LIPSCHITZ MAPS ON TREES

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Abstract. We introduce and study a metric notion for trees and relate it to a conjecture of Shelah [10] about the existence of a finite basis for a class of linear orderings.

Contents

0. Introduction

The notion of a *tree* in this note is to be interpreted in its order-theoretic sense, i.e. a partially ordered set $(T, \leq_T)$ with the property that the predecessors of every point form a well-ordered chain[1]. If we think of trees as natural generalizations of ordinals, then we would put a tree S to be *smaller* than a tree T, or $S \leq T$ in short, if there is a strictly increasing map $f : S \longrightarrow T$. We say that S and T are *equivalent*, and write $S \equiv T$, whenever $S \leq T$ and $T \leq S$. We shall write $S < T$, whenever $S \leq T$ and $T \nleq S$. Note that if $f : S \longrightarrow T$ is strictly increasing, then $g : S \longrightarrow T$ defined by $g(t) = f(t) \restriction \mathrm{ht}(t)$ is

[1]The order-type of $\{x \in T : x <_T t\}$ is called the height of t in T and denoted by $\mathrm{ht}(t)$. The αth level of T is the set $T_\alpha = \{t \in T : \mathrm{ht}(t) = \alpha\}$. If $\alpha \leq \mathrm{ht}(t)$, then $t \restriction \alpha$ denotes the $s \leq_T t$ such that $\mathrm{ht}(s) = \alpha$. We make the implicit assumption that different nodes of the same level of T have different sets of predecessors. This allows us to define $s \wedge t = \{x \in T : x \leq s \text{ and } x \leq t\}$.

also strictly increasing. So, without loss of generality, we may restrict ourselves to strictly increasing maps that are also *level-preserving*. It turns out that there is a more natural and more general way to introduce these maps. Consider $\Delta : T^2 \longrightarrow \mathrm{Ord}$, defined by

$$\Delta(s,t) = \mathrm{otp}\{x \in T : x <_T s \text{ and } x <_T t\}.$$

One should view Δ as some sort of distance function on T by interpreting inequalities like $\Delta(x,y) > \Delta(x,z)$ as saying that x is closer to y than to z. A partial map g from a tree S into a tree T is *Lipschitz*, if g is level-preserving and

$$\Delta(g(x), g(y)) \geq \Delta(x,y)$$

for all $x, y \in \mathrm{dom}(g)$. Note that this notion is equivalent to the notion of a strictly increasing level-preserving map, when the domain of g is a downward closed subset of S, but is otherwise more general. Any partial Lipschitz map g from S into T, however, naturally extends to a strictly increasing level-preserving map $\hat{g}$ on the downward closure of $\mathrm{dom}(g)$ in S, defined by letting $\hat{g}(x) = g(s) \restriction \mathrm{ht}(x)$ for some (equivalently, for all) $s \in \mathrm{dom}(g)$ extending x. With the order $\leq$ and the corresponding equivalence relation $\equiv$, the class $\mathcal{T}$ of all trees has been successfully used as a source of invariants in places where ordinals are not sufficient (see e.g. [16]). Unfortunately, the structure of $(\mathcal{T}, \leq)$ is immense in comparison with the structure of the ordinals, so one is naturally constrained to study smaller subclasses of $\mathcal{T}$. A natural way to split $\mathcal{T}$ into subclasses that can be studied separately is to consider the class $\mathcal{B}$ of all trees which are comparable with any other tree. Clearly, every ordinal belongs to $\mathcal{B}$ and the ordinals split $\mathcal{T}$ into classes of trees according to their heights. But $\mathcal{B}$ is much more extensive than the class of ordinals and therefore a much finer partition of $\mathcal{T}$ is possible. For example, Ohkuma [9] showed that $\mathcal{B}$ includes the class $\mathcal{S}$ of all scattered trees, the class of trees that do not contain isomorphic copies of the tree of all finite binary sequences. In [8], Ohkuma also showed that $\mathcal{S}$ is well-ordered under $\leq$. It is unknown however whether $\mathcal{B}$ is included in the class $\mathcal{S}_\sigma$ of all sigma-scattered trees, the class of trees for which the analogues of Ohkuma's results can be proved. The most natural class of trees for testing the extent of $\mathcal{B}$ relative to $\mathcal{S}_\sigma$ would be the class $\mathcal{A}$ of all uncountable trees with no uncountable chains nor levels, the class of so-called Aronszajn trees. This class of trees is of independent interest as there is a growing number of problems in combinatorial set theory that have their reformulation inside $\mathcal{A}$. In fact, the results of this note were obtained during the course of working on one such problem, the basis problem for uncountable linear orderings (see e.g. [14]). From experience one learns that any naturally defined member of $\mathcal{A}$ has plenty of Lipschitz self-maps. Motivated by this, we isolate the notion of a *Lipschitz tree*, an uncounable tree T with the property that every level-preserving map from an uncountable subset of T into T has an uncountable Lipschitz restriction. It turns out that with the exceptions of a few pathological examples, every Lipschitz tree belongs to $\mathcal{A}$ and conversely, that any naturally defined member of $\mathcal{A}$ is Lipschitz. It was therefore quite unexpected to find out that it is impossible to define two incomparable Lipschitz trees. In fact, for quite a long time we did not know if the whole class $\mathcal{A}$ is a chain or not, and the comparability problem for $\mathcal{A}$ seemed to us a closely related to a conjecture of Shelah [10] about a basis for a class of linear orderings. We shall show that while $\mathcal{A}$ inherits lots of structure from the chain $\mathcal{C}$ of Lipschitz trees, it is possible to construct large families of pairwise incomparable members of $\mathcal{A}$. As one might have expected, the construction is based on a particular infinite strictly decreasing sequence of members of $\mathcal{C}$. Both results can be used to solve an old problem of Laver [6], who asked whether the class $\mathcal{A}$ is well-quasi-ordered under a stronger quasi-order than $\leq$. Curiously, the analogy between the pair $(\mathcal{S}, \mathcal{S}_\sigma)$ and the pair $(\mathcal{C}, \mathcal{A})$ still stands, though perhaps this could not have been guessed at the time Laver was writting his papar. A considerable part of our paper concerns the structure that one finds inside the class $\mathcal{C}$ of Lipschitz trees. For example, we prove that the class of uniformly branching Lipschitz trees has the Schroeder-Bernstein property, that there is a reduction of $\mathcal{C}$ into the space of uniform ultrafilters on ω_1, and that there is a natural shift operation on $\mathcal{C}$ which respects the quasi-ordering $\leq$. It turns out that the shift $T^{(1)}$ of a tree $T \in \mathcal{C}$ is the immediate successor of T in $\mathcal{C}$,[2] so every $T \in \mathcal{C}$ belongs to the $\mathbb{N}$-chain $T^{(n)}$ $(n \in \mathbb{N})$ of positive shifts. We shall also determine when a negative shift is defined and in particular, when a Lipschitz tree T belongs to a $\mathbb{Z}$-chain $T^{(n)}$ $(n \in \mathbb{Z})$ of its shifts. It should be pointed out that we always have the equation $T^{(n+m)} \equiv (T^{(n)})^{(m)}$ whenever these shifts are properly defined.

As it will precisely be explained in §8, further development of the structure theory for $\mathcal{C}$ and $\mathcal{A}$ requires a natural hypothesis about the profusion of partial Lipschitz maps between members of $\mathcal{A}$ which turns out to be closely related to a well-known basis problem for uncountable linear orderings. We shall use the standard approach one takes when

[2]The shift $T^{(1)}$ is also an immediate successor of T in the whole class $(\mathcal{A}, \leq)$ if we use a natural conjecture about Lipschitz maps, a conjecture which turn out to be equivalent to an old conjecture of Shelah[10, Conjecture 1]; see §8.

working on this kind of problem (see e.g. [14] for more explanations). On one hand, one tries to identify all members of the basis, i.e., to list all the critical structures of a given class. These are the canonical structures whose description should not rely on any additional set-theoretical assumption. On the other hand, to show that a given list of critical structures is in some sense complete, it is frequently necessary to use additional set-theoretic assumptions that postulate some sort of logical completeness of the natural framework where all the structure of our class can be found. For example, for the problems involving the structure $(\mathcal{A}, \leq)$, the natural framework is the class $\mathrm{H}(\omega_2)$ of sets whose transitive closures have cardinalities not larger than $\aleph_1$. In fact, all problems involving the structure of $(\mathcal{C}, \leq)$ and $(\mathcal{A}, \leq)$ are sentences of $(\mathrm{H}(\omega_2), \in)$ with only one change of quantifiers. The fact that there is a quite robust such theory of $(\mathrm{H}(\omega_2), \in)$ (see [17]) gives us some reassurance to this approach. For studying $(\mathcal{C}, \leq)$ and $(\mathcal{A}, \leq)$ we shall in fact need a rather weak form of this logical completeness, appearing as postulates about possible extensions of the Baire category theorem (see e.g. [3] or [13]). More precisely, for all our purposes here, we need only to postulate the existence of filters that would meet given families of $\aleph_1$-many dense sets in proper partial orderings of size at most $\aleph_1$. Since there seem to be no commonly accepted notation for this set-theoretic postulate, we will denote it here by (*) and use this symbol to mark any result in the paper that makes use of it. As a matter of fact, a large majority of our results needs (*) for partial orderings satisfying the countable chain condition, so most of the time, we shall be working in an even older framework (see [3]). It should also be noted that the two sides of work on a given basis problem described above usually benefit from each other and there is no exception in the problem we choose to study here. For example, the fact that under (*), $\mathcal{C}$ is a chain in $(\mathcal{A}, \leq)$ which is both cofinal and coinitial, indicates that to describe a tree from $\mathcal{A}$ with certain properties, it is likely that some Lipschitz trees will be used in the construction. This indeed was the hint behind the construction of §3 of a pair of incomparable trees from $\mathcal{A}$. The fact that under (*), $\mathcal{C}$ has no minimal nor maximal elements, was the hint towards the definition of the shift operation that gave us a $\mathbb{Z}$-chain of Lipschitz trees thus solving the old problem of Laver [6]. Conversely, the definition of the shift operation suggests the very natural question of whether there is any tree strictly between T and its shift $T^{(1)}$ leading us towards the result that under (*), there is no Lipschitz tree S such that $T < S < T^{(1)}$. Whether $T^{(1)}$ is an immediate successor of T in the larger class $\mathcal{A}$ as well, lies at the very heart of the conjecture of Shelah [10, Conjecture 1].

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1. Lipschitz trees

We have remarked above that the notion of Lipschitz map captures the quasi-ordering $\leq$ we wish to study here. The following derived notion is then quite natural.

1.1 Definition. A *Lipschitz tree* is any Aronszajn tree T with the property that every level-preserving map from an uncountable subset of T into T is Lipschitz on an uncountable subset of its domain.

1.2 Remark. It should be noted that if we restrict ourselves to countably branching trees of height ω_1, there are essentially no Lipschitz trees outside the class of Aronszajn trees. A tree of height ω_1 in which every node has extensions to all higher levels cannot have an uncountable chain if it is to satisfy the requirement of Definition 1.1. Similarly, with (*) such a tree must have all levels countable as well.

The following property of Lipschitz trees will be frequently used below:

1.3 Lemma. *Suppose T is a Lipschitz tree, n is a positive integer, and that A is an uncountable subset of the nth power of T.*[3]*. Then there exists an uncountable $B \subseteq A$ such that $\Delta(a_i, b_i) = \Delta(a_j, b_j)$ for all $a \neq b$ in B and $i, j < n$. It follows that the nth power of T is a Lipschitz tree as well.*

Proof. Fix $i, j < n$ and apply the Definition 1.1 to the partial map $a_i \longmapsto a_j$ $(a \in A)$ obtaining an uncountable $A_0 \subseteq A$ such that $\Delta(a_i, b_i) \leq \Delta(a_j, b_j)$. Applying Definition 1.1 to the inverse map $a_j \longmapsto a_i$ $(a \in A_0)$ will give us an uncountable $A_1 \subseteq A$ such that $\Delta(a_i, b_i) = \Delta(a_j, b_j)$. Repeating this procedure successively for every pair $i, j < n$, we reach the conclusion of Lemma 1.3. □

1.4 Lemma. *Every uncountable subset of a Lipschitz tree T contains an uncountable antichain. More generally, every family A if pairwise disjoint finite subsets of T contains an uncountable subfamily B such that $\cup B$ is an antichain of T.*

[3] $T \otimes T \otimes \cdots \otimes T$ (n times), is defined as the set of all n-tuples of elements of T of equal heights equipped with the coordinatewise ordering.

