the form $\check{\alpha} \in \dot{V}_N$ and $\check{\alpha} \in \dot{U}_N$ for $\alpha \in N \cap \nu$. Let $\delta = M \cap \omega_1$. Note that p^0 will not be able to force that S is not a Souslin tree once we show that for every choice of $s_0 \in S_\delta$ the pair (q^0, s_0) is an $(M, \mathcal{P} \times S)$-generic condition. To this end, consider a dense-open subset $\mathcal{D}$ of $\mathcal{P} \times S$ such that $\mathcal{D} \in M$ and consider an arbitrary extension (q, s) of (q_0, s_0). We need to find $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$ compatible with (q, s). We may assume that s has height bigger than the height of any node in the range of q^0. Choose $\gamma_0 < \delta$ that is bigger than any height of a node in the range of $q^0 \cap M$ and any ordinal of the form $q^1(N)$ for $N \in \mathcal{N}_q \setminus M$ provided of course that this ordinal is $< \delta$. Moreover, as before, we assume that γ_0 is large enough such that

$$(\forall N, N' \in \mathcal{N}_q \setminus M)\ q^0(N) \restriction [\gamma_0, \delta) = q^0(N') \restriction [\gamma_0, \delta),\ \text{and so}$$

$$(\forall N, N' \in \mathcal{N}_q \setminus M)[q^0(N) \restriction \delta \neq q^0(N') \restriction \delta \Rightarrow q^0(N) \restriction \gamma_0 \neq q^0(N') \restriction \gamma_0].$$

Let $N_0, ..., N_{n-1}$ be the increasing enumeration of all $N \in \mathcal{N}_q \setminus M$ such that $q^0(N)$ and s agree on all ordinals $\geq \gamma_0$ in common domain. As before, for each $i < n$, we fix an automorphism $\sigma_i \in M$ of S which maps $q(M_i) \restriction \gamma_0$ to $s \restriction \gamma_0$, and therefore maps $q(M_i)$ below s.

As before, let $\mathcal{F}$ be the collection of all sequences $\{(t_i, \xi_i) : i < n\}$ of elements of $S \times \nu$ for which we can find $(\bar{q}, \bar{s}) \in \mathcal{D}$ end extending $(p_0, s \restriction \gamma_0)$, and in fact isomorphic with (q, s) over p_0, γ_0, and σ_i $(i < n)$ such that $t_i = \sigma_i(\bar{q}^0(\bar{N}_i))$ and $\bar{q}^0(\bar{N}_i) \Vdash \check{\xi}_i = x_{\bar{N}_i}$ for all $i < n$, where $\bar{N}_0, ..., \bar{N}_{n-1}$ is the increasing enumeration of those $\bar{N} \in \mathcal{N}_{\bar{q}} \setminus \mathcal{N}_{p_0}$ for which $\bar{q}(\bar{N})$ and $\bar{s}$ agree on ordinals in common domain that are $\geq \gamma_0$. For $\upsilon = \{(t_i, \xi_i) : i < n\}$ in $\mathcal{F}$ let $f(\upsilon)$ be the minimal node $\bar{s}$ extending t_{n-1} and appearing as the second coordinate of some pair $(\bar{q}, \bar{s}) \in \mathcal{D}$ witnessing $\upsilon \in \mathcal{F}$. Clearly $f, \mathcal{F} \in M \cap H_\theta = M_0$ and $\{(\sigma_0(q(N_0)), \xi_{N_0}), ..., (\sigma_{n-1}(q(N_{n-1})), \xi_{N_{n-1}})\} \in \mathcal{F}$. Note that we have arranged that $\{(\sigma_0(q(N_0)), \xi_{N_0}), ..., (\sigma_{n-1}(q(N_{n-1})), \xi_{N_{n-1}})\}$ is separated by the elementary submodels $N_0 \in N_1 \in ... \in N_{n-1}$. For a sequence $\upsilon = \{(t_i, \xi_i) : i < n-1\}$, let $\dot{X}_\upsilon$ be the S-name for the collection of all ordinals $\xi < \nu$ for which we can find a node $t \in S$ such that $\upsilon^\frown(t, \xi) \in \mathcal{F}$ and such that $\sigma_{n-1}^{-1}(f(\upsilon^\frown(t, \xi)))$ belonging to the generic branch. Let $\partial\mathcal{F}$ be the collection of sequences $\upsilon = \{(t_i, \xi_i) : i < n-1\}$ for which there is $u \in S$ extending t_{n-1} such that

$$\sigma_{n-1}^{-1}(u) \Vdash \overline{\dot{X}_\upsilon}^{\check{u}} \in \dot{\mathcal{F}}.$$

Extend the mapping f from $\mathcal{F}$ to $\partial\mathcal{F}$ that for a given $\upsilon \in \partial\mathcal{F}$, $f(\upsilon)$ is the minimal such a node u. Clearly $\partial\mathcal{F}$ and the extended function still belong to M_0 and therefore to all the models of the chain $N_0 \in N_1 \in ... \in N_{n-1}$. So a simple elementarity argument over N_{n-1} shows that

$$\{(\sigma_0(q^0(N_0)), \xi_{N_0}), ..., (\sigma_{n-2}(q^0(N_{n-2})), \xi_{N_{n-2}})\} \in \partial\mathcal{F}.$$

Similarly for a sequence υ of length $n-2$ we let $\dot{X}_\upsilon$ be the S-name for the collection of all ordinals $\xi < \nu$ for which we can find a node $t \in S$ such that such that $\upsilon^\frown(t, \xi) \in \partial\mathcal{F}$ and such that $\sigma_{n-1}^{-1}(f(a^\frown(t, \xi)))$ belonging to the generic branch. Let $\partial^2\mathcal{F}$ be the collection of all sequences a for which there is $u \in S$ extending t_{n-2} such that

$$\sigma_{n-2}^{-1}(u) \Vdash \overline{\dot{X}_\upsilon}^{\check{u}} \in \dot{\mathcal{F}}.$$

We extend f to $\partial^2\mathcal{F}$ by letting $f(\upsilon)$ be the minimal such node u. We still have that $\partial^2\mathcal{F}$ and the extended f belongs to M_0 and therefore to all submodels $N_0 \in N_1 \in ... \in N_{n-1}$. So a simple elementarity argument over N_{n-2} shows that

$$\{(\sigma_0(q^0(N_0)), \xi_{N_0}), ..., (\sigma_{n-3}(q^0(N_{n-3})), \xi_{N_{n-3}})\} \in \partial^2\mathcal{F}.$$

Proceeding in this way we arrive at $\partial^{n-1}\mathcal{F} \in M_0$ and the extended mapping $f \in M_0$ such that $\{(\sigma_0(q^0(N_0)), \xi_{N_0})\} \in \partial^{n-1}\mathcal{F}$. Let $\dot{X}_\emptyset$ be the S-name for the collection of all $\xi < \nu$ for which we can find a node t extending $s \restriction \gamma_0$ such that $\{(t,\xi)\} \in \partial^{n-1}\mathcal{F}$ and such that $\sigma_0^{-1}(f(\{(t,\xi)\}))$ belong to the generic branch. Using the elementarity of N_0, the choice of the name $\dot{x}_{N_0}$ and the fact that $q^0 \Vdash \dot{x}_{N_0} = \dot{\xi}_{N_0}$ one easily shows that there must be a node $u_0 \in S \cap M$ such that $s \restriction \gamma_0 \leq u_0 <_S s$ and such that

$$\sigma_0^{-1}(u_0) \Vdash \overline{\dot{X}_\emptyset}^{\check{\mathcal{U}}} \in \dot{\mathcal{F}}.$$