Proof. Let X be a given uncountable subset of some Lipschitz tree T. It suffices to find an uncountable antichain in the downwards closure of X in T, so we may in fact assume that X is already downward closed in T. We may also assume that no $x \in X$ is an end-node of T and in fact that every splitting node $x \in X$ has two successors x_0 and x_1 in X that split at x and have the same height in T. Applying the fact that T is a Lipschitz tree to the partial map $f(x_0) = x_1$ (x a splitting node of X) gives us an uncountable set Y of splitting nodes of X such that f is Lipschitz on $Y_0 = \{y_0 : y \in Y\}$. It follows that Y_0 is an uncountable antichain of T. The second part of the lemma follows from the first and the fact that finite powers of T are Lipschitz as well (see 1.3). This finishes the proof. $\square$

1.5 Definition. An Aronszajn tree T is *irreducible*, if $T \leq U$ for every uncountable downward closed subset U of T.

The following Lemma gives us the first piece of information about the structure of the class of Lipschitz trees.

1.6 Lemma (*). *Every Lipschitz tree is irreducible.*

Proof. Let T be a given rooted Lipschitz tree and let U be an uncountable downwards closed subset of T. We assume that every $t \in U$ has extensions in U in all levels above $\mathrm{ht}(t)$ and will use (*) to produce a map witnessing $T \leq U$. Let C be the set of all countable limit ordinals λ with the property that every node t of U of height $< \lambda$ has infinitely many extensions in the λth level U_λ of U. Note that C is closed and unbounded and that $0 \in C$.

Let $\mathcal{P}$ be the poset of all partial finite level-preserving Lipschitz maps from T into $U \restriction C$[4]. Let us show that $\mathcal{P}$ satisfies the countable chain condition. So let p_ξ ($\xi < \omega_1$) be a given sequence of elements of $\mathcal{P}$. We may assume that for every $\xi < \eta < \omega_1$, p_ξ and p_η agree on the intersection of their domains. Let a_ξ be the projection of $\mathrm{dom}(p_\xi) \setminus (T \restriction \xi)$ [5] on the ξth level T_ξ of T, and let b_ξ be the projection of $\mathrm{rang}(p_\xi) \setminus (T \restriction \xi)$ on the ξth level of T. Note that if m_ξ is the cardinality of a_ξ and n_ξ the cardinality of b_ξ, then $m_\xi \geq n_\xi$. Let

$$h(\xi) = \max\{\Delta(s,t) + 1 : s,t \in a_\xi \cup b_\xi, s \neq t\}.$$

Note that if ξ is a limit ordinal, $h(\xi)$ is smaller than ξ, since we are working under the implicit assumption that our trees have the property that different nodes on the same limit level have different sets of predecessors. By the Pressing Down Lemma, there is a stationary set

[4] $U \restriction C = \bigcup_{\lambda \in C} U_\lambda$.

[5] $T \restriction \xi$ is the union of the first ξ levels of T.

Γ of countable limit ordinals such that h is constantly equal to α on Γ. Refining Γ, we may assume that for some $a, b \subseteq T_\alpha$, $a_\xi \restriction \alpha = a$ and $b_\xi \restriction \alpha = b$ for all $\xi \in \Gamma$. Moreover, we may assume that for some m and n and all $\xi \in \Gamma$, $m_\xi = m$ and $n_\xi = n$. Let $a_\xi(i)$ $(i < m)$ and $b_\xi(i)$ $(i < n)$ be a fixed enumeration for all $\xi \in \Gamma$. Note that p_ξ naturally extends to a map from a_ξ into b_ξ and this in turn induces a map from m onto n. We assume that this map from m onto n does not depend on ξ in Γ. Applying Lemma 1.4, we go to an uncountable set Ω such that for all $\xi \neq \eta$ in Ω, every node of $a_\xi \cup b_\xi$ is incomparable to every node of $a_\eta \cup b_\eta$. Now, we apply the assumption that T is a Lipschitz tree via Lemma 1.3 to obtain an uncountable subset Σ of Ω such that for all $\xi \neq \eta$ in Σ and all $i < m$ and $k < n$,

$$\Delta(a_\xi(i), a_\eta(i)) = \Delta(b_\xi(k), b_\eta(k)).$$

We claim that $r = p_\xi \cup p_\eta$ is a Lipschitz map for all $\xi, \eta \in \Sigma$. For, if $x, y \in \mathrm{dom}(p_\xi) \cup \mathrm{dom}(p_\eta)$ are either in the root, or they both come from $\mathrm{dom}(p_\xi)$ or $\mathrm{dom}(p_\eta)$, or they project to $a_\xi(i)$ and $a_\eta(j)$ with $i \neq j$, then the inequality $\Delta(r(x), r(y)) \geq \Delta(x, y)$ follows from the corresponding inequality for p_ξ and p_η. If they project to $a_\xi(i)$ and $a_\eta(j)$ respectively with $i = j$, then we in fact have the equality $\Delta(r(x), r(y)) = \Delta(x, y)$.

For $t \in T$ let $\mathcal{D}_t$ be the collection of all $p \in \mathcal{P}$ such that $t \in \mathrm{dom}(p)$. By the choice of C it follows easily that each $\mathcal{D}_t$ is a dense-open subset of $\mathcal{P}$. Applying (*) to $\mathcal{P}$ and the collection $\mathcal{D}_t (t \in T)$ of dense-open subsets of $\mathcal{P}$, one gets a total Lipschitz map $f : T \longrightarrow U \restriction C$. This shows that $T \leq U$ and completes the proof. □

The following fact shows that Lipschitz trees have a representation that is quite convenient to work with.

1.7 Lemma (*). *Every Lipschitz tree T is isomorphic to a downward closed subtree U of the tree of all maps from countable ordinals into ω such that $\{\xi \in \mathrm{dom}(x) \cap \mathrm{dom}(y) : x(\xi) \neq y(\xi)\}$ is finite for all $x, y \in U$.*

Proof. Define $\mathcal{P}$ to be the poset of all finite partial functions p from $T \times \omega_1$ into ω such that the following holds for all $x \neq y$ in $\mathrm{dom}_0(p)$[6]:

(1) $p(x, \xi) = p(y, \xi)$ for $\xi < \Delta(x, y)$,
(2) $p(x, \xi) \neq p(y, \xi)$ for $\xi = \Delta(x, y)$[7].

We let p extend q, if p extends q as a function, and

[6]Here and below, $\mathrm{dom}_0(p) = \{t \in T : (t, \alpha) \in \mathrm{dom}(p)$ for some $\alpha\}$ and $\mathrm{dom}_1(p) = \{\xi \in \omega_1 : (t, \xi) \in \mathrm{dom}(p)$ for some $t \in T\}$.

[7]In (1) and (2) we are making the implicit requirement that for every $\xi \in \mathrm{dom}_1(p)$ and $x \neq y \in \mathrm{dom}_0(p)$, if $\xi \leq \Delta(x, y)$, then $(x, \xi) \in \mathrm{dom}(p)$ iff $(y, \xi) \in \mathrm{dom}(p)$, and moreover, that always $(x, \Delta(x, y)$ and $(y, \Delta(x, y))$ belong to $\mathrm{dom}(p)$.

(3) $p(x,\xi) = p(y,\xi)$ for all $x, y \in \mathrm{dom}_0(q)$ and $\xi < \mathrm{ht}(x), \mathrm{ht}(y)$ with the property that $\xi \notin \mathrm{dom}_1(q)$.

It is clear that a sufficiently generic filter will give us the desired embedding, and so we concentrate on showing that $\mathcal{P}$ satisfies the countable chain condition.

So let p_δ $(\delta < \omega_1)$ be a given sequence of elements of $\mathcal{P}$. Let a_δ be the projection of $\mathrm{dom}_0(p_\delta)$ onto the δth level of T and let

$$h(\delta) = \max\{\Delta(s,t) + 1 : s, t \in a_\delta, s \neq t\}.$$

Then there is a stationary set Γ of countable limit ordinals on which the mapping h, as well as the mapping $\delta \longmapsto \mathrm{dom}(p_\delta) \cap (T \restriction \delta) \times \delta$ is constant. Let α and F be these constant values respectively. We may assume that all a_δ's project to the same set a on the αth level and all p_δ's generate isomorphic structures over α, a and F. Thus, in particular we want that the isomorphism between the p_δ's $(\delta \in \Gamma)$ respects a fixed enumeration $a_\delta(i)$ $(i < n)$ of a_δ, where n is the common cardinality of these sets. As in the previous proof, we find an uncountable subset Σ of Γ such that for all $\gamma \neq \delta$ in Σ:

(4) $a_\gamma(i)$ and $a_\delta(j)$ are incomparable for all $i, j < n$,
(5) $\Delta(a_\gamma(i), a_\delta(i)) = \Delta(a_\gamma(j), a_\delta(j))$ for all $i, j < n$.

We claim that if $\gamma \neq \delta$ are in Σ then p_γ and p_δ are compatible in $\mathcal{P}$. By (5), we have an ordinal β smaller than both γ and δ such that

$$\Delta(a_\gamma(i), a_\delta(i)) = \beta \text{ for all } i < n.$$

Define $r \in \mathcal{P}$ by letting its domain be

$$\mathrm{dom}(p_\gamma) \cup \mathrm{dom}(p_\delta) \cup \{(t,\beta) : t \in \mathrm{dom}_0(p_\gamma) \cup \mathrm{dom}_0(p_\delta), \mathrm{ht}(t) > \beta\}$$

and letting $r(t,\beta) = 0$ if $t \in \mathrm{dom}_0(p_\gamma)$, $\mathrm{ht}(t) > \beta$ and $r(t,\beta) = 1$ if $t \in \mathrm{dom}_0(p_\delta)$, $\mathrm{ht}(t) > \beta$. Note that r is indeed a member of $\mathcal{P}$, as it clearly satisfies the conditions (1) and (2) above. It is also easily checked that r extends both p and q, i.e. $r(x,\xi) = r(y,\xi)$ for all $x, y \in \mathrm{dom}_0(p)$ or $x, y \in \mathrm{dom}_0(q)$ and ξ is equal to β, the only new member of $\mathrm{dom}_1(r)$. This finishes the proof. □

1.8 Definition. A *coherent tree* is a tree that allows to be represented as a family T of functions from countable ordinals into ω such that $\{\xi \in \mathrm{dom}(s) \cap \mathrm{dom}(t) : s(\xi) \neq t(\xi)\}$ is finite for all $s, t \in T$.

Lemma 1.7 tells us that in the context of (*) the class of Lipschits trees coincides with the class of coherent Aronszajn trees giving us an explanation of why Lipschitz trees that one can define without appealing to additional axioms of set theory are almost always coherent.[8] It follows that the class of coherent trees is less special than one might have expected so we shall spend some time in examining this class. The profusion of Lipschitz maps on coherent trees is perhaps best explained by the following general fact.

1.9 Lemma. *The square of every special*[9] *coherent tree T can be covered by the graphs of countably many Lipschitz functions.*

Proof. It suffices to show that if $f : D \longrightarrow T$ is a level-preserving map from a subset D of T which intersects a given level of T in at most one point, then D can be decomposed into countably many sets on which f is Lipschitz. Let $a : T \longrightarrow \omega$ be a fixed map such that $a(s) \neq a(t)$, whenever $s <_T t$ and such that a is one-to-one on levels of T. For $t \in D$, let

$$F_t = \{\xi \in \omega_1 : \xi = \mathrm{ht}(t) \text{ or } \xi < \mathrm{ht}(t) \text{ and } t(\xi) \neq f(t)(\xi)\}.$$

Then F_t is a finite set of ordinals. Let $p_t : F_t \longrightarrow \omega$ and $q_t : F_t \longrightarrow \omega$ be defined by

$$p_t(\xi) = a(t \restriction \xi) \text{ and } q_t(\xi) = a(f(t) \restriction \xi).$$

Since the family of all finite partial functions from ω_1 into ω can be decomposed into countably many subfamilies of pairwise compatible functions, it suffices to establish the inequality

$$\Delta(f(s), f(t)) \geq \Delta(s,t)$$

for $s, t \in D$ with the following properties: $a(s) = a(t)$, $a(f(s)) = a(f(t))$ and both p_s and p_t as well as q_s and q_t are compatible functions.

To this end, let $\alpha = \Delta(s,t)$. Then $\alpha < \mathrm{ht}(s), \mathrm{ht}(t)$, since s and t (as well as $f(s)$ and $f(t)$) are incomparable in T, as they are mapped by a to the same integer. By the properties of our parameters s and t, we have that $F_s \cap \alpha = F_t \cap \alpha$ and that $f(s)$ and $f(t)$ agree on this set. From the definition of F_s and F_t, we conclude that $f(s)$ and $f(t)$ must agree below α, and so $\Delta(f(s), f(t)) \geq \alpha$. This finishes the proof. □

Note also the following fact which follows from the proof of Lemma 1.9.