Let k_0 be the integer such that $\dot{X}_\emptyset = a^{k_0}_{M_0}$. Let $\mathcal{M}_0$ be the collection of all $N \in \mathcal{N}_q \setminus M$ with the property that $q^1(N) < \delta$ and $\sigma_0(q^0(N)) \restriction \delta = s \restriction \delta$. Fix an $N \in \mathcal{M}_0$. Then from the definition of $\dot{x}_{M_0}$ and the fact that $q^0(N) \Vdash \dot{x}_{M_0} \in \dot{V}_N$ we get that

$$A_N = \{k < \omega : q^0(N) \Vdash \dot{y}^{M_0}_k \in \dot{V}_N\} \in \mathcal{U}$$

So we can choose an integer $k \geq k_0$ in the intersection of all sets A_N for $N \in \mathcal{M}_0$. Since $\dot{y}^{M_0}_k$ belongs to M_0 and it is a name for an element of $\dot{X}_\emptyset = a^{k_0}_{M_0}$, we can find $v_0 \in S \cap M_0$, $u_0 \leq_S v_0 <_S s$, a well founded $\mathcal{U}$-tree T, an assignment $\{x_\iota : \iota \in \partial T\} \subseteq M_0 \cap \nu$, all in M, such that

$$\sigma_0^{-1}(v_0) \Vdash \{x_\iota : \iota \in \partial T\} \subseteq \dot{X}_\emptyset \text{ and } \dot{y}^{M_0}_k = \lim_{\iota \in \partial T} x_\iota.$$

It follows that for all $N \in \mathcal{M}_0$, the set

$$B_N = \{\iota \in \partial T : q^0(N) \Vdash x_\iota \in \dot{V}_N\}$$

belongs to the ultrafilter $\mathcal{U}_T$ living on the boundary of the tree T. So, we can pick $\iota \in \partial T$ belonging to all sets B_N for $N \in \mathcal{M}_0$. Pick $(t_0, \xi_0) \in \partial^{n-1}F$ such that $\iota = \xi_0$. Since v_0 forces that $\xi \in \dot{X}_\emptyset$, we can find a node $u_1 \geq v_0$ in the generic branch $s \restriction \delta$ such that u_1 forces that $\sigma_0^{-1}(f(t_0,\xi_0))$ belongs to the generic branch. So in particular, $u_1 \geq_S \sigma_0^{-1}(f(t_0,\xi_0))$, and therefore,

$$\sigma_1^{-1}(u_1) \Vdash \overline{\dot{X}_{\{(t_0,\xi_0)\}}}^{\check{\mathcal{U}}} \in \dot{\mathcal{F}}.$$

Note that this means not only that we continue picking $v_1 \geq u_1$, (t_1, ξ_1), etc, but also that we have achieved part of the compatibility requirement between (q, s) and the desired copy $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$, the requirement that for all $N \in \mathcal{M}_0$ we will have that $\bar{q}^0(\bar{N}_0) <_S q^0(N)$ and that

$$q^0(N) \Vdash \dot{x}_{\bar{N}_0} \in \dot{V}_N.$$

So, it should be clear that continuing this procedure we can pick a sequence $\{(t_i, \xi_i) : i < n\} \in \mathcal{F} \cap M$ so that its witnessing condition $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$ is compatible with (q, s). This finishes the proof that $\mathcal{P}$ preserves S. □

The generic filter $\dot{\mathcal{G}}$ introduces an uncountable $\in$-chain $\dot{\mathcal{N}}$ of countable elementary submodels of H_θ and a mapping $\dot{g} : \mathcal{N} \to S \times \omega_1$ such that among other things $g^1(N) < N \cap \omega_1$ for all $N \in \dot{\mathcal{N}}$. Note that the above proof of properness of $\mathcal{P}$ and the preservation of S gives that the set $\dot{D} = \{N \cap \omega_1 : N \in \dot{\mathcal{N}}\}$ is stationary. Note also that $N \mapsto \dot{g}^1(N)$ induces a regressive mao $N \cap \omega_1 \mapsto \dot{g}^1(N)$ on $\dot{D}$, so we can find a condition p_0 in $\mathcal{P}$, an ordinal $\gamma_0 < \omega_1$, and a $\mathcal{P}$-name $\dot{E}$ for a stationary subset of $\dot{D}$ such that

$$p_0 \Vdash (\forall N \in \dot{\mathcal{N}})[N \cap \check{\omega}_1 \in \dot{E} \Rightarrow \dot{g}^1(N) = \check{\gamma}_0].$$

It follows that for all $\alpha < \omega_1$, the set

$$\mathcal{D}_\alpha = \{p \in \mathcal{P} : (\exists N \in \mathcal{N}_p)\ [\alpha \in N \text{ and } p^1(N) \leq \gamma_0]\}$$

is an open subset of $\mathcal{P}$ which is dense below p_0. Take a filter $\mathcal{G}$ of $\mathcal{P}$ intersecting all these sets and let $\bigcup \mathcal{H} = g : \mathcal{N} \to S \times \omega_1$. Then

$$\mathcal{M} = \{N \in \mathcal{N} : g^1(N) \leq \gamma_0\}$$

is an uncountable $\in$-subchain of $\mathcal{N}$. Note that for every pair $M \in N$ of elements of $\mathcal{M}$,

$$g^0(M) <_S g^0(N) \text{ implies } g^0(N) \Vdash \dot{x}_M \in \dot{V}_N \text{ and } \dot{x}_N \notin \dot{U}_M.$$

Find an $s \in S$ such that $\{g^0(N) : N \in \mathcal{M}\}$ is dense above s. Let $\dot{\mathcal{M}}_0$ be the S-name for the set of all $N \in \mathcal{M}$ such that $g^0(N)$ belongs to the generic branch. Then s forces that $\dot{\mathcal{M}}_0$ is uncountable, and

$$s \Vdash (\forall N \in \dot{\mathcal{M}}_0)\{\dot{x}_M : M \in \dot{\mathcal{M}}_0, M \in N\} \subseteq \dot{V}_N$$

and

$$s \Vdash (\forall N \in \dot{\mathcal{M}}_0)\{\dot{x}_M : M \in \dot{\mathcal{M}}_0, N \in M\} \subseteq \dot{K} \setminus \dot{U}_N.$$

So, in particular, s forces that $\{\dot{x}_N : N \in \dot{\mathcal{M}}_0\}$ is an uncountable sequence of elements of $\dot{X}$ that is free in K. This finishes the proof. □

8.3 Corollary. *Assume $PFA(S)$. The coherent Souslin tree S forces that every separable countably tight compactum has cardinality at most continuum.*

The following is a corollary of the proof of Theorem 8.2.