1.10 Lemma. *A coherent tree T is Lipschitz if and only if every uncountable subset of T contains an uncountable antichain.*

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[8] The only Lipschitz tree known to us, which is not given in this way, is the tree $T(\rho_0)$ of [12].

[9] A tree is *special*, if it can be covered by countably many antichains.

2. Shifts of trees

In this and following sections we assume that trees T are represented in such a way that its elements on a given level α are simply functions from α into ω. A tree is *uniform* if it contains all finite changes of its elements. More precisely, if there is a $k \leq \omega$ such that $\mathrm{rang}(t) \subseteq k$ for all $t \in T$, then one can define the *uniform k-closure* T^* of T as the set of all $s : \alpha \longrightarrow k$ for which we can find $t \in T_\alpha$ such that $\{\xi < \alpha : s(\xi) \neq t(\xi)\}$ is finite. Thus, a tree is uniform, if it is equal to its ω-closure, but we shall use this word to even cover the case when the tree is equal to its k-closure for k not necessarily equal to ω. In fact, it will be clear from below that the crucial information found in this paper is not changed if we assume that we are working with uniform trees only, an assumption which we choose not to make.

Let Λ denote the set of all countable limit ordinals including 0, and for a positive integer n, let $\Lambda + n = \{\lambda + n : \lambda \in \Lambda\}$.

2.1 Definition. For an integer m and a tree T, we let $T^{(m)}$ be its mth *shift*, the downward closure of $\{t^{(m)} : t \in T \restriction \Lambda\}$, where for a limit node t of T, we let $t^{(m)}$ be the function with the same domain λ as t defined by,

$$t^{(m)}(\xi) = t(\xi - m),$$

when $\xi - m$ exists; otherwise (i.e., when m is positive and the largest limit ordinal $\leq \xi$ is less than m steps away), we let $t^{(m)}(\xi) = 0$.

2.2 Remark. Note that a positive shift $T^{(m)}$ of any Lipschitz tree T is Lipschitz and that the map $t \longmapsto t^{(m)}$ is a strictly increasing map from T into $T^{(m)}$. It follows that $T \leq T^{(m)}$ for all $m \geq 0$. Note also that for nonnegative integers m and n,

$$T^{(m+n)} = (T^{(m)})^{(n)}$$

holds. Therefore we have that $T^{(m)} \leq T^{(n)}$ for every pair of nonnegative integers m and n such that $m \leq n$.

2.3 Lemma. *If T is a Lipschitz tree, then $T^{(m)} < T^{(n)}$ for every pair of nonnegative integers m and n such that $m < n$.*

Proof. Suppose $m < n$ are nonnegative integers and consider a level-preserving map $f : T^{(n)} \longrightarrow T^{(m)}$. For each countable ordinal δ, pick a representative t_δ from the δth level of T and let $s_\delta \in T_\delta$ be such that

$$f(t_\delta^{(n)}) = s_\delta^{(m)}.$$

By Lemma 1.3, there is an uncountable set Γ of countable limit ordinals such that $\Delta(t_\gamma, t_\delta) = \Delta(s_\gamma, s_\delta)$ for all $\gamma, \delta \in \Gamma$, $\gamma \neq \delta$. Choose $\gamma \neq \delta$

in Γ such that $\alpha = \Delta(t_\gamma, t_\delta) = \Delta(s_\gamma, s_\delta)$ is smaller than both γ and δ (i.e. t_γ is incomparable to t_δ and s_γ is incomparable to s_δ). Then

$$\Delta(t_\gamma^{(n)}, t_\delta^{(n)}) = \alpha + n > \alpha + m = \Delta(s_\gamma^{(m)}, s_\delta^{(m)}).$$

This shows that f is not a Lipschitz map, finishing the proof. □

The following Lemma reveals that in the realm of Lipschitz trees, $T^{(1)}$ is indeed the minimal tree above T.

2.4 Lemma (*). *For every pair S and T of Lipschitz trees, $S < T$ implies $S^{(1)} \leq T$.*

Proof. Choose representatives $s_\delta \in S_\delta$ $(\delta < \omega_1)$ and $t_\delta \in T_\delta$ $(\delta < \omega_1)$. In §5 (Lemmas 5.2 and 5.3) we show that there is an uncountable set Γ of countable limit ordinals such that $\Delta(s_\gamma, s_\delta) < \Delta(t_\gamma, t_\delta)$ for all $\gamma \neq \delta$ in Γ. We have already seen above in the proof of Lemma 2.3 that

$$\Delta(s_\gamma^{(1)}, s_\delta^{(1)}) = \Delta(s_\gamma, s_\delta) + 1 \leq \Delta(t_\gamma, t_\delta)$$

for all $\gamma \neq \delta$ in Γ. Applying again Lemma 5.2, we conclude that $S^{(1)} \leq T$. □

It follows that (under (*)) for every Lipschitz tree T, the chain $T^{(n)}$ $(n \in \mathbb{N})$ of positive shifts is really an $\mathbb{N}$-chain, i.e. its convex closure inside the class of Lipschitz trees is isomorphic to $\mathbb{N}$ as an ordered set. Assuming a natural conjecture in this context (which turn out to be equivalent to Shelah's conjecture [10]), in §8, we shall show that for every Lipschitz tree T, the gap $(T, T^{(1)})$ is actually a gap in the class of all Aronszajn trees, so in this context the convex closure of $T^{(n)}$ $(n \in \mathbb{N})$ in the class of all Aronszajn trees is isomorphic to $\mathbb{N}$. The case of negative shifts is a bit more subtle though we shall see that they do behave as expected.

2.5 Definition. A tree T is *orthogonal* to a set of ordinals Γ, if there is an uncountable subset X of T such that $\Delta(x, y) \notin \Gamma$ for all $x, y \in X$, $x \neq y$.

2.6 Lemma. *Suppose that $n < m \leq 0$ and that T is a Lipschitz tree which is orthogonal to $\Lambda + k$ for all $0 \leq k \leq |n|$. Then $T^{(m)}$ and $T^{(n)}$ are also Lipschitz and $T^{(m)} \not\leq T^{(n)}$.*

Proof. We shall only show that $T^{(m)} \not\leq T^{(n)}$, since the argument will also contain the proof that $T^{(m)}$ and $T^{(n)}$ are Lipschitz. For each $0 \leq k \leq |n|$, we choose an uncountable $X_k \subseteq T$ such that $\Delta(x, y) \notin \Lambda + k$ for all $x \neq y$. We may assume that for some fixed $\Gamma \subseteq \Lambda$, each set X_k for $k \leq |n|$ is enumerated as level-sequence of the form $t_\delta(k)$ $(\delta \in \Gamma)$. Applying Lemma 1.3 to $\langle t_\delta(k) : 0 \leq k \leq |n| \rangle$ $(\delta \in \Gamma)$, we get an uncountable set $\Gamma_0 \subseteq \Gamma$ such that for all $\gamma \neq \delta$ in Γ and $0 \leq j, k \leq |n|$, we have

$$\Delta(t_\gamma(j), t_\delta(j)) = \Delta(t_\gamma(k), t_\delta(k)).$$

Let $t_\delta = t_\delta(0)$ for $\delta \in \Gamma_0$. This gives us an uncountable level-sequence with the property that $\Delta(t_\gamma, t_\delta) \notin \Lambda + k$ for all $\gamma \neq \delta$ in Γ_0 and $0 \leq k \leq |n|$.

Consider a level-preserving map $f : T^{(m)} \longrightarrow T^{(n)}$ and for each $\gamma \in \Gamma_0$ find $s_\gamma \in T_\gamma$ such that

$$f(t_\gamma^{(m)}) = s_\gamma^{(n)}.$$

Applying Lemma 1.3 again, we find an uncountable $\Gamma_1 \subseteq \Gamma_0$ such that $\Delta(s_\gamma, s_\delta) = \Delta(t_\gamma, t_\delta)$ whenever $\gamma \neq \delta$ are chosen from Γ_1. Consider $\gamma \neq \delta$ in Γ_0 such that $\alpha = \Delta(s_\gamma, s_\delta) = \Delta(t_\gamma, t_\delta)$ is smaller than both γ and δ. Let λ be the maximal limit ordinal below α. Then $\alpha > \lambda + |n|$, so the ordinals $\alpha + m$ and $\alpha + n$ are well-defined and $> \lambda$. Note that

$$\Delta(t_\gamma^{(m)}, t_\delta^{(m)}) = \alpha + m > \alpha + n = \Delta(s_\gamma^{(n)}, s_\delta^{(n)}).$$

This shows that f is not a Lipschitz map and finishes the proof. $\square$

The following result summarizes what has been shown so far about the shift operation on the class of Lipschitz trees.

2.7 Theorem (*). *For every Lipschitz tree T which is orthogonal to $\Lambda + k$ for all $k \geq 0$, the shifts $T^{(n)}$ ($n \in \mathbb{Z}$) form a family of Lipschitz trees with the following properties:*

1. $T^{(m+n)} \equiv (T^{(m)})^{(n)}$,
2. $T^{(n)} < T^{(m)}$ *iff* $n < m$,
3. *there is no Lipschitz tree S such that $T^{(n)} < S < T^{(n+1)}$ for some $n \in \mathbb{Z}$.*

2.8 Lemma. *There exists a coherent Lipschitz tree that is orthogonal to $\Lambda + k$ for every nonnegative integer k.*

Proof. Let $C_0 = \emptyset$, $C_{\alpha+1} = \{\alpha\}$ and for $\alpha \in \Lambda \setminus \{0\}$, choose $C_\alpha \subseteq \alpha$ of order-type ω such that if $\xi \in C_\alpha$ and $k = |C_\alpha \cap \xi|$, then $\xi = \lambda + k + 1$, where λ is the maximal limit ordinal $\leq \xi$. The sequence C_α ($\alpha < \omega_1$) is then used to define the *walk* from any countable ordinal β to some smaller ordinal α as follows: $\beta = \beta_0 \to \beta_1 \to \ldots \to \beta_n = \alpha$, where

$$\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha) \ (i < n).$$

The *weight* $w(\beta_i, \beta_{i+1})$ of the step $\beta_i \to \beta_{i+1}$ in the walk from β to α is defined to be equal to the cardinality of the set $C_{\beta_i} \cap \alpha = C_{\beta_i} \cap \beta_{i+1}$. This allows us to define, for every $\beta < \omega_1$, a function $t_\beta : \beta \longrightarrow 2$ by letting $t_\beta(\alpha) = 1$ iff the last step of the walk from β to α is of maximal weight.

The following facts are easily established by induction on $\alpha < \beta < \omega_1$ (see [15, 1.67-1.72] where the corresponding tree is denoted by $T(\rho_3)$):

(a) $\mathrm{supp}(t_\beta) \cap (\Lambda + k)$ is finite for all $k < \omega$[10],

(b) $t_\alpha =^* t_\beta \restriction \alpha$[11], whenever $\alpha < \beta < \omega_1$,

(c) there is no $t : \omega_1 \longrightarrow 2$ such that $t \restriction \alpha =^* t_\alpha$ for all $\alpha < \omega_1$.

It follows that if we let T be the uniform 2-closure of the collection $\{t_\beta \restriction \alpha : \alpha \leq \beta < \omega_1\}$, we get a tree satisfying the conclusion of Lemma 2.8. □

2.9 Theorem. *There is a uniform coherent Lipschitz tree T so that the corresponding shifts $T^{(n)}$ $(n \in \mathbb{Z})$ are all uniform, coherent and Lipschitz and moreover $T^{(m)} < T^{(n)}$ holds whenever $m < n$.*

Proof. By Lemmas 2.3,2.6 and 2.8 it remains to show that if T is the tree of Lemma 2.8, then $T^{(m)} \leq T^{(n)}$ whenever $m < n \leq 0$. Let

$$S = \{t \in T : t(\xi) = 0 \text{ for all } \xi \in \mathrm{dom}(t) \cap \bigcup_{k \leq |m|} \Lambda + k\}.$$

Then S is a downwards closed subset of T with the property that $S^{(m)} = T^{(m)}$. So it suffices to observe that $t^{(m)} \longmapsto t^{(n)}$ $(t \in S \restriction \Lambda)$ is a (partial) Lipschitz map from $S^{(m)}$ into $S^{(n)} \subseteq T^{(n)}$. □

2.10 Remark. This shows that the class $\mathcal{C}$ of Lipschitz trees is not well-quasi-ordered under $\leq$. So this gives a solution to an old problem of Laver in [6], who asked whether the larger class $\mathcal{A}$ is well-quasi-ordered under even a stronger quasi-ordering. The question can also be negatively answered, if one produces an infinite antichain in $(\mathcal{A}, \leq)$. This will be done in the next section.