8.4 Theorem. *Assume $PFA(S)$. The coherent Souslin tree S forces that every countably tight compactum is sequentially compact.*

Proof. Fix an S-name $\dot{K}$ for a countably tight compactum and fix a nonprincipal ultrafilter $\mathcal{U}$ on ω. We may assume that S forces that $\dot{K}$

is a subset of some Tychonoff cube $[0,1]^{\check{\nu}}$ and that for some sequence $\{\dot{d}_n : n < \omega\}$ of S-names for elements of $[0,1]^{\check{\nu}}$,

$$\Vdash_S \overline{\{\dot{d}_n : n < \check{\omega}\}} = \dot{K}$$

and that our job is to show that there is $s \in S$ which forces that some subsequence of $\{\dot{d}_n : n < \omega\}$ converges to some point of $\dot{K}$. Let $\mathcal{C}$ be the collection of countable partial functions from ω_1 into $\{0,1\}$. Recall, that $\mathcal{C}$ forces CH. It should also be clear that $\mathcal{C}$ preserves S. By Corollary 8.3, we know that $\mathcal{C}$ forces that CH is true even after forcing by S and therefore $\mathcal{C}$ forces that S forces that $\dot{K}$ has cardinality at most $\aleph_1$. If $\mathcal{C}$ forces that S forces that $\dot{K}$, moreover, remains compact, we would be able to find an increasing sequence $\{n_k : k < \omega\}$ of integers and some $(p,s) \in \mathcal{C} \times S$ such that

$$(p,s) \Vdash \{\dot{d}_{n_k}; k < \check{\omega}\} \text{ is convergent in } \dot{K}.$$

So, in particular s forces that the sequence $\{\dot{d}_{n_k}; k < \check{\omega}\}$ is convergent in $\dot{K}$, as required. So, we are left with the alternative that $\mathcal{C}$ forces that $\dot{K}$ is an $\check{S}$-name for a $\mathcal{U}$-sequentially closed non closed subset of $[0,1]^{\check{\nu}}$. Then from the proof of Theorem 8.2, we can conclude that there is a $\mathcal{C}$-name for a proper poset $\dot{\mathcal{P}}$ which introduces a sequence $\{(U_t, \dot{(V)}_t) : t \in \dot{B}\}$ of pairs of basic open subsets of $[0,1]^{\check{\nu}}$ with common finite supports in ν and with open intervals with rational end-points as their values on supports, where $\dot{B}$ is an uncountable subset of S such that for each $t \in \dot{X}$ the closure of V_t is included in U_t and such that whenever c is a finite chain of elements of $\dot{X}$ and $c = a \cup b$ is its decomposition such that $s <_S t$ for all $s \in a$ and $t \in b$, then every $u \in \dot{B}$ which dominates c forces that

$$(1) \qquad (\textstyle\bigcap_{s\in a}([0,1]^{\check{\nu}} \setminus U_s)) \cap (\bigcap_{t\in b} \overline{V_t}) \cap \dot{K} \neq \emptyset.$$

Applying $PFA(S)$ to the poset $\mathcal{C} * \mathcal{P}$ and appropriately chosen set of $\aleph_1$ dense-open sets, we get a filter of $\mathcal{C} * \mathcal{P}$ whose interpretation gives us a sequence $\{(U_t, \dot{(V)}_t) : t \in A\}$ of pairs of basic open subsets of $[0,1]^{\check{\nu}}$ indexed by an uncountable subset A of S such that $\overline{V_t} \subseteq U_t$ for all $t \in A$ and such that for every finite chain $c = a \cup b$ of A decomposed into two chains $a <_S b$, every $u >_S c$ in A forces that the above intersection with $\dot{K}$ is not empty. Let $\dot{A}_G$ be the S-name for the intersection of A with the generic branch then there is an $s \in S$ forcing that

$$\{(\overline{\check{V}_t} \cap \dot{K}, \check{U}_t \cap \dot{K}) : t \in \dot{A}_G\}$$

is an uncountable free sequence of regular pairs of $\dot{K}$ (see [24]). So in particular s forces that $\dot{K}$ is not countably tight, a contradiction. □

While we will not need the following in the rest of the paper, we note that the above proof gives the following slightly more general result.

8.5 Theorem. *Assume $PFA(S)$. The coherent Souslin tree S forces that every countably tight compactum has a G_δ-point.*

Note that also the following consequence of Theorems 7.5 and 8.2.

8.6 Theorem. *Assume $PFS(S)$. The coherent Souslin tree S forces that every non-Lindelöf subspace $\dot{X}$ of a compact countably tight space $\dot{K}$ has an uncountable discrete subspace.*

Proof. We give a sketch of our first proof of this result as it is of independent interest. We assume that $\dot{K}$ lives on $\check{\nu}$ for some ordinal $\nu \geq \omega_1$ and that $\dot{X}$ lives on $\check{\omega}_1$. As before, for each $\alpha < \omega_1$ we fix S-names $\dot{U}_\alpha$ and $\dot{V}_\alpha$ for open subsets of $\dot{K}$ such that

$$\Vdash_S \check{\alpha} \in \dot{V}_\alpha \subseteq \overline{\dot{V}_\alpha} \subseteq \dot{U}_\alpha \text{ and } \check{\beta} \notin \dot{U}_\alpha$$

whenever $\alpha < \beta < \omega_1$. We will find a proper poset $\mathcal{P}$ which preserves S and forces the existence of an S-name $\dot{\Gamma}$ for a subset of ω_1 such that for some $s \in S$,

$$s \Vdash |\dot{\Gamma}| = \aleph_1 \text{ and } (\forall \alpha, \beta \in \dot{\Gamma})[\alpha < \beta \Rightarrow \alpha \notin \dot{V}_\beta].$$

So, in particular, s forces that $\dot{X}$ contains an uncountable discrete subspace.

As before, we fix a sufficiently large regular cardinal θ and consider only countable elementary submodels of the expanded structure $(H_\theta, \in, <_\theta)$, where $<_\theta$ is a fixed well-ordering of H_θ. For a countable elementary submodel N of $(H_\theta, \in, <_\theta)$, let $\delta_N = N \cap \omega_1$ and let $\dot{V}_N = \dot{V}_{\delta_N}$ and $\dot{U}_N = \dot{U}_{\delta_N}$. Let $\mathcal{P}$ be the collection of all mappings $p : \mathcal{N}_p \to S$ where $\mathcal{N}_p$ is a finite $\in$-chains of countable elementary submodels of $(H_\theta, \in, <_\theta)$ containing all relevant objects such that

(1) for all $N \in \mathcal{N}_P$, $p(N)$ decides all statements of the form $\check{\xi} \in \dot{U}_N$ and $\check{\xi} \in \dot{V}_N$ for $\xi < \delta_N$,
(2) for all $M, N \in \mathcal{N}_P$, if $M \in N$ then $p(M) \in N$.

For $p, q \in \mathcal{P}$, let $p \leq q$ if

(3) $\mathcal{N}_p \supseteq \mathcal{N}_q$ and $q = p \restriction \mathcal{N}_q$, and
(4) $(\forall N \in \mathcal{N}_q)(\forall M \in (\mathcal{N}_p \setminus \mathcal{N}_q) \cap N)[p(M) <_S p(N) \Rightarrow p(N) \Vdash \check{\delta}_M \notin \dot{V}_N]$.

As before the crucial part of the proof is contained in the following claim.