3. Incomparable trees

In this section, we shall utilize the following slightly stronger version of Theorem 2.9:

3.1 Lemma. *There is a sequence $S^{(n)}$ $(n \in \mathbb{Z})$ of uniform coherent Lipschitz trees such that if $n < m$ and U is an uncountable downwards closed subtree of $S^{(m)}$, then $U \not\leq S^{(n)}$.*

Proof. Note that in the proof of Lemma 2.6, the representative t_δ of the δth level can be chosen in the subtree U. □

[10] $\mathrm{supp}(t_\beta) = \{\alpha < \beta : t_\beta(\alpha) = 0\}$.

[11] Here, $=^*$ denotes the equality modulo a finite set.

From now on, we fix a sequence $S^{(n)}$ $(n \in \mathbb{Z})$ of uniform coherent Lipschitz trees satisfying Lemma 3.1 and for each $n \in \mathbb{N}$ and $\delta \in \Lambda$, we pick a representative x^n_δ from the δth level of $S^{(n)}$. For each $\delta \in \Lambda \setminus \{0\}$, we fix a decomposition $\delta = \bigcup_{k=0}^{\infty} I^k_\delta$ such that for each k, I^k_δ is a nonempty closed-open interval of ordinals with

$$\sup(I^k_\delta) = \min(I^{k+1}_\delta),$$

the ordinal that we are going to denote by δ_k. Thus, $\{\delta_n\}$ is a strictly increasing sequence converging to δ.

3.2 Definition. For $\Gamma \subseteq \Lambda$, let $T(\Gamma)$ be the collection of all mappings $t : \alpha \longrightarrow \omega$, where $\alpha < \omega_1$ such that either $t \in S^{(n)}$ for some $n \in \mathbb{N}$, or there exist $\delta \leq \alpha$ in Γ, integers $l, m \in \mathbb{N}$, nodes $y_l \in S^{(l)}_\alpha$ and $z_k \in S^{(k)}_{\delta_k}$ $(k < m)$ such that

$$t = (\bigcup_{k=0}^{m-1} z_k \restriction I^k_\delta) \cup (\bigcup_{k=m}^{\infty} x^k_{\delta_k} \restriction I^k_\delta) \cup (y_l \restriction [\delta, \alpha)).$$

Clearly, $T(\Gamma)$ is an Aronszajn tree for all $\Gamma \subseteq \Lambda$.

3.3 Lemma. *If $\Sigma, \Omega \subseteq \Lambda$ are such that the difference $\Sigma \setminus \Omega$ is a stationary subset of ω_1, then $T(\Sigma) \nleq T(\Omega)$.*

Proof. Suppose $\Sigma \setminus \Omega$ is stationary, yet there is a strictly increasing mapping $f : T(\Sigma) \longrightarrow T(\Omega)$. As before, we may assume that f is also level-preserving. For $\delta \in \Sigma \setminus \Omega$, we choose a representative t_δ of the δth level of $T(\Sigma)$ of the form

$$t_\delta = (\bigcup_{k=0}^{m-1} z^k_\delta \restriction I^k_\delta) \cup (\bigcup_{k=m}^{\infty} x^k_{\delta_k} \restriction I^k_\delta)$$

for some choice of $m = m(\delta) \in \mathbb{N}$ and $z^k_\delta \in S^{(k)}_{\delta_k}$ $(k < m)$. Since $f(t_\delta)$ is a member of the δth level of $T(\Omega)$, it allows a decomposition according to Definition 3.2:

$$f(t_\delta) = (\bigcup_{k=0}^{\bar{m}-1} \bar{z}^k_{\xi_k} \restriction I^k_\xi) \cup (\bigcup_{k=\bar{m}}^{\infty} x^k_{\xi_k} \restriction I^k_\xi) \cup (y^l_\delta \restriction [\xi, \delta))$$

for some choices of $\bar{m} = \bar{m}(\delta) \in \mathbb{N}$, $l = l(\delta) \in \mathbb{N}$, $y^l_\delta \in S^{(l)}_\delta$, $\xi = \xi(\delta)$ in $\Omega \cap (\delta + 1) = \Omega \cap \delta$, $\bar{z}^k_{\xi_k} \in S^{(k)}_{\xi_k}$ $(k < \bar{m})$, and $y^l_\delta \in S^{(l)}_\delta$. By putting some z^k_δ's and $\bar{z}^k_{\xi_k}$'s to be equal to x^k_δ's and $x^k_{\xi_k}$'s respectively, we may assume that $m(\delta) = \bar{m}(\delta) > l(\delta)$, and that $\delta_{m(\delta)-1} \geq \xi(\delta)$. By the Pressing Down Lemma, there is a stationary set $\Gamma \subseteq \Sigma \setminus \Omega$ so that all the mappings are constant on Γ. Let m, l, ξ, δ_k $(k < m), z^k$ $(k < m)$ and $\bar{z}^k$ $(k < m)$ be the corresponding constant values. Since Γ is a stationary set, we can find an integer $n > m$ and an uncountable set $\Gamma_0 \subseteq \Gamma$ such that:

(i) $\delta \longmapsto \delta_k$ $(k < n)$ are all constant on Γ_0.

(ii) $\delta \longmapsto \delta_n$ is strictly increasing on Γ_0.

Shrinking Γ_0, we assume that

$$\delta \longmapsto x^n_{\delta_n} \restriction \delta_{n-1} \text{ and } \delta \longmapsto y^l_\delta \restriction \delta_{n-1}$$

are constant mappings. Let U be the downward closed subtree of $S^{(n)}$ generated by the set $\{x^n_{\delta_n} : \delta \in \Gamma_0\}$ and let V be the downward closed subtree of $S^{(l)}$ generated by $\{y^l_\delta : \delta \in \Gamma_0\}$. Define $g : U \longrightarrow V$ by

$$g(t) = f(x^n_{\delta_n} \restriction \mathrm{dom}(t))$$

for some choice of $\delta \in \Gamma_0$ such that $x^n_{\delta_n}$ extends t. Since f is increasing,

$$f(x^n_{\delta_n} \restriction \mathrm{dom}(t)) = f(x^n_{\gamma_n} \restriction \mathrm{dom}(t))$$

for every pair $\gamma, \delta \in \Gamma_0$ for which $x^n_{\gamma_n}$ and $x^n_{\delta_n}$ both extend t, so the definition does not depend on the choice of the extension $x^n_{\delta_n}$ of t. The same argument shows that g, defined in this way, is strictly increasing. Since $n > l$, this contradicts the fact that the sequence $S^{(k)}$ $(k \in \mathbb{N})$ satisfies the conclusion of Lemma 3.1. This finishes the proof. □

3.4 Theorem. *There is a family of size $2^{\aleph_1}$ of pairwise incomparable Aronszajn trees.*

Proof. Choose a family $\mathcal{F} \subseteq \mathcal{P}(\Lambda)$ of size $2^{\aleph_1}$ such that $\Sigma \setminus \Omega$ is stationary, whenever Σ and Ω are distinct members of $\mathcal{F}$. Then, $T(\Sigma) \not\leq T(\Omega)$ and $T(\Omega) \not\leq T(\Sigma)$ for every pair $\Sigma \neq \Omega$ in $\mathcal{F}$. □

Starting from a family $\mathcal{F}$ of $\aleph_1$-many pairwise disjoint stationary sets, there is a natural way (see e.g. [11, §5]) to fuse $T(\Sigma)$ $(\Sigma \in \mathcal{F})$ into a single Aronszajn tree with the following extreme property that can be considered as a culmination of one line of work on Kurepa's isomorphism problem for Aronszajn trees (see [5, §10.4] and [4]).

3.5 Theorem. *There is an Aronszajn tree with no nontrivial strictly increasing self-maps.* □

4. An ultrafilter from a Lipschitz tree

Recall the definition of the distance function $\Delta : T^2 \longrightarrow \omega$ on a tree T of height ω_1. For $X \subseteq T$, let

$$\Delta(X) = \{\Delta(x, y) : x, y \in X, x \neq y\}.$$

We use this notation to describe a family of subsets of ω_1 as follows:

$$\mathcal{U}(T) = \{\Gamma \subseteq \omega_1 : \Gamma \supseteq \Delta(X) \text{ for some uncountable } X \subseteq T\}.$$

4.1 Lemma. *The family $\mathcal{U}(T)$ is a uniform filter on ω_1 for every Lipschitz tree T.*

Proof. Given two uncountable subsets X and Y of T, we need to find an uncountable subset Z of T such that

$$\Delta(X) \cap \Delta(Y) \supseteq \Delta(Z).$$

By the assumption about T, it is clear that we may replace X and Y by two level-sequences x_δ $(\delta \in \Gamma)$ and y_δ $(\delta \in \Gamma)$ indexed by the same uncountable set Γ of limit ordinals. Moreover, we may assume that the x_δ's and y_δ's are all pairwise incomparable (see Remark 1.2(2)). Apply Lemma 1.3 to the subset (x_δ, y_δ) $(\delta \in \Gamma)$ of $T \otimes T$ and obtain an uncountable set $\Sigma \subseteq \Gamma$ such that

$$\Delta(x_\gamma, x_\delta) = \Delta(y_\gamma, y_\delta) \text{ for all } \gamma, \delta \in \Sigma, \gamma \neq \delta.$$

So we can take Z to be any of the sets $\{x_\delta : \delta \in \Sigma\}$ or $\{y_\delta : \delta \in \Sigma\}$. This finishes the proof. □

Using (*), we can say a bit more about $\mathcal{U}(T)$:

4.2 Theorem (*). *For every Lipschitz tree T, the filter $\mathcal{U}(T)$ is in fact an ultrafilter.*

Proof. Let Γ be a given subset of ω_1. We need to find an uncountable subset X of T such that $\Delta(X)$ is included in either Γ or its complement. Let $\mathcal{P}_\Gamma$ be the poset of all finite subsets p of T that take at most one point from a given level of T such that

$$\Delta(p) = \{\Delta(x,y) : x, y \in p, x \neq y\} \subseteq \Gamma.$$

If $\mathcal{P}_\Gamma$ satisfies the countable chain condition, then a straightforward application of (*) gives an uncountable $X \subseteq T$ such that $\Delta(X) \subseteq \Gamma$. So let us consider the alternative that $\mathcal{P}_\Gamma$ fails to satisfy this condition. Let p_δ $(\delta < \omega_1)$ be a sequence of pairwise incomparable members of $\mathcal{P}_\Gamma$. Reenumerating, we may assume that every node of a given p_δ has height at least δ, so we can define $a_\delta =$ the projection of p_δ on the δth level of T. For $\delta \in \Lambda$, let

$$h(\delta) = \{\Delta(x,y) + 1 : x, y \in a_\delta, x \neq y\}.$$

Find stationary $\Omega \subseteq \Lambda$ such that h is constant on Ω. Let $\bar{\xi}$ be the constant value of h. Going to an uncountable subset of Ω, we may assume that all a_δ $(\delta \in \Omega)$ are of equal size n and that they are given with an enumeration $a_\delta(i)$ $(i < n)$. Moreover, we may assume that $a_\gamma(i) \restriction \bar{\xi} = a_\delta(i) \restriction \bar{\xi}$ for all $\gamma, \delta \in \Omega$. Applying Lemmas 1.3 and 1.4, we obtain an uncountable $\Sigma \subseteq \Omega$ such that for all $\gamma \neq \delta$ in Σ:

(1) $a_\gamma(i)$ and $a_\delta(j)$ are incomparable for all $i, j < n$,

(2) $\Delta(a_\gamma(i), a_\delta(i)) = \Delta(a_\gamma(j), a_\delta(j))$ for all $i, j < n$.

It follows that for all $\gamma \neq \delta$ in Σ:

(3) $\Delta(p_\gamma \cup p_\delta) = \Delta(p_\gamma) \cup \Delta(p_\delta) \cup \{\Delta(a_\gamma(0), a_\delta(0))\}$.

Since $\Delta(p_\gamma)$ and $\Delta(p_\delta)$ are subsets of Γ, $\Delta(a_\gamma(0), a_\delta(0)) \notin \Gamma$ must hold. This gives rise to an uncountable set $X = \{a_\delta(0) : \delta \in \Sigma\}$ with the property that $\Delta(X) \cap \Gamma = \emptyset$. This finishes the proof. □

4.3 Remark. Note that we have just shown that $\mathcal{P}_\Gamma$ satisfies the countable chain condition for every set Γ which belongs to the co-ideal $\mathcal{U}(T)^+ = \{\Sigma : \Sigma \cap \Delta \neq \emptyset \text{ for every } \Delta \in \mathcal{U}(T)\}$. Since for disjoint Γ_0 and Γ_1 the product $\mathcal{P}_{\Gamma_0} \times \mathcal{P}_{\Gamma_1}$ fails the countable chain condition unless one of the two posets is in fact countable, this show that in Theorem 4.2 we have used only the productiveness of the countable chain condition rather than the full strength of (*).

5. Comparability of Lipschitz trees

Throughout this section, we assume (*) and examine the behaviour of the quasi-ordering $\leq$ in the domain of Lipschitz trees. For the time being, fix a pair S and T of Lipschitz trees, and for each $\delta < \omega_1$, fix representatives s_δ and t_δ from the δth level of S and T respectively. This gives us two mappings

$$\Delta_s : [\omega_1]^2 \longrightarrow \omega_1 \text{ and } \Delta_t : [\omega_1]^2 \longrightarrow \omega_1$$

defined as follows:

$$\Delta_s(\alpha, \beta) = \Delta(s_\alpha, s_\beta) \text{ and } \Delta_t(\alpha, \beta) = \Delta(t_\alpha, t_\beta).$$

For $\Gamma \subseteq \omega_1$, we let $\Delta_s(\Gamma) = \{\Delta_s(\alpha, \beta) : \alpha, \beta \in \Gamma, \alpha \neq \beta\}$ and $\Delta_t(\Gamma) = \{\Delta_t(\alpha, \beta) : \alpha, \beta \in \Gamma, \alpha \neq \beta\}$. The argument from the previous section yields the following useful description of $\mathcal{U}(T)$ which in fact does not need (*) nor any other additional set-theoretic assumption.

5.1 Lemma. *For every $\Gamma \in \mathcal{U}(T)$ and uncountable $\Omega \subseteq \omega_1$ there is an uncountable $\Sigma \subseteq \Omega$ such that $\Delta_t(\Sigma) \subseteq \Gamma$.* □

It follows that $\Delta_t(\Sigma)$ ($\Sigma \subseteq \omega_1$ uncountable) generates the filter $\mathcal{U}(T)$.

5.2 Lemma (*). *The following are equivalent for every pair S and T of Lipschitz trees:*

(a) $S \leq T$,

(b) *there is an uncountable $\Gamma \subseteq \omega_1$ such that $\Delta_s(\alpha, \beta) \leq \Delta_t(\alpha, \beta)$ for all $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$,*

(c) *for every uncountable $\Sigma \subseteq \omega_1$ there is an uncountable $\Gamma \subseteq \Sigma$ such that $\Delta_s(\alpha, \beta) \leq \Delta_t(\alpha, \beta)$ for all $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$.*

Proof. To deduce (b) from (a), suppose we are given a strictly increasing level-preserving map $f : S \longrightarrow T$. Apply Lemma 1.3 to $(t_\delta, f(s_\delta))$ $(\delta \in \omega_1)$ and obtain an uncountable $\Gamma \subseteq \omega_1$ such that for all $\gamma \neq \delta$:

(1) t_γ and t_δ are incomparable,
(2) $f(s_\gamma)$ and $f(s_\delta)$ are incomparable,
(3) $\Delta(t_\gamma, t_\delta) = \Delta(f(s_\gamma), f(s_\delta))$.

Clearly, this Γ satisfies (b). Similarly one shows that (a) implies (c). Note that clause (b) simply says that the map $s_\delta \longmapsto t_\delta$ $(\delta \in \Gamma)$ is Lipschitz and it therefore extends to a strictly increasing map from the downward closure S_0 of the set $\{s_\delta : \delta \in \Gamma\}$ in S. By Lemma 1.6, $S \leq S_0 \leq T$. This shows that (b), and therefore the stronger (c), implies (a) and finishes the proof. $\square$

The next lemma gives us a convenient reformulation of the inequality $T \not\leq S$.

5.3 Lemma (*). *The following are equivalent for every pair S and T of Lipschitz trees:*

(a) $T \not\leq S$,
(b) *there is an uncountable $\Gamma \subseteq \omega_1$ such that $\Delta_s(\alpha, \beta) < \Delta_t(\alpha, \beta)$ for all $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$,*
(c) *for every uncountable $\Sigma \subseteq \omega_1$ there is an uncountable $\Gamma \subseteq \Sigma$ such that $\Delta_s(\alpha, \beta) < \Delta_t(\alpha, \beta)$ for all $\alpha, \beta \in \Gamma$.*

Proof. To see that (a) implies (b), let $\mathcal{P}$ be the poset of all finite $p \subseteq \omega_1$ such that

(1) $\Delta_t(\alpha, \beta) \leq \Delta_s(\alpha, \beta)$ for all $\alpha, \beta \in p$, $\alpha \neq \beta$.

If $\mathcal{P}$ would satisfy the countable chain condition, an application of (*) would give us an uncountable set $\Gamma \subseteq \omega_1$ such that $\Delta_t(\alpha, \beta) \leq \Delta_s(\alpha, \beta)$ for all $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$ which by Lemma 5.2 would give us $T \leq S$, contradicting (a). So let p_δ $(\delta \in \omega_1)$ be a given sequence of pairwise incompatible members of $\mathcal{P}$. We may assume that $\min(p_\delta) \geq \delta$ for all $\delta \in \omega_1$. For $\delta \in \omega_1$, let

$$a_\delta = \{s_\xi \upharpoonright \delta : \xi \in p_\delta\} \text{ and } b_\delta = \{t_\xi \upharpoonright \delta : \xi \in p_\delta\}.$$

For $\delta \in \Lambda$, let $h(\delta)$ be the maximum of all ordinals that have the form $\Delta(x, y) + 1$, $x, y \in a_\delta$, $x \neq y$ and $\Delta(x, y) + 1$, $x, y \in b_\delta$, $x \neq y$. Find a stationary $\Omega \subseteq \Lambda$ such that h is constant on Ω and let $\bar{\xi}$ be the constant value. Shrinking Ω, we may assume that all a_δ $(\delta \in \Omega)$ are of some fixed size m, and that all b_δ $(\delta \in \Omega)$ are of some fixed size n. Let $a_\delta(i)$ $(i < m)$ and $b_\delta(j)$ $(j < n)$ be fixed enumerations. Applying

Lemma 1.3, we can find an uncountable $\Sigma \subseteq \Omega$ such that for all $\gamma \neq \delta$ in Σ:

(2) $a_\gamma(i) \restriction \bar{\xi} = a_\delta(i) \restriction \bar{\xi}$ for all $i < m$,
(3) $b_\gamma(i) \restriction \bar{\xi} = b_\delta(i) \restriction \bar{\xi}$ for all $i < n$,
(4) $a_\gamma(i)$ and $a_\delta(j)$ are incomparable for all $i, j < m$,
(5) $b_\gamma(i)$ and $b_\delta(j)$ are incomparable for all $i, j < m$,
(6) $\Delta(a_\gamma(i), a_\delta(i)) = \Delta(a_\gamma(j), a_\delta(j))$ for all $i, j < m$,
(7) $\Delta(b_\gamma(i), b_\delta(i)) = \Delta(b_\gamma(j), b_\delta(j))$ for all $i, j < n$.

Consider $\gamma \neq \delta$ in Σ. Then $p_\gamma \cup p_\delta$ fails to satisfy condition (1), i.e. there exist $\xi \in p_\gamma$ and $\eta \in p_\delta$ such that $\Delta(s_\xi, s_\eta) < \Delta(t_\xi, t_\eta)$. Let $s_\xi \restriction \gamma = a_\gamma(i)$, $s_\eta \restriction \delta = a_\delta(j)$, $t_\xi \restriction \gamma = b_\gamma(k)$ and $t_\eta \restriction \delta = b_\delta(l)$. Using (2),(3),(4) and the fact that p_γ and p_δ satisfy (1), we conclude that $i = j$ and $k = l$ and therefore, by (6) and (7), we have the following:

(8) $\Delta(a_\gamma(0), a_\delta(0)) = \Delta(a_\gamma(i), a_\delta(i)) < \Delta(b_\gamma(k), b_\delta(k)) =$
$= \Delta(b_\gamma(0), b_\delta(0))$.

Of course, we may assume that the enumerations of a_δ and b_δ are given in a way such that if $\xi(\delta) = \min(p_\delta)$, then $s_{\xi(\delta)} \restriction \delta = a_\delta(0)$ and $t_{\xi(\delta)} \restriction \delta = b_\delta(0)$ for all δ in Σ. Let $\Gamma = \{\xi(\delta) : \delta \in \Gamma\}$. Then Γ satisfies clause (b), finishing thus the proof that (a) implies (b). Similarly, one proves that (a) in fact implies (c). The implication from (b) to (a) follows from Lemma 5.2. This finishes the proof. □

Finally, we are in a situation to state the main result of this section.

5.4 Theorem (*). *Every two Lipschitz trees are comparable.*

Proof. Suppose we are given a pair S and T of Lipschitz trees such that $T \not\leq S$. By Lemma 5.3, there is an uncountable $\Gamma \subseteq \omega_1$ such that $\Delta_s(\alpha, \beta) < \Delta_t(\alpha, \beta)$ for all $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$. By Lemma 5.2, we conclude that $T \leq S$. This completes the proof. □

5.5 Remark. This result and Remark 1.8 show that under (*) coherent Aronszajn trees are totally ordered by $\leq$. This in turn hints that the construction of §3 which combines a number of different coherent trees into a single Aronszajn tree in order to produce a pair of incomparable trees is in some sense necessary. More on the necessity of the approach of §3 will be seen below (see Theorem 7.7).

Recall the definition of the functor $T \longmapsto \mathcal{U}(T)$, whose domain is the class of Lipschitz trees and whose range is the space of all uniform filters on ω_1 (see §3). The following result shows that $T \longmapsto \mathcal{U}(T)$ is a reduction of the equivalence relation $\equiv$ on the class of all Lipschitz trees to the equality relation on the space $\mathcal{U}(\omega_1)$ of uniform ultrafilters on ω_1.

5.6 Theorem (*). *The following are equivalent:*

(a) $S \equiv T$,
(b) $\mathcal{U}(S) = \mathcal{U}(T)$.

Proof. Assume $S \equiv T$. Applying Lemma 5.2 twice, we get an uncountable set $\Gamma \subseteq \omega_1$ such that $\Delta_s(\alpha,\beta) = \Delta_t(\alpha,\beta)$ for all $\alpha,\beta \in \Gamma$, $\alpha \neq \beta$. Now, the equality $\mathcal{U}(S) = \mathcal{U}(T)$ follows from Lemma 5.1. Assume now that, say, $T \not\leq S$. By Lemma 5.3, there is an uncountable set $\Gamma \subseteq \omega_1$ such that $\Delta_s(\alpha,\beta) < \Delta_t(\alpha,\beta)$ for all $\alpha,\beta \in \Gamma$, $\alpha \neq \beta$. Let $\mathcal{P}$ be the poset of all finite $p \subseteq \Gamma$ such that $\Delta_s(p) \cap \Delta_t(p) = \emptyset$. It suffices to show that $\mathcal{P}$ satisfies the countable chain condition. To see this, consider a sequence p_δ $(\delta < \omega_1)$ of elements of $\mathcal{P}$. Going to a Δ-system and noting that the root can't contribute to the incomparability, we may assume that all p_δ's are pairwise disjoint and in fact that $\min(p_\delta) \geq \delta$ for all δ. Working as in the proof of Lemma 5.3, we define a_δ, b_δ and h and obtain Ω, $\bar{\xi}$ and then an uncountable $\Sigma \subseteq \Omega$ such that (2)-(7) hold. It follows that for all $\gamma \neq \delta$ in Σ, we have:

(8) $\Delta_s(p_\gamma \cup p_\delta) = \Delta_s(p_\gamma) \cup \Delta_s(p_\delta) \cup \{\Delta(a_\gamma(0), a_\delta(0))\}$,
(9) $\Delta_t(p_\gamma \cup p_\delta) = \Delta_t(p_\gamma) \cup \Delta_t(p_\delta) \cup \{\Delta(b_\gamma(0), b_\delta(0))\}$.

We may assume that our enumerations are chosen in such a way that if $\xi(\delta) = \min(p_\delta)$, then

(10) $s_{\xi(\delta)} \restriction \delta = a_\delta(0)$ for all $\delta \in \Sigma$,
(11) $t_{\xi(\delta)} \restriction \delta = b_\delta(0)$ for all $\delta \in \Sigma$.