8.6.1 Claim. *$\mathcal{P}$ is proper and it preserves S.*

Proof. The proof is essentially the same as the proof of Claim 6.1.1 above but we give some details in order to expose its possible further applications. In order to show that given $p_0 \in \mathcal{P}$ does not force that S is not Souslin, we take a countable elementary submodel M of $H_{(2^\theta)^+}$, containing p_0 and other relevant object and show that if $M_0 = M \cap H_\theta$ and $q_0(M_0)$ is any node of S deciding all the statements of the form $\check{\xi} \in \dot{U}_{M_0}$ and $\check{\xi} \in \dot{V}_{M_0}$ for $\xi < \delta_{M_0}$, then for every $s_0 \in S_{\delta_{M_0}}$, the pair (q_0, s_0) is an $(M, \mathcal{P} \times S)$-generic condition, where

$$q_0 = p_0 \cup \{(M_0, q_0(M_0))\}.$$

To this end, choose a dense-open subset $\mathcal{D}$ of $\mathcal{P} \times S$ such that $\mathcal{D} \in M$ and choose an arbitrary extension (q, s) of (q_0, s_0). We may assume $(q, s) \in \mathcal{D}$ and we need to find $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$ compatible with (q, s). We may assume that the height of s is bigger than the height of any node in the range of q. Let $\delta = \delta_{M_0} = M \cap \omega_1$ and let $\gamma_0 < \delta$ be an ordinal bigger than the height of all nodes of the range of q that have heights $< \delta$ such that for every $u, v \in \{s\} \cup \text{range}(q) \setminus M$,

$$u \restriction [\gamma_0, \delta) = v \restriction [\gamma_0, \delta) \text{ and } u \restriction \delta \neq v \restriction \delta \Rightarrow u \restriction \gamma_0 \neq v \restriction \gamma_0.$$

Let $N_0, ..., N_{n-1}$ be the increasing enumeration of all elements N of $\mathcal{N}_p \setminus M$ with the property that $q(N)$ agrees with s on all ordinals $\geq \gamma_0$ in common domain. Then for each $i < n$, we can fix an automorphism σ_i of S such that $\sigma_i \in M$ and such that if ξ_i is the height of $q(N_i)$ then

$$\sigma_i(s) \restriction \xi_i = q(N_i).$$

Let $\mathcal{F}$ be the collection of all sequences $\upsilon = \{(\delta_0, t_0), ..., (\delta_{n-1}, t_{n-1})\}$ of elements of $\omega_1 \times S$ for which we can find $(\bar{q}, \bar{s}) \in \mathcal{D}$ end-extending $(q \cap M, s \restriction \gamma_0)$ and isomorphic to (q, s) over $(q \cap M, s \restriction \gamma_0)$ and γ_0 such that if $\bar{N}_0, ..., \bar{N}_{n-1}$ from $\mathcal{N}_{\bar{q}}$ correspond to $N_0, ..., N_{n-1}$ in the isomorphism, then for all $i < n$,

(5) $\delta_i = \delta_{\bar{N}_i}$,
(6) $t_i <_S \bar{s}$ and $\sigma_i(t_i) = \bar{q}(\bar{N}_i)$.

Then $\mathcal{F} \in M \cap H_\theta = M_0$ and therefore $\mathcal{F}$ belongs to all the models $N_0, ..., N_{n-1}$. Moreover, we have that

$$\{(\delta_{N_0}, \sigma_0^{-1}(q(N_0))), ..., (\delta_{N_{m-1}}, \sigma^{-1}(q(N_{n-1})))\} \in \mathcal{F},$$

For $\upsilon = \{(\delta_i, t_i) : i < n\}$ in $\mathcal{F}$ let $f(\upsilon)$ be the minimal node $\bar{s}$ extending t_{n-1} and appearing as the second coordinate of some pair $(\bar{q}, \bar{s}) \in \mathcal{D}$ witnessing $\upsilon \in \mathcal{F}$. Clearly $f \in M \cap H_\theta = M_0$ and so in particular $f \in N_i$ for all $i < n$. For a sequence $\upsilon = \{(\delta_0, t_0), ..., (\delta_{n-2}, t_{n-2})\}$ of length $n - 1$ let $\dot{X}_\upsilon$ be the S-name for the collection of all $\xi < \omega$ for which we can find (δ_{n-1}, t_{n-1}) such that $\tau = \upsilon^\frown(\delta_{n-1}, t_{n-1}) \in \mathcal{F}$ and such that $f(\tau)$ belongs to the generic branch. Let $\partial\mathcal{F}$ be the collection of all such sequences $\upsilon = \{(\delta_0, t_0), ..., (\delta_{n-2}, t_{n-2})\}$ of length $n-1$ for which some extension u of t_{n-2} forces that $\dot{X}_\upsilon$ is uncountable. For such υ let $f(\upsilon)$ be the minimal such u. Working in the model N_{n-1} we easily conclude that

$$\{(\delta_{N_0}, \sigma_0^{-1}(q(N_0))), ..., (\delta_{N_{n-2}}, \sigma_{n-2}^{-1}(q(N_{n-2})))\} \in \partial\mathcal{F}.$$

Proceeding in this way we arrive at a node $u_0 <_S s$ in M such that $\sigma_0(u_0)$ forces that the set $\dot{X}_\emptyset$ is uncountable. Let $\dot{x}_0 \in M$ be an S-name for a complete accumulation point of $\dot{X}_\emptyset$ and fix a non-principal ultrafilter $\mathcal{U}$ on ω such that $\mathcal{U} \in M$. Working in M and applying Theorem 8.2, we can find a well-founded $\mathcal{U}$-tree T, an extension v_0 of u_0 still in the (M, S)-generic branch $s \restriction \delta$, and an assignment $\{\alpha_\iota : \iota \in \partial T\} \subseteq \delta$ such that

$$\sigma_0(v_0) \vDash \{\alpha_\iota : \iota \in \partial T\} \subseteq \dot{X}_\emptyset \text{ and } \dot{x}_0 = \lim_{\iota\in\partial T} \alpha_\iota.$$

Let $\mathcal{M}_0$ be the collection of all $N \in \mathcal{N}_q \setminus M$ such that

$$q(N) \restriction \delta = q(N_0) \restriction \delta.$$

Given a $N \in \mathcal{M}_0$, we know that $q(N)$ forces $\dot{x}_0 \notin \overline{\dot{V}_N}$, so we can find a set $B_N \in \mathcal{U}_T$ such that

$$q(N) \Vdash \check{B}_N \cap \dot{V}_N = \emptyset.$$

Pick a point ι in the intersection of the sets B_N $(N \in \mathcal{M}_0)$. We may assume that v_0 decides the reason for $\check{\alpha}_\iota \in \dot{X}_\emptyset$. In other words, we may assume that there is $\tau = \{(\delta_0, t_0)\} \in \partial^{n-1}\mathcal{F}$ such that $v_0 >_S f(\tau)$. So, in particular, $\sigma_0(v_0)$ forces that $\dot{X}_\tau$ is an uncountable subset of ω_1, and so we can proceed on the second stage of the construction of the required copy $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$ of (q, s) that is compatible with it. This will finish the proof of the Claim. $\square$

For $\alpha < \omega_1$, let

$$\mathcal{D}_\alpha = \{p \in \mathcal{P} : (\exists N \in \mathcal{N}_p)\ \alpha \in N\}.$$

It is easily seen that each $\mathcal{D}_\alpha$ is a dense open subset of $\mathcal{P}$ and that if $\mathcal{G}$ is a filter of $\mathcal{P}$ that intersect all these sets then $g = \bigcup \mathcal{G}$ is a function from an uncountable $\in$-chain $\mathcal{N} = \bigcup_{p\in\mathcal{G}} \mathcal{N}_p$ into S. Note also that by the definition of the ordering of our forcing notion $\mathcal{P}$, we can go to an uncountable subchain $\mathcal{M}$ of $\mathcal{N}$ such that

$$(\forall M, N \in \mathcal{M})[M \in N \Rightarrow g(N) \Vdash \check{\delta}_M \notin \dot{V}_N].$$

Let $\dot{\Gamma}$ be the S-name for the collection of all countable ordinals γ for which we ban find $M \in \mathcal{M}$ such that $\gamma = \delta_M$ and such that $g(M)$

belongs to the generic branch. Clearly, any node s of S above which the image $g[\mathcal{M}]$ is dense forces that $\dot\Gamma$ is an uncountable discrete subspace of $\dot X$. This finishes the proof.

□

9. Countably tight compacta are sequential

Before we step towards the main result of this section, we need to develop a notion that captures sequential closure operator in a quite similar manner to that developed in the previous section. A *well-founded tree* is a collection T of finite subsets of ω such that

(1) $\emptyset \in T$ and $t \in T$ implies that all initial segments of t are in T,
(2) if $t \cup \{m\} \in T$ for some $m > \max(t)$ then

$$\{n < \omega : n > \max(t) \text{ and } t \cup \{n\} \in T\} \text{ is infinite },$$

(3) there is no infinite subset M of ω such that all finite initial segments of M belong to T.

As before, we let ∂T be the collection of all terminal nodes of T, the collection of nodes of T with no proper end-extensions in T. Then, given a Hausdorff space X, an assignment $\{x_t : t \in \partial T\} \subseteq X$ can sometimes be extended to a global assignment $\{x_t : t \in T\}$ defined recursively on the rank of T as follows. If T has a rank 1, i.e., $T = \{\emptyset\} \cup \{\{n\} : n \in M\}$ for some infinite $M \subseteq \omega$, we let $x_\emptyset = \lim_{n \to \mathcal{U}} x_{\{n\}}$, provided, of course, this limit exists. If T has rank > 1, for each $\{n\} \in T$, we apply, whenever possible, the inductive hypothesis and extend the assignment $\{t \in \partial T : n = \min(t)\}$ to the assignment $\{t \in T : n = \min(t)\}$ and again let $x_\emptyset = \lim_{n \to \mathcal{U}} x_{\{n\}}$ provided this limit exists in the surrounding space X. We call $x_\emptyset$ *the limit of the assignment* $\{x_t : t \in \partial T\}$ and write

$$x_\emptyset = \lim_{t \in \partial T} x_t.$$

Given a well-founded tree T, a FIN-subtree is any subset T_0 with the following properties

(4) $\emptyset \in T_0$,
(5) if $t \in T_0$ and if $t \notin \partial T$ then

$$\{n < \omega : t \cup \{n\} \in T \text{ but } t \cup \{n\} \notin \partial T\} \text{ is a finite set.}$$

Note that the boundary ∂T_0 of any FIN-subtree T_0 of T is a subset of the boundary ∂T of the tree T. Note also that for every open subset U which contains the limit point $x_\emptyset = \lim_{t \in \partial T} x_t$ there is a FIN-subtree T_0 of T such that $\partial T_0 \subseteq U$. Note, moreover, that the collection

$$\{\partial T_0 : T_0 \text{ a FIN -subtree of } T\}$$

generates a nontrivial filter of subsets of ∂T. We call this filter FIN^T. Again, the purpose of this definition is to give a more manageable representation of the sequential closure operator given in the following lemma whose proof is almost identical to that of Lemma 8.1.

9.1 Lemma. *Fix a subset Z of some Hausdorff space X. Let $\overline{Z}^{\omega_1}$ be the sequential closure of the set Z in the space X. Then*

$$\overline{Z}^{\omega_1} = \{\lim_{t\in\partial T} x_t : T \text{ a well-founded tree and } \{x_t : t \in \partial T\} \subseteq Z\}.$$

We are finally ready to state and prove the main result of this section.

9.2 Theorem. *Assume $PFA(S)$. The coherent Souslin tree S forces that every countably tight compactum is sequential.*

Proof. The proof will follow the lines of the proof of Theorem 8.2, so we shall only indicate changes and supplement arguments in places that are new. So as before, we fix an S-name $\dot{X}$ for a sequentially closed non-closed subset of some countably tight compact space $\dot{K}$ and we work for a contradiction. We shall show that S forces that $\dot{K}$ is not countably tight. As before, we fix an S-name for an assignment

$$x \in V_x \subseteq \overline{V_x} \subseteq U_x \ (x \in \dot{X})$$

of open sets of $\dot{K}$ such that S forces that $\dot{X} \setminus \bigcup_{x\in Y} U_x \neq \emptyset$ for all countable subset Y of $\dot{X}$ and e find a proper poset $\mathcal{P}$ which preserves S and forces a sequence $\dot{x}_\xi$ $(\xi < \omega_1)$ of S-names of elements of $\dot{X}$ such that for all $\alpha < \omega_1$, S forces that

$$\{\dot{x}_\xi : \xi \leq \alpha\} \subseteq V_{\dot{x}_\alpha} \text{ and } \{x_\xi : \xi > \alpha\} \subseteq \dot{X} \setminus U_{\dot{x}_\alpha}.$$

This will show that, in fact, $\dot{K}$ cannot be countably tight giving us the desired contradiction. As before, we assume that S forces that $\dot{K}$ and $\dot{X}$ live on some ordinal $\check{\nu}$ and we shall fix a sufficiently large regular cardinal θ and consider only countable elementary submodels of the expanded structure $(H_\theta, \in, <_\theta)$, where $<_\theta$ is a fixed well-ordering of H_θ. As before, we fix an enumeration

$$N = \{a_i^N : i < \omega\}$$

for every countable elementary submodel N of $(H_\theta, \in, <_\theta)$. We also fix an S-name $\dot{\mathcal{F}}$ for a maximal filter of sequentially closed subsets of $\dot{X}$ such that

$$(\forall Y \in [\dot{X}]^{\leq\aleph_0})\ \dot{X} \setminus \bigcup_{x\in Y} U_x \in \dot{\mathcal{F}}.$$

Then for every countable elementary submodel N of $(H_\theta, \in, <_\theta)$, we can associate an S-name $\dot{x}_N$ for an element of $\dot{X}$ as follows. For each $k < \omega$, we first let $\dot{z}_k^N$ be the S name for the first element of the intersection of all subsets of $\dot{X}$ that are interpretations of all S-names appearing in the finite sets $\{a_i^N : i \leq k\}$, or in other words an S-name with the property that S forces $\dot{z}_k^N = \min(\bigcap(\dot{\mathcal{F}} \cap \{a_i^N : i \leq k\}))$. Then, in particular, S forces that $\{\dot{z}_k^N : k < \omega\} \subseteq \dot{X} \cap N[G] \subseteq N \cap \check{\nu}$. Working still with the countable elementary submodel N and using Theorem 8.4, we define recursively as long as possible a sequence t_α $(\alpha < \omega_1)$ of pairwise incompatible elements of S and a sequence A_α $(\alpha < \omega_1)$ of infinite subsets of ω such that:

(a) $A_\beta \subseteq A_\alpha$ whenever $\alpha < \beta$,
(b) $t_\alpha \Vdash \{\dot{z}_k^N : k \in \check{A}_\alpha\}$ is a convergent sequence in $\dot{K}$.

Clearly, this process must stop at some countable ordinal α, so we arrive at an infinite set $A^N \subseteq \omega$ such that every element of S forces that $\{\dot{z}_k^N : k \in \check{A}^N\}$ is a convergent sequence in $\dot{K}$. Let

$$A^N = \{k_n : n < \omega\}$$

be the increasing enumeration of the set A^N, and we simplify the notation and let $\dot{y}_n^N = \dot{z}_{k_n}^N$. Since $\dot{X}$ is forced by S to be sequentially closed there is an S-name $\dot{x}_N$ for an element of $\dot{X}$ such that

$$\Vdash_S \dot{x}_N = \lim_{n\to\infty} \dot{y}_n^N.$$

This, in particular, gives us the S-names $V_{\dot{x}_N}$ and $U_{\dot{x}_N}$ for open subsets of $\dot{K}$, which, as before, we simply denote as $\dot{V}_N$ and $\dot{U}_N$, respectively.