It follows that for all $\gamma \neq \delta$ in Σ:

(12) $\Delta(a_\gamma(0), a_\delta(0)) = \Delta_s(\xi(\gamma), \xi(\delta)) < \Delta_t(\xi(\gamma), \xi(\delta)) =$
$\Delta(b_\gamma(0), b_\delta(0))$.

This together with (2)-(7) and (8),(9) gives the desired conclusion that p_γ and p_δ are compatible in $\mathcal{P}$ for some (all) $\gamma \neq \delta$ in Σ. This finishes the proof. $\square$

5.7 Remark. Note that the implication from $S \equiv T$ to $\mathcal{U}(S) = \mathcal{U}(T)$ needs neither (*) nor any other additional assumption.

5.8 Theorem (*). *Two uniform Lipschitz trees which have the same branching degrees are isomorphic if and only if they are equivalent.*

Proof. If $S \equiv T$, then by Lemma 5.2, there is an uncountable set $\Gamma \subseteq \omega_1$ such that Δ_s and Δ_t agree on pairs from Γ. Let $\mathcal{P}$ be the poset of all partial finite level-preserving isomorphisms p from S into T which extend to isomorphisms between downward closures of $\mathrm{dom}(p)$ in S and $\mathrm{rang}(p)$ in T. Using the agreement of Δ_s and Δ_t on Γ and the argument from the proof of Lemma 5.3, one shows that $\mathcal{P}$ satisfies the countable chain condition. The uniformity of the trees S and T are used to show that all sets of the form

$$\mathcal{D}_x = \{p \in \mathcal{P} : x \in \mathrm{dom}(p)\} \ (x \in S),$$
$$\mathcal{E}_y = \{p \in \mathcal{P} : y \in \mathrm{rang}(p)\} \ (y \in T)$$

are dense open in $\mathcal{P}$. A filter of $\mathcal{P}$ which intersects all these sets gives us the required isomorphism between S and T. □

5.9 Remark. The Schroeder-Bernstein phenomenon encountered in Theorem 5.8 does not hold in the wider class of all Aronszajn trees. To see this, let $T^{(n)}$ $(n \in \mathbb{Z})$ be the sequence of Lipschitz trees from Theorem 2.9 and let S be the free sum of $T^{(2k+1)}$ $(k \in \mathbb{Z})$ with a root attached to them, and let T be the free sum of $T^{(2k)}$ $(k \in \mathbb{Z})$ again with a root attached to them. It is clear that $S \leq T$ and $T \leq S$ and it is easily checked that S and T are not isomorphic.

6. Equivalence of Ultrafilters

In this section, we give a more detailed analysis of the functor $T \longmapsto \mathcal{U}(T)$ on the class $\mathcal{C}$ of Lipschitz trees. In particular, we show that using (*) that all ultrafilters of the form $\mathcal{U}(T)$ for $T \in \mathcal{C}$ are equivalent.

6.1 Definition. Two Lipschitz trees S and T are Δ-*equivalent* on some set of ordinals Γ, if there are level-sequences $s_\alpha \in S_\alpha$ $(\alpha \in \Gamma)$ and $t_\alpha \in T_\alpha$ $(\alpha \in \Gamma)$ such that for every unordered triple α, β, γ of elements of Γ,

$$\Delta(s_\alpha, s_\beta) > \Delta(s_\alpha, s_\gamma) \text{ iff } \Delta(t_\alpha, t_\beta) > \Delta(t_\alpha, t_\gamma). \tag{1}$$

6.2 Lemma (*). *Every pair of Lipschitz trees S and T is Δ-equivalent on some uncountable set of levels.*

Proof. Fix level-sequences $s_\alpha \in S_\alpha$ $(\alpha \in \omega_1)$ and $t_\alpha \in T_\alpha$ $(\alpha \in \omega_1)$. Let $\mathcal{P}$ be the set of all pairs $p = (\Gamma_p, C_p)$ of finite subsets of ω_1 such that S and T are equivalent on Γ_p as witnessed by the fixed level-sequences and such that for all $\alpha, \beta \in \Gamma_p$ and $\xi \in C_p$,

$$\Delta(s_\alpha, s_\beta) < \xi \text{ iff } \Delta(t_\alpha, t_\beta) < \xi. \tag{2}$$

We let $\mathcal{P}$ be ordered by coordinatewise inclusions. To see that $\mathcal{P}$ is a proper partial ordering, note that if M is a countable elementary submodel of some large-enough structure of the form $(\mathrm{H}(\theta), \in)$ then for every $p \in \mathcal{P} \cap M$, the extension $q = (\Gamma_p, C_p \cup \{M \cap \omega_1\})$ is an M-generic condition of $\mathcal{P}$. Note also that a condition of the form $(\{\xi\}, \{\xi\})$, where $\xi = M \cap \omega_1$ for such a model M forces that the unions of both projections of the generic filter are uncountable. Hence an application of (*) gives us the conclusion of Lemma 6.2. □

6.3 Definition. Suppose that g is a partial map from ω_1 into ω_1 and that T is a given tree represented as a family of functions as in §2. Then the *g-shift* of T, denoted by $T^{(g)}$ is the downwards closure of $\{t^{(g)} : t \in T \upharpoonright \Omega\}$, where $\Omega = \{\delta < \omega_1 : g''\delta \subseteq \delta\}$ and $t^{(g)}$ is defined by

$$t^{(g)}(\xi) = t(g(\xi))$$

if $\xi \in \mathrm{dom}(g)$; otherwise $t^{(g)}(\xi) = 0$.

In §2 we considered the shifts $T^{(g)}$ for maps of the form $g(\xi) = \xi - m$, but it should be clear that the arguments from that section are sufficient to give us the following facts:

6.4 Lemma. *If g is a partial strictly increasing map on ω_1 and if* $\mathrm{rang}(g) \in \mathcal{U}(T)$ *for some Lipschitz tree T, then the g-shift $T^{(g)}$ is also a Lipschitz tree.* □

6.5 Lemma. *Suppose that T is a Lipschitz tree and g is a strictly increasing partial map on ω_1 such that* $\mathrm{rang}(g) \in \mathcal{U}(T)$. *If g is regressive*[12] *then $T^{(g)} \not\leq T$. On the other hand, if g is expanding*[13] *then $T \not\leq T^{(g)}$.* □

6.6 Remark. Note that if g is a strictly increasing regressive partial map on ω_1 such that $\mathrm{rang}(g) = \omega_1$, then $T < T^{(g)}$ holds for every Lipschitz tree T. This observation can be used to construct both strictly increasing and strictly decreasing ω_1-sequences of Lipschitz trees.

6.7 Theorem (*). *For every pair S and T of Lipschitz trees, there is a strictly increasing partial map g on ω_1 such that $S \equiv T^{(g)}$.*

Proof. By Lemma 6.2, we have an uncountable $\Gamma \subseteq \omega_1$ on which S and T are Δ-equivalent for some choice of level-sequences $s_\alpha \in S_\alpha$ ($\alpha \in \Gamma$) and $t_\alpha \in T_\alpha$ ($\alpha \in \Gamma$). Define

$$g : \{\Delta(s_\alpha, s_\beta) : \alpha, \beta \in \Gamma, \alpha \neq \beta\} \longrightarrow \{\Delta(t_\alpha, t_\beta) : \alpha, \beta \in \Gamma, \alpha \neq \beta\}$$

by letting $g(\Delta(s_\alpha, s_\beta)) = \Delta(t_\alpha, t_\beta)$. By (1), this is a well-defined strictly increasing partial map on ω_1 such that $\mathrm{dom}(g) \in \mathcal{U}(S)$ and $\mathrm{rang}(g) \in \mathcal{U}(T)$. Let $\Omega = \{\delta < \omega_1 : g''\delta \subseteq \delta\}$. Assuming that $\{s_\alpha : \alpha \in \Gamma\}$ and $\{t_\alpha : \alpha \in \Gamma\}$ form antichains, then replacing each s_α and t_α by their extensions on the $\delta(\alpha)$th level of S and T respectively, where $\delta(\alpha) = \min(\Omega \setminus \alpha + 1)$, and finally replacing Γ with $\{\delta(\alpha) : \alpha \in \Gamma\}$, we may assume that Γ is actually a subset of Ω. By Lemma 6.4, the shift $T^{(g)}$ is a Lipschitz tree and $t_\alpha^{(g)}$ ($\alpha \in \Gamma$) is its level-sequence such that

$$\Delta(t_\alpha^{(g)}, t_\beta^{(g)}) = \Delta(s_\alpha, s_\beta) \text{ for all } \alpha, \beta \in \Gamma, \alpha \neq \beta. \tag{3}$$

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[12] I.e. $g(\xi) < \xi$ for all $\xi \in \mathrm{dom}(g)$.

[13] I.e. $g(\xi) > \xi$ for all $\xi \in \mathrm{dom}(g)$.

By Lemma 5.2 we conclude that $S \equiv T^{(g)}$. This finishes the proof. $\square$

6.8 Corollary (*). *For every pair S and T of Lipschitz trees, there is a strictly increasing partial map g on ω_1 which maps $\mathcal{U}(S)$ into $\mathcal{U}(T)$.*

Proof. By Theorems 5.6 and 6.7 we may assume that in fact $S = T^{(g)}$ for some strictly increasing partial map g on ω_1 with $\mathrm{rang}(g) \in \mathcal{U}(T)$. Now note that $\mathrm{dom}(g) \in \mathcal{U}(T^{(g)})$ and that $g''A \in \mathcal{U}(T)$ for every $A \in \mathcal{U}(T^{(g)})$. $\square$

7. Coinitiality and Cofinality

In this section we show that the chain $\mathcal{C}$ of Lipschitz trees is cofinal as well as coinitial in $(\mathcal{A}, \leq)$.

7.1 Lemma (*). *For every Aronszajn tree S, there is a Lipschitz tree T such that $S \leq T$.*

Proof. Let $\mathcal{P}$ be the set of all finite partial functions p from $S \times \omega_1$ into ω such that:

(1) $\xi < \mathrm{ht}(x)$ for all $(x, \xi) \in \mathrm{dom}(p)$,
(2) $p(x, \xi) = p(y, \xi)$ for all $(x, \xi), (y, \xi) \in \mathrm{dom}(p)$ with $\xi < \Delta(x, y)$.

We let p extend q if p extends q as a function and

(3) $p(x, \xi) = p(y, \xi)$ for all $(x, \xi), (y, \xi) \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$ such that $x, y \in \mathrm{dom}_0(q)$[14],
(4) $p(x, \xi) \neq q(x, \eta)$ for all $(x, \eta) \in \mathrm{dom}(q)$ and $(x, \xi) \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$.

A simple Δ-system argument (contained in the proofs of Lemmas 1.6 and 1.7 above) shows that $\mathcal{P}$ satisfies the countable chain condition, so an application of (*) gives us a map g from $S \times \omega_1$ into ω so that its fibers $g_x(\xi) = g(x, \xi)$ are total maps from $\mathrm{ht}(x)$ into ω for all $x \in S$ and such that:

(5) $g_x : \mathrm{ht}(x) \longrightarrow \omega$ is a finite-to-one map for all $x \in S$,
(6) $\Delta(x, y) \leq \Delta(g_x, g_y)$ for all $x, y \in S$,
(7) $\{\xi : g_x(\xi) \neq g_y(\xi)\}$ is finite for all $x, y \in S$.

It follows that the downwards closure T of $\{g_x : x \in S\}$ is a coherent Aronszajn tree and that $x \longmapsto g_x$ is a Lipschitz map from S into T. This finishes the proof. $\square$

7.2 Lemma (*). *For every Aronszajn tree S, there is a Lipschitz tree T such that $T \leq S$.*

[14] Recall the notations $\mathrm{dom}_0(p) = \{x : (x, \eta) \in \mathrm{dom}(p) \text{ for some } \eta\}$ and $\mathrm{dom}_1(p) = \{\eta : (x, \eta) \in \mathrm{dom}(p) \text{ for some } x\}$.

Proof. Let $\mathcal{P}$ be the set of all finite partial functions p from $S \times \omega_1$ into ω such that:

(8) $\xi < \mathrm{ht}(x)$ for all $(x, \xi) \in \mathrm{dom}(p)$,

(9) for every pair x and y of incomparable nodes from $\mathrm{dom}_0(p)$, there is $\xi \leq \Delta(x, y)$ with $(x, \xi), (y, \xi) \in \mathrm{dom}(p)$ and $p(x, \xi) \neq p(y, \xi)$.

We let p extend q if p extends q as a function and

(10) $p(x, \xi) = p(y, \xi)$ for all $(x, \xi), (y, \xi) \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$ such that $x, y \in \mathrm{dom}_0(q)$,

(11) $p(x, \xi) \neq q(x, \eta)$ for all $(x, \eta) \in \mathrm{dom}(q)$ and $(x, \xi) \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$.

To prove that $\mathcal{P}$ satisfies the countable chain condition, we start with an uncountable subset $\mathcal{X}$ of $\mathcal{P}$ and perform the Δ-system argument from the proof of Lemma 1.6, obtaining two conditions p and q in $\mathcal{X}$ such that for some $\bar{\xi} < \alpha < \beta$, $v_0, \ldots, v_n \in S_{\bar{\xi}}$, $s_0, \ldots, s_n \in S_\alpha$ and $t_0, \ldots t_n \in S_\beta$ we have

(12) every node of $\mathrm{dom}_0(p)$ is either of height $< \bar{\xi}$ or it extends some s_i $(i \leq n)$,

(13) every node of $\mathrm{dom}_0(q)$ is either of height $< \bar{\xi}$ or it extends some t_i $(i \leq n)$,

(14) $\mathrm{dom}_1(p) \subseteq \bar{\xi} \cup (\alpha, \beta)$ and $\mathrm{dom}_1(q) \subseteq \bar{\xi} \cup (\beta, \omega_1)$,

(15) $v_i \neq v_j$ for $i \neq j \leq n$,

(16) s_i and t_i extend v_i but are incomparable for all $i \leq n$,

(17) p and q are isomorphic conditions via an isomorphism that is the identity on $\bar{\xi}$, v_i $(i \leq n)$ and maps s_i to t_i for all $i \leq n$.

We claim that such p and q can be amalgamated into a condition r of $\mathcal{P}$ that extends them both. Let $\xi = \min\{\Delta(s_i, t_i) : i \leq n\}$. Then $\bar{\xi} \leq \xi < \alpha$. Let $k = \max(\mathrm{rang}(p)) = \max(\mathrm{rang}(q))$. Let $\mathrm{dom}(r)$ be equal to the union of $\mathrm{dom}(p)$, $\mathrm{dom}(q)$ and the following two sets:

$$D = \{(x, \xi) : x \in \mathrm{dom}_0(p), \mathrm{ht}(x) \geq \alpha\},$$

$$E = \{(y, \xi) : y \in \mathrm{dom}_0(q), \mathrm{ht}(y) \geq \beta\}.$$

Define r by giving it constant value $k+1$ on D and constant value $k+2$ on E. Note that r satisfies (8) and (9) as well as conditions (10) and (11) for extending both p and q.

Applying (*) to $\mathcal{P}$ gives us a partial map $g : S \times \omega_1 \longrightarrow \omega$ so that if $g_x(\xi) = g(x, \xi)$ then:

(18) g_x is a finite-to-one map from $\mathrm{ht}(x)$ into ω for all $x \in S$,

(19) $\Delta(g_x, g_y) \leq \Delta(x, y)$ for all $x, y \in S$,

(20) $\{\xi : g_x(\xi) \neq g_y(\xi)\}$ is finite for all $x, y \in S$.

It follows that the downwards closure T of $\{g_x : x \in S\}$ is a Lipschitz tree and that $g_x \longmapsto x$ is a partial Lipschitz map from T into S, witnessing the relation $T \leq S$. This finishes the proof. □

7.3 Theorem (*). *There is no maximal Aronszajn tree.*

Proof. Given an Aronszajn tree S by Lemma 7.1 we find a Lipschitz tree T such that $S \leq T$. By Lemma 2.3, $T < T^{(1)}$, so in particular $S < T^{(1)}$. □

7.4 Theorem (*). *For every Lipschitz tree T there is a Lipschitz tree S such that $S < T$.*

Proof. Fix a level-sequence $t_\alpha \in T_\alpha$ $(\alpha \in \omega_1)$ in a given Lipschitz tree T. Let $\Delta_t : [\omega_1]^2 \longrightarrow \omega_1$ be the corresponding distance function $\Delta_t(\alpha, \beta) = \Delta(t_\alpha, t_\beta)$. For $\Gamma \subseteq \omega_1$, let

$$\Delta_t(\Gamma) = \{\Delta_t(\alpha, \beta) : \alpha, \beta \in \Gamma, \alpha \neq \beta\}.$$

Let $\mathcal{P}$ be the set of all pairs $p = (f_p, \Gamma_p)$ such that:

(21) Γ_p is a finite subset of ω_1,
(22) f_p is a finite partial strictly increasing map from ω_1 into ω_1 which can be extended to a total strictly increasing and continuous map $f : \omega_1 \longrightarrow \omega_1$ so that $\mathrm{rang}(f)$ is disjoint from $\Delta_t(\Gamma_p)$ and separates[15] the points of $\Delta_t(\Gamma_p)$.

We order $\mathcal{P}$ by coordinatewise inclusion. To show that $\mathcal{P}$ is proper, consider a countable elementary submodel M of some large-enough structure of the form $(\mathrm{H}(\theta), \in)$ such that M contains $\mathcal{P}, T$ and the level-sequence t_α $(\alpha \in \omega_1)$. For a given $p \in \mathcal{P} \cap M$ let $q = (f_p \cup \{\langle \delta, \delta \rangle\}, \Gamma_p)$, where $\delta = M \cap \omega_1$. We claim that q is an M-generic condition of $\mathcal{P}$. To show this, consider a dense-open subset $\mathcal{D}$ of $\mathcal{P}$ such that $\mathcal{D} \in M$ and an extension r of q. We need to show that r is compatible with a member of $\mathcal{D} \cap M$. Extending r, we may assume $r \in \mathcal{D}$. Let v_i $(i \leq n)$ be a one-to-one enumeration of $\{t_\alpha \upharpoonright \delta : \alpha \in \Gamma_p \setminus \delta\}$. Let $\bar{p} = r \upharpoonright M$. Then $\bar{p} \in \mathcal{P} \cap M$ and so we can find an extension $\bar{f} : \omega_1 \longrightarrow \omega_1$ of $f_{\bar{p}}$ satisfying (22) for $\bar{p}$ such that $\bar{f} \in M$. Let $\bar{\xi} \in (\max(\Gamma_{\bar{p}}), \delta)$ be a fixed point of $\bar{f}$. Find a copy $\bar{r}$ of r in $\mathcal{D} \cap M$ such that if we let $\bar{\delta}$ and $\bar{v}_i$ $(i \leq n)$ be its versions of δ and v_i $(i \leq n)$, then

(23) $v_i \upharpoonright \bar{\xi} = \bar{v}_i \upharpoonright \bar{\xi}$ for all $i \leq n$,
(24) v_i and $\bar{v}_i$ are incomparable for all $i \leq n$,
(25) $\Delta(v_0, \bar{v}_0) = \ldots = \Delta(v_n, \bar{v}_n)$.

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[15] i.e. between every two members of $\Delta_t(\Gamma_p)$, there is a member of $\mathrm{rang}(f)$.

It is clear that we can combine the function $\bar{f}$ with the normal functions witnessing (22) for $\bar{r}$, r and obtain a strictly increasing continuous function $f : \omega_1 \longrightarrow \omega_1$ which fixes $\bar{\xi}$ and witnesses (22) simultaneously for $\bar{r}$, r and moreover, the ordinal $\Delta(v_0, \bar{v}_0)$ is not in its range. Since

$$\Delta_t(\Gamma_{\bar{r}} \cup \Gamma_r) = \Delta_t(\Gamma_{\bar{r}}) \cup \Delta_t(\Gamma_r) \cup \{\Delta(v_0, \bar{v}_0)\},$$

this shows that $(f_{\bar{r}} \cup f_r, \Gamma_{\bar{r}} \cup \Gamma_r)$ is a member of $\mathcal{P}$ witnessing the compatibility of $\bar{r}$ and r.

Applying (*) to $\mathcal{P}$ gives us an uncountable $\Gamma \subseteq \omega_1$ and a closed unbounded set $C \subseteq \omega_1$ such that $C \cap \Delta_t(\Gamma) = \emptyset$ and C separates the points of $\Delta_t(\Gamma)$. For $\delta < \omega_1$, let δ^+ be the minimal point of C above δ. Define

$$C_0 = \{\delta \in C : (\delta, \delta^+) \cap \Delta_t(\Gamma) \neq \emptyset\}.$$

Note that for $\delta \in C$, there is only one point of $\Delta_t(\Gamma)$ in the interval (δ, δ^+). Call this point $g(\delta)$. This defines a strictly increasing map g from C_0 onto $\Delta_t(\Gamma)$. So in particular, $\mathrm{rang}(g) \in \mathcal{U}(T)$. Let $S = T^{(g)}$ (see Definition 6.3). From Lemma 6.4 we conclude that S is a Lipschitz tree. Since clearly $g(\delta) > \delta$ for all $\delta \in C_0$, we conclude that $T \not\leq S$ from Lemma 6.5. This finishes the proof. $\square$

7.5 Corollary (*). *There is no minimal Aronszajn tree.*

Proof. This follows from Lemma 7.2 and Theorem 7.4. $\square$

7.6 Corollary (*). *The class $\mathcal{C}$ of Lipschitz trees is a chain which is coinitial and cofinal in $(\mathcal{A}, \leq)$ and moreover, it has neither a maximum nor minimum.* $\square$

In §3, we have seen many pairs of Aronszajn trees S and T such that $S \not\leq T$ and $T \not\leq S$. The trees S and T have been constructed by using members of some fixed sequence of coherent trees as building blocks. Can one improve the construction by making S and T 'coherently incompatible'? The following result answers this question.

7.7 Theorem (*). *For every pair S and T of Aronszajn trees there is a coherent tree U such that $U \leq S$ and $U \leq T$.*

Proof. By Lemma 7.2, find two coherent trees U_0 and U_1 such that $U_0 \leq S$ and $U_1 \leq T$. By Theorem 5.4 we have that either $U_0 \leq U_1$ or else $U_1 \leq U_0$. So $U = \min\{U_0, U_1\}$ will satisfy the conclusion of the Theorem. $\square$

This finishes our analysis of the structure theory of the chain $\mathcal{C}$ of Lipschitz trees inside the class $\mathcal{A}$ of Aronszajn trees under the assumption of (*). In the next section we shall see that under a natural strengthening of (*) some more structure theory can be developed.

8. The Lipschitz-map conjecture

We have seen above that under (*), for every one-to-one level-preserving map f from an uncountable subset of one Lipschitz tree into another Lipschitz tree, f or its inverse f^{-1} is Lipschitz on an uncountable subset of its domain. In the previous section, we have seen that in this context, Lipschitz trees appear as subtrees of any other Aronszajn tree. It is therefore quite natural to pose the following conjecture about Lipschitz maps on arbitrary Aronszajn trees:

Lipschitz-map conjecture:[16] If f is a one-to-one level-preserving map from an uncountable subset of an Aronszajn tree into another, then f or f^{-1} has an uncountable Lipschitz restriction.

Note that LMC immediately gives that every two irreducible Aronszajn trees are comparable under $\leq$, so one can view LMC as some sort of linearity conjecture for the class $\mathcal{A}$. The following result shows that combining LMC with (*), one has considerably more comparabilites inside $(\mathcal{A}, \leq)$.