We are now finally ready to define our poset $\mathcal{P}$ which is essentially the same as before, the collection of all mappings $p : \mathcal{N}_p \to S \times \omega_1$ such that for all $N \in \mathcal{N}$ the pair $p(N) = (p_0(N), p_1(N))$ has the following properties:

(1) $p_0(N) \in S \setminus N$ and $p_1(N) \in N \cap \omega_1$,
(2) $p_0(N)$ decides the values of $\dot{x}_N$ and $\dot{z}_k^N$'s as well as all the statements of the forms $\check{\alpha} \in \dot{V}_N$ and $\check{\alpha} \in \dot{U}_N$ for $\alpha \in N \cap \nu$.

Moreover, we require that for all $M \in N$ from $\mathcal{N}_p$, we have the following two conditions:

(3) $p_0(M) \in N$, and
(4) $p_0(M) <_S p_0(N)$ and $M \cap \omega_1 > p_1(N)$ imply $p_0(N) \Vdash \dot{x}_M \in \dot{V}_N$.

We let $\mathcal{P}$ be ordered by the inclusion. We again have to prove the following crucial fact.

9.2.1 Claim. *$\mathcal{P}$ is proper and it preserves S.*

Proof. This follows closely the proof of the corresponding Claim in the proof of Theorem 8.2. So we only explain the corresponding initial (and, therefore, also the inductive) step of the procedure whose result is a compatible copy $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$ of the given condition (q, s). Thus we have in M, the S-name $\dot{X}_\emptyset$ for the collection of all $\xi < \nu$ for which we can find a node t extending $s \restriction \gamma_0$ such that $\{(t, \xi)\} \in \partial^{n-1}\mathcal{F}$ and such that $\sigma_0^{-1}(f(\{(t, \xi)\}))$ belongs to the generic branch and a node $u_0 \in S \cap M$ such that $s \restriction \gamma_0 \leq u_0 <_S s$ and such that

$$\sigma_0^{-1}(u_0) \Vdash \overline{\dot{X}_\emptyset}^{\omega_1} \in \dot{\mathcal{F}}.$$

Let k^* be the integer such that $\dot{X}_\emptyset = a_{M_0}^{k^*}$. Let $\mathcal{M}_0$ be the collection of all $N \in \mathcal{N}_q \setminus M$ with the property that

$$q^1(N) < \delta \text{ and } \sigma_0(q^0(N)) \restriction \delta = s \restriction \delta.$$

Fix an $N \in \mathcal{M}_0$. Then from the definition of $\dot{x}_{M_0}$ and the facts that $q^0(N) \Vdash \dot{x}_{M_0} \in \dot{V}_N$ and that $q^0(N)$ decides the intersection of $\dot{V}_N$ with the set $N \cap \nu$, there is $i_N < \omega$ such that

$$(\forall i \geq i_N)\ q^0(N) \Vdash \dot{y}_i^{M_0} \in \dot{V}_N.$$

Pick an integer i greater than all the i_N for $N \in \mathcal{M}_0$ and such that moreover $k_i \geq k^*$ (recall, that $\dot{y}_i = \dot{z}_{k_i}$. Since $\dot{y}_i^{M_0}$ belongs to M_0 and it is a name for an element of $\dot{X}_\emptyset = a_{M_0}^{k^*}$, we can find $v_0 \in S \cap M_0$, $u_0 \leq_S v_0 <_S s$, a well founded FIN-tree T, an assignment

$$\{x_\iota : \iota \in \partial T\} \subseteq M_0 \cap \nu,$$

all elements of M, such that

$$\sigma_0^{-1}(v_0) \Vdash \{x_\iota : \iota \in \partial T\} \subseteq \dot{X}_\emptyset \text{ and } \dot{y}_i^{M_0} = \lim_{\iota \in \partial T} x_\upsilon.$$

It follows that for all $N \in \mathcal{M}_0$

$$B_N = \{\iota \in \partial T : q^0(N) \Vdash x_\iota \in \dot{V}_N\}$$

contains the boundary of an FIN-subtree of T. So, we can pick $\iota \in \partial T$ belonging to all sets B_N for $N \in \mathcal{M}_0$. Pick $(t_0, \xi_0) \in \partial^{n-1}F$ such that $\iota = \xi_0$. Since v_0 forces that $\xi \in \dot{X}_\emptyset$, we can find a node $u_1 \geq v_0$ in the generic branch $s \restriction \delta$ such that u_1 forces that $\sigma_0^{-1}(f(t_0, \xi_0))$ belongs to the generic branch. So in particular, $u_1 \geq_S \sigma_0^{-1}(f(t_0, \xi_0))$, and therefore,

$$\sigma_1^{-1}(u_1) \Vdash \dot{X}_{\{(t_0, \xi_0)\}} \in \dot{\mathcal{F}}.$$

This means not only that we continue picking $v_1 \geq u_1$, (t_1, ξ_1), etc, but also that we have achieved part of the compatibility requirement between (q, s) and the desired copy $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$, the requirement that for all $N \in \mathcal{M}_0$ we will have that $\bar{q}^0(\bar{N}_0) <_S q^0(N)$ and that

$$q^0(N) \Vdash \dot{x}_{\bar{N}_0} \in \dot{V}_N.$$

So, it should be clear that continuing this procedure we can pick a sequence $\{(t_i, \xi_i) : i < n\} \in \mathcal{F} \cap M$ so that its witnessing condition $(\bar{q}, \bar{s}) \in \mathcal{D} \cap M$ is compatible with (q, s). This finishes the proof. □

Given the Claim, applying $PFA(S)$ as in the corresponding part of the proof of Theorem 8.2, we arrive at an S-name for an uncountable free sequence of $\dot{K}$ all of whose terms are actually elements of the sequentially closed non-closed subset $\dot{X}$. This is in contradiction with our initial assumption that S forces $\dot{K}$ to be countably tight. This finishes the proof. □

We finish this section with the following corollary of the results accumulated so far.

9.3 Corollary. *Assume $PFA(S)$. The coherent Souslin tree S forces that hereditarily separable subspaces of compact countably tight spaces are hereditarily Lindelöf.*

9.4 Remark. The statements from the conclusions of Theorem 9.2 and Corollary 9.3 are known to follow also from PFA and $MA_{\aleph_1}$, respectively (see [1] and [18]). The reader is encouraged to compare the corresponding proofs.

10. Chain Conditions in Topology

In this section we use the results accumulated so far to shed some light on the following conjecture.

10.1 Conjecture. *If K is a compact space satisfying the separation axiom T_5 then either,*

(1) *K contains an uncountable family of open subsets, or*
(2) *there is a continuous map $f : K \to M$ where M is a metric space and $f^{-1}(x)$ has at most two points for every $x \in M$.*

The origin of this conjecture is a result of [25] (Theorems 3.5 and 4.1) which transfers a chain-condition pathology from the class of compact T_5 spaces into the class of metrizably fibered compacta, i.e., the class of compact spaces K that admit continuous maps $f : K \to M$ into metric spaces M with the property that every fiber $f^{-1}(x)$ for $x \in M$ is metrizable. In this section, we shall see that forcing with the coherent Souslin trees provides us with a weak form of Conjecture 10.1. We start with the following general fact true in the forcing extension of S.

10.2 Theorem. *Assume $PFA(S)$. The coherent Souslin tree S forces that if Y is a nonseparable subspace of some regular space X, then either*

Y contains an uncountable discrete subspace or a subset Z with the property that its closure $\overline{Z}$ in X has no point of countable π-character[3].

Proof. We work in the forcing extension of S and fix a regular space X and its nonseparable subspace Y. We may assume that Y admits a well-ordering $<$ of order-type ω_1 with closed initial segments. So for each $y \in Y$ we fix a pair V_y and U_y of open subsets of X such that

$$y \in V_y \subseteq \overline{V_y} \subseteq U_y$$

and such that $x \notin U_y$ for all $x < y$. Consider the following symmetric irreflexive binary relation E on $Y : (x, y) \in E$ if and only if $x \notin V_y$ and $y \notin V_x$. Thus, the complete subgraphs of (Y, E) are discrete subspaces of Y, so we get the conclusion of the theorem if (Y, E) contains an uncountable complete subgraph. By the rectangular graph axiom for the vertex set ω_1, true in our context by Theorem 5.4, we may consider the alternative that there exist two uncountable subsets A and B of Y such that we can't find uncountable $A' \subseteq A$ and $B' \subseteq B$ such that $(a, b) \in E$ for all $a \in A'$ and $b \in B'$ such that $a < b$. Let $\mathcal{U}$ be the collection of all open subsets U of X with countable intersection with A. If $(\bigcup \mathcal{U}) \cap A$ is uncountable it would contain an uncountable discrete subspace, as required. So we may assume this set is countable and consider its complement $Z = A \setminus \bigcup \mathcal{U}$ in A. We claim that if $\overline{Z}$ is the closure of Z in X, every point of $\overline{Z}$ has uncountable π-character. Suppose not, and fix a point $z \in \overline{Z}$ and a sequence W_n $(n < \omega)$ of nonempty relatively open subsets of $\overline{Z}$ such that every open set U containing z must contain one of the W_n's. So, in particular, for every $y \in B$ with $y > z$, we can fix an $n_y < \omega$ such that $W_{n_y} \cap \overline{V_y} = \emptyset$. Pick an $n < \omega$ for which the set $B' = \{y \in B : n_y = n\}$ is uncountable. Let $A' = W_n \cap Z$. Then A' is also uncountable and $(a, b) \in E$ for all $a \in A'$, $b \in B'$, $a \neq b$. This completes the proof.

□

10.3 Corollary. *Assume $PFA(S)$. The coherent Souslin tree S forces that if a compact space contains a nonseparable subspace with no uncountable discrete subspace then it maps onto the cube $[0, 1]^{\omega_1}$.*

Proof. This follows from Theorem 10.2 and the well-known result from [16] saying that a compact space K maps onto the Tychonoff cube $[0, 1]^{\omega_1}$ if and only if it contains a closed subset F such that every point x of F has uncountable character in F. □

[3]Recall that a π-character of a point x in some topological space is the minimal cardinality of a family $\mathcal{B}$ of nonempty open subsets of X with the property that every neighborhood of x in X includes at least one set from $\mathcal{B}$.

10.4 Corollary. *Assume $PFA(S)$. The coherent Souslin tree S forces that the following three conditions on subspaces X of compact countably tight spaces K are equivalent:*

(a) *X is hereditarily separable.*
(b) *X is hereditarily Lindelöf.*
(c) *X is hereditarily ccc.*

Proof. This follows from Corollaries 10.3 and 9.3. □

We now concentrate on analysis of compact T_5 spaces after forcing by S having in mind the Conjecture 10.1.

10.5 Theorem. *Assume $PFA(S)$. Then the coherent Souslin tree S forces that every separable T_5 compactum is countably tight.*

Proof. This is a consequence of Corollary 5.3 and the main result of [15] saying that the statement about open graphs appearing in Corollary 5.3 implies that every separable compact T_5 space is countably tight. □

We are now ready to state and prove the main result of this section which shows that in Conjecture 10.1 the countable chain condition can be replaced by considerably stronger chain condition provided we work in the forcing extension of S.

10.6 Theorem. *Assume $PFA(S)$. Then the coherent Souslin tree S forces that the following three conditions on a T_5 compactum K are equivalent:*

(a) *K is ccc.*
(b) *K is hereditarily Lindelöf.*
(c) *K is hereditarily separable.*

Proof. We work in the generic extension of S and fix a T_5 compactum K satisfying the countable chain condition. We shall show that K contains no uncountable discrete subpace, giving us the other two chain conditions by Corollary 10.4. We first work towards proving that K is separable. By Theorems 9.2 and 10.5, the closure of every countable subset of K is sequential. So for every point x of K that is an accumulation point of a countable subset of $K \setminus \{x\}$, we can find a sequence $\{x_n : n < \omega\}$ converging to x. Since $K \setminus \{x\}$ is a normal space, we can find a sequence $V_n(x)$ $(n < \omega)$ such that $x_n \in V_n(x)$ and such that $\{V_n : n < \omega\}$ is a discrete family in $K \setminus \{x\}$. It follows that the sequence of open sets conveges to x in the sense that for every open set U containing x there is $m < \omega$ such that $V_n(x) \subset U$. We fix such a sequence $V_n(x)$ of open sets for every $x \in K$ that is in the closure of a countable subset of $K \setminus \{x\}$. Let D denotes the set of such $x \in K$. Recursively over the set of countable ordinals, we now choose, if possible, a sequence D_α $(\alpha < \omega_1)$ of countable subsets of D such that

(1) $D_\alpha \subseteq D_{\alpha+1}$ and $D_{\alpha+1} \nsubseteq \overline{D_\alpha}$.
(2) if $V_m(x) \cap V_n(y) \neq \emptyset$ for some $x, y \in D_\alpha$ and $m, n < \omega$, then

$$V_m(x) \cap V_n(y) \cap D_{\alpha+1} \neq \emptyset.$$

Clearly, the construction will go over all countable ordinals if we never reach an $\alpha < \omega_1$ such that $\overline{D_\alpha} = K$. Since this will give us the required conclusion that K is separable, we work towards proving that the existence of such a sequence D_α $(\alpha < \omega_1)$ leads to a contradiction. Let $D_{\omega_1} = \bigcup_{\alpha<\omega_1} D_\alpha$. Note that the closure of D_{ω_1} is a ccc space and that

$$\{V_n(x) \cap \overline{D_{\omega_1}} : x \in D_{\omega_1}, n < \omega\}$$

is a π-basis of $\overline{D_{\omega_1}}$. Replacing K, we may assume that $K = \overline{D_{\omega_1}}$. Applying Theorem 10.2 to D_{ω_1} as a subspace of K, we conclude that it mist have an uncountable discrete subspace $\{d_\xi : \xi < \omega_1\}$. We simplify the notation by letting $V_n(\xi) = V_n(d_\xi)$ for $\xi < \omega_1$ and $n < \omega$. We now go and work in the ground model fixing the names $\dot{d}_\xi$ $(\xi < \omega_1)$ and $\dot{V}_n(\xi)$ $(\xi < \omega_1, n < \omega)$ for these objects.