8.1 Proposition (*). *The Lipschitz-map conjecture implies that every irreducible Aronszajn tree is comparable with any other Aronszajn tree.*

Proof. Let T be a given irreducible Aronszajn tree and let S be any other Aronszajn tree. Let $\mathcal{P}$ be the poset of all finite partial level-preserving Lipschitz maps from S into T. If $\mathcal{P}$ satisfies the countable chain condition, then an application of (*) would give us a total Lipschitz map $f : S \longrightarrow T$ witnessing thus the relation $S \leq T$. So we are left with the alternative that $\mathcal{P}$ fails to satisfy the countable chain condition. Let $\mathcal{X}$ be an uncountable family of pairwise incomparable members of $\mathcal{P}$. We may assume that $\mathcal{X}$ forms a Δ-system. Removing the root, which obviously does not contribute to incomparability, we may in fact assume that $\mathrm{dom}(p)$ $(p \in \mathcal{X})$ is a family of pairwise disjoint finite subsets of S, all of some fixed size n. For $p \in \mathcal{X}$, let $s_i(p)$ $(i < n)$ enumerate $\mathrm{dom}(p)$ and let $t_i(p) = p(s_i(p))(i < n)$. Applying the Δ-Lemma again, we obtain an uncountable subfamily $\mathcal{Y}$ of $\mathcal{X}$ such that

$$\{\Delta(s_i(p), s_j(p) : i < j < n)\}\ p \in \mathcal{Y}),$$

and

$$\{\Delta(t_i(p), t_j(p) : i < j < n)\}\ p \in \mathcal{Y})$$

both form Δ-systems. Shrinking $\mathcal{Y}$ further, we may assume that if α is the least upper bound of the roots of these two Δ-systems, then for

[16] LMC in short.

all $p \neq q$ in $\mathcal{Y}$:

$$s_i(p) \restriction \alpha = s_i(q) \restriction \alpha \text{ and } t_i(p) \restriction \alpha = t_i(q) \restriction \alpha \text{ for all } i < n.$$

Since any $p \neq q$ from $\mathcal{Y}$ are incompatible in $\mathcal{P}$, it follows that there exists $i = i(p,q) < n$ such that:

$$\Delta(s_i(p), s_i(q)) > \Delta(t_i(p), t_i(q)).$$

Apply LMC n times successively and obtain an uncountable subset $\mathcal{Z}$ of $\mathcal{Y}$ such that for all $i < n$, the map

$$s_i(p) \longmapsto t_i(p) \ (p \in \mathcal{Z})$$

or its inverse is Lipschitz. If for every $i < n$, the map $s_i(p) \longmapsto t_i(p)$ $(p \in \mathcal{Z})$ is Lipschitz, we would obtain two compatible members of $\mathcal{X}$ contradicting our initial assumption about $\mathcal{X}$. So, there must be $i < n$ so that the inverse map

$$g : t_i(p) \longmapsto s_i(p) \ (p \in \mathcal{Z})$$

is Lipschitz. Let T_0 be the downward closure of $\{t_i(p) : p \in \mathcal{Z})$ in T. Then T_0 is uncountable and g extends to a Lipschitz map $\hat{g} : T_0 \longrightarrow S$. It follows that $T_0 \leq S$. Since T is irreducible, we have that $T \leq T_0$, and therefore $T \leq S$. This finishes the proof. □

It follows that under LMC, the chain $\mathcal{A}_{\text{irr}}$ of irreducible Aronszajn trees, and therefore the chain $\mathcal{C}$ of Lipschitz trees, naturally splits the whole class $\mathcal{A}$ into a family of pairwise comparable convex blocks which can then be studied independently of each other. It would be interesting to find out more about the cuts of $(\mathcal{C}, \leq)$, where members of $\mathcal{A} \setminus \mathcal{C}$ can appear. The following result, which strengthens Lemma 2.6, gives some information on this.

8.2 Proposition (*). *The Lipschitz-map conjecture implies that no Aronszajn tree is strictly between a Lipschitz tree T and its shift $T^{(1)}$.*

Proof. Suppose there is an Aronszajn tree S such that $T < S < T^{(1)}$. During the course of the proof of Proposition 8.1 we have seen that since the poset of all finite Lipschitz maps from S into T fails to satisfy the countable chain condition, an application of LMC to an uncountable family of pairwise incomparable members of this poset will provide us with an uncountable set $\Gamma \subseteq \Lambda$ and level-sequences $s_\gamma \in S_\gamma$ $(\gamma \in \Gamma)$, $t_\gamma \in T_\gamma$ $(\gamma \in \Gamma)$ such that

$$\Delta(s_\gamma, s_\delta) > \Delta(t_\gamma, t_\delta) \text{ for all } \gamma \neq \delta \text{ in } \Gamma.$$

Let S_0 be the downward closure of $\{s_\gamma : \gamma \in \Gamma\}$ in S. Consider now the poset of all finite partial Lipschitz maps from $T^{(1)}$ into S_0. By (*), this poset cannot satisfy the countable chain condition either. So applying

LMC to an uncountable family of pairwise incompatible members of this poset would give us uncountable level-sequences $r_\gamma \in S_\gamma$ ($\gamma \in \Sigma$), $q^{(1)}_\gamma \in T^{(1)}_\gamma$ ($\gamma \in \Sigma$) such that

$$\Delta(q^{(1)}_\gamma, q^{(1)}_\delta) > \Delta(r_\gamma, r_\delta) \text{ for all } \gamma \neq \delta \text{ in } \Sigma.$$

For each $\gamma \in \Sigma$, choose $\xi(\gamma) \in \Gamma$ such that

$$\xi(\gamma) \geq \gamma \text{ and } r_\gamma = s_{\xi(\gamma)} \restriction \gamma.$$

We may assume that $\{q_\gamma : \gamma \in \Sigma\}$, $\{q^{(1)}_\gamma : \gamma \in \Sigma\}$ and $\{r_\gamma : \gamma \in \Sigma\}$ are all antichains in their respective trees. For each $\gamma \in \Sigma$, find $p_{\xi(\gamma)} \in T_{\xi(\gamma)}$ such that $p_{\xi(\gamma)} \restriction \gamma = q_\gamma$ and therefore

$$p^{(1)}_{\xi(\gamma)} \restriction \gamma = q^{(1)}_\gamma.$$

Applying Lemma 1.3, we get an uncountable $\Sigma_0 \subseteq \Sigma$ such that

$$\Delta(t_{\xi(\gamma)}, t_{\xi(\delta)}) = \Delta(p_{\xi(\gamma)}, p_{\xi(\delta)}) \text{ for all } \gamma \neq \delta \text{ in } \Sigma_0.$$

Combining all this, we get the following for all $\gamma \neq \delta$ in Σ_0:

$$\Delta(p^{(1)}_{\xi(\gamma)}, p^{(1)}_{\xi(\delta)}) = \Delta(q^{(1)}_\gamma, q^{(1)}_\delta) > \Delta(r_\gamma, r_\delta) = \Delta(s_{\xi(\gamma)}, s_{\xi(\delta)}) >$$
$$> \Delta(t_{\xi(\gamma)}, t_{\xi(\delta)}) = \Delta(p_{\xi(\gamma)}, p_{\xi(\delta)}).$$

It follows that for $\gamma \neq \delta$ in Σ_0:

$$\Delta(p^{(1)}_{\xi(\gamma)}, q^{(1)}_{\xi(\delta)}) \geq \Delta(p_{\xi(\gamma)}, p_{\xi(\delta)}) + 2,$$

a contradiction. This finishes the proof of Proposition 8.2. □

We finish this section by showing that the Lipschitz-map conjecture is equivalent to a well-known conjecture about a basis for a class of uncountable linear orderings.

Shelah's conjecture:[17] Every uncountable linear ordering contains either an uncountable well-ordered or conversely well-ordered subset, an uncountable separable ordering, or an uncountable ordering whose cartesian square is the union of countably many chains.

It will be more convenient to connect LMC with the following statement appearing in a paper of Abraham and Shelah [1, p.79] that came after that of Shelah [10] where SC first appeared.

Coloring Axiom for Aronszajn trees:[18] For any partition $T = K_0 \cup K_1$ of an Aronszajn tree T, there is an uncountable set $X \subseteq T$ and $i < 2$ such that $x \wedge y \in K_i$ for all $x, y \in X$, $x \neq y$.[19]

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[17]SC in short. That SC is a consistent postulate has been conjectured by S. Shelah [10, Conjecture 1].

[18]CAT in short.

[19]Recall that $x \wedge y$ denotes the maximal $z \in T$ such that $z \leq_T x$ and $z \leq_T y$.

8.3 Remark. It clearly makes sense considering this coloring statement for an arbitrary uncountable tree T. However, a simple analysis involving some standard results from this area shows that the full weight of CAT is reached only in the class of Aronszajn trees. In fact, it turns out that, modulo (*), the postulate CAT for the whole class of Aronszajn trees is in fact equivalent to its version for a single Aronszajn tree T with no difference in the choice of T. This fact will essentially be established below during the course of proving the claimed equivalence of these statements, and it is this fact that explains our choice for the notation for this postulate.

8.4 Proposition (*). *The Lipschitz-map conjecture is equivalent to the Coloring Axiom for Aronszajn trees.*

Proof. In order to show that LMC implies CAT, let $T = K_0 \cup K_1$ be a given partition of some Aronszajn tree T. For $i < 2$, let T^i be the tree obtained from T by inserting a point between every node $u \in K_i$ and its immediate successors in T. Considering T as a subset of T^0 and T^1, we apply LMC successively to the identity function *id* restricted to $T \upharpoonright \Lambda$ viewed as a partial map from T^0 into T^1 and vice versa. This will give us uncountable $X \subseteq T \upharpoonright \Lambda$ and $i < 2$ such that $id \upharpoonright X$, as a partial map from T^i to T^{1-i}, is Lipschitz. Then $x \wedge y \notin K_{1-i}$ for all $x \neq y$ in X.

To prove that CAT implies LMC, we consider a one-to-one level-preserving map from an uncountable subset X of an Aronszajn tree S into an Aronszajn tree T. Clearly, we may assume that X is a level-antichain of S and that $f''X$ is a level-antichain of T.[20] Applying (*) and shrinking X, we may assume that the downward closure U of the graph of f in the product tree $S \otimes T$ is binary. We are using here the easily checked fact that the poset of all finite subsets of an Aronszajn tree with (at most) binary downward closures satisfies the countable chain condition. Let K_0 be the collection of all splitting nodes $u \in U$ with the property that $x_0 \neq y_0$ where (x_0, x_1) and (y_0, y_1) are the two immediate successors of u in U. An application of CAT gives us an uncountable subset Z of Y such that all splittings between elements of the set $(x, f(x))$ $(x \in Z)$ inside the tree U either all belong to K_0 or all fall outside of K_0. In the first case, the inverse of $f \upharpoonright Z$ is Lipschitz and in the second case, $f \upharpoonright Z$ itself is Lipschitz. $\square$

Before going further let us recall the coherent Aronszajn tree $T(\rho_3)$ constructed above for proving Lemma 2.8, and let $C(\rho_3)$ be the linearly

[20] A *level-antichain* is an antichain of a given tree which takes at most one point from any level of the tree.

ordered set obtained by lexicographically ordering $T(\rho_3)$. We shall need the following two facts about $C(\rho_3)$ (see [15, Section 1] for proofs):

8.5 Lemma. *The square of $C(\rho_3)$ with the product partial ordering can be decomposed into countably many chains.* □

8.6 Lemma (*). *For every uncountable $X \subseteq C(\rho_3)$, we have that $C(\rho_3) \leq X$.*[21] □

The claim that SC and CAT are 'consistency-wise equivalent' appeared without proof in [1, p.79]. It was perhaps the following fact that the authors of [1] had in mind when they were writing those lines.

8.7 Proposition (*). (see [1, p.79] and [14]) *Shelah's conjecture is equivalent to the Coloring Axiom for Aronszajn trees.*

Proof. To prove that CAT implies SC, let L be a given uncountable linear ordering with neither an uncountable well-ordered nor conversely well-ordered subset, nor an uncountable separable subset. It is well-known and easily shown (see [11, §5]), that there is an Aronszajn tree T and a lexicographical ordering $<_{lex}$ of T such that L is isomorphic to a subset of $(T, <_{lex})$. Going to a subset of T, we may assume that T is binary. (This can be achieved either by using (*) or by constructing a binary tree T that would represent an uncountable subset of L.) By the result of Abraham and Shelah[1] which can be seen to use only (*), we can choose a closed and unbounded set $C \subseteq \omega_1$ and an isomorphism $f : T \restriction C \longrightarrow T(\rho_3) \restriction C$ (see also [11]). Going to a subtree of T, we may assume that C separates the heights of every pair of comparable splitting nodes of T. This can be achieved by applying (*) to the poset of all finite subsets of T whose downward closures are separated by C in this way, a poset that is easily seen to satisfy the countable chain condition. Thus, to every splitting node $u = x \wedge y$ of T, there corresponds a unique splitting node $v = f(x) \wedge f(y)$ of $T(\rho_3)$ and the association does not depend on which x and y we pick in T to represent u. Let K_0 be the collection of all splitting nodes $u = x \wedge y$ of T such that $x <_{lex} y$ iff $f(x) <_{lex} f(y)$. Note again that the definition does not depend on the x and y in T we choose to represent u. By OCA_T, there is an uncountable antichain X of T such that $\{x \wedge y : x, y \in X, x \neq y\}$ is either included in K_0 or is disjoint from it. In the first case, $f \restriction X$ is strictly increasing and in the second case, it is strictly decreasing, relative to the lexicographical orderings on T and $T(\rho_3)$.

To prove that SC implies CAT, let $T = K_0 \cup K_1$ be a given partition of some Aronszajn tree T. For each splitting node u of T, we

[21] For a pair K and L of linearly ordered sets