We first choose a closed and unbounded set $C \subseteq \omega_1$ such that for every $\delta \in C$, every node of S_δ forces the statement $\dot{V}_m(\xi) \cap \dot{V}_n(\eta) = \emptyset$ or its negation for all $\xi, \eta < \delta$ and $m, n < \omega$. For $\delta \in C$, let δ^+ be the minimal element of C above δ. For $i < 2$, let $\dot{H}_i$ be the S-name for the set of all $\dot{d}_\delta$ for $\delta \in C$ for which the evaluation of the generic branch at the ordinal δ^+ is equal to i. Since $\{\dot{d}_\xi : \xi < \omega_1\}$ is forced to be a discrete subspace of $\dot{K}$, we conclude that, in particular, S forces that

$$(\overline{\dot{H}_0} \cap H_1) \cup (H_0 \cap \overline{\dot{H}_1}) = \emptyset.$$

Since $\dot{K}$ is forced to be T_5, we can find an S-names $\dot{O}_0$ and $\dot{O}_1$ for a regular-open subsets of $\dot{K}$ such that

$$\Vdash_S \dot{H}_0 \subseteq \dot{O}_0, \dot{H}_1 \subseteq \dot{O}_1 \text{ and } \dot{O}_0 \cap \dot{O}_1 = \emptyset.$$

Since the collection $\{\dot{V}_n(\xi) : \xi < \omega_1, n < \omega\}$ is forced to be a π-basis of the ccc compactum $\dot{K}$, and since S does not add new countable sets of ordinals, we can find a $s \in S$ and two sequences $\{(\xi_k, m_k) : k < \omega\}$ and $\{(\eta_k, n_k) : k < \omega\}$ such that

$$(2) \qquad s \Vdash \dot{O}_0 = \operatorname{Int}(\overline{\textstyle\bigcup_{k<\omega} \dot{V}_{m_k}(\xi_k)}) \text{ and } \dot{O}_1 = \operatorname{Int}(\overline{\textstyle\bigcup_{k<\omega} \dot{V}_{n_k}(\eta_k)}).$$

We may assume that s belongs to the level S_δ of S such that $\xi_k, \eta_k < \delta$ for all $k < \omega$. Pick $t \in S_{\delta^+}$ extending s. Let

$$M_0 = \{n < \omega : (\exists k < \omega)\ t \Vdash \dot{V}_n(\delta) \cap \dot{V}_{m_k}(\xi_k) \neq \emptyset\},$$

and

$$M_1 = \{n < \omega : (\exists k < \omega)\ t \Vdash \dot{V}_n(\delta) \cap \dot{V}_{n_k}(\eta_k) \neq \emptyset\}.$$

Note that since S forces that the sequence $\dot{V}_n(\delta)$ converges to $\dot{d}_\delta$, if for some $i < 2$ the set M_i is infinite, we would get that $t \Vdash \dot{d}_\delta \in \dot{O}_i$ which we know is not true. Thus, both sets M_0 and M_1 are finite. Let $u = t^\frown 0$. Then

$$u \Vdash \dot{d}_\delta \in \dot{H}_0 \subseteq \dot{O}_0,$$

so we can find $n > \max(M_0 \cup M_1)$, $k < \omega$ and $v \geq_S u$ such that

$$v \Vdash \dot{V}_n(\delta) \cap \dot{V}_{m_k}(\xi_k) \neq \emptyset.$$

By the choice of the club C, we have that also

$$v \restriction \delta^+ = u \Vdash \dot{V}_n(\delta) \cap \dot{V}_{m_k}(\xi_k) \neq \emptyset.$$

So $n \in M_0$, a contradiction. This finishes the proof that S forces that the T_5 compactum $\dot{K}$ that satisfies the countable chain condition must, in fact, be separable. Applying Theorems 9.2 and 10.5, we conclude that$S$ forces that $\dot{K}$ is sequential. We shall finish the proof of Theorem 10.6 by showing that S forces that $\dot{K}$ contains no uncountable discrete subspaces (see Corollary 10.4).

Work in the forcing extension of S and pick a sequence d_n $(n < \omega)$ of S-names for elements of K such that $\overline{\{d_n : n < \omega\}} = K$. Working as before, for every non-isolated $x \in K$ we can pick a sequence $V_n(x)$ $(n < \omega)$ of open subsets of $K \setminus \{x\}$ such that every open neighborhood of x contains all but finitely many sets $V_n(x)$. If x is isolated in K (then it must be one of the d_n's), we let $V_n(x) = \{x\}$ for all $n < \omega$. Suppose K has an uncountable discrete subspace x_ξ $(\xi < \omega_1)$. We may assume that $x_\xi \neq d_n$ for all $\xi < \omega_1$ and $n < \omega$. We simplify the notation by letting

$$V_n(\xi) = V_n(x_\xi)\ (\xi < \omega_1, n < \omega) \text{ and } V(n,k) = V_n(d_k)\ (n,k < \omega).$$

We now move to the ground model and work with a fixed S-names for these objects. Pick a closed unbounded set C in ω_1 such that for every $\delta \in C$, every $s \in S_\delta$ decides all the statements of the form

$$\dot{V}_m(\xi) \cap \dot{V}_n(\eta) = \emptyset \text{ and } \dot{V}_m(\xi) \cap \dot{V}(n,k) = \emptyset\ (\xi, \eta < \delta)(m,n,k < \omega).$$

As before, for $i < 2$, let $\dot{H}_i$ be the S-name for the set of all $\dot{d}_\delta$ for $\delta \in C$ for which the evaluation of the generic branch at the ordinal δ^+ is equal to i. Fix S-names $\dot{O}_0$ and $\dot{O}_1$ for pairwise disjoint regular-open subsets of $\dot{K}$ such that $\Vdash_S \dot{H}_i \subseteq \dot{O}_i$ for $i < 2$. Choose an $s \in S$ and sets $I_i \subseteq \omega \times \omega$ $i < 2$ such that s forces that $\dot{O}_i$ is equal to the interior of the closure of the union $\bigcup_{(n,k)\in I_i} \dot{V}(n,k)$. We may assume that $s \in S_\delta$ for some $\delta \in C$ and we choose an arbitrary $t \in S_{\delta^+}$ extending s. As before, for $i < 2$, we let M_i be the set of all $m < \omega$ for which we can find $(n,k) \in I_i$ such that

$$t \Vdash \dot{V}_m(\delta) \cap \dot{V}(n,k) \neq \emptyset.$$

Since t does not decide the value of the generic branch at δ^+ both sets M_0 and M_1 must be finite. As before, we find $v \in S$ extending $t^\frown 0$ an integer $m > \max(M_0 \cup M_1)$ and a pair $(n,k) \in I_0)$ such that $v \Vdash \dot{V}_m(\delta) \cap \dot{V}(n,k) \neq \emptyset$, and therefore $t \Vdash \dot{V}_m(\delta) \cap \dot{V}(n,k) \neq \emptyset$. It follows that $n \in M_0$, a contradiction. This finishes the proof. □

11. Concluding remarks

The research of this paper was originally motivated by a problem asking whether $\mathrm{MA}_{\aleph_1}$ can be reformulated as a purely Ramsey-theoretic statement about colorings of $[\omega_1]^2$, or equivalently, as a statements identifying two classical chain conditions.

11.1 Problem. *Is* $\mathrm{MA}_{\aleph_1}$ *equivalent to the statement that every ccc poset has the Knaster property?*[4]

Since $\mathrm{MA}_{\aleph_1}$ fails in any forcing extension of a Souslin tree, this problem naturally leads to the following question.

11.2 Problem. *Assume $PFA(S)$. Does the Souslin tree S force that every ccc poset has the Knaster property?*

The following might be a good topological test question that could help us resolve this problem which is moreover of clear independent interest in view of the topological results gathered in the previous sections of this paper.

11.3 Problem. *Assume $PFA(S)$. Does the Souslin tree S force that every regular hereditarily separable topological spaces must be Lindelöff?*

There are of course many more consequences of PFA that forcing with a Souslin tree could separate. For example, the paper [6] shows that not all $\aleph_1$-dense sets of reals are isomorphic in a forcing extension by a Souslin tree. So one can ask which other consequences of PFA of this type could pass through a forcing extension by a coherent Souslin tree.

[4]Recall that a poset $\mathcal{P}$ has the *Knaster property* if every uncountable subset of $\mathcal{P}$ contain an uncountable subsets of pairwise compatible conditions.

Finally we mention the technology of $\mathbb{P}_{max}$-forcing relativized to the existence of a Souslin tree initiated by Woodin [31] that could also be relevant to some of these problems and which should therefore seriously be investigated.

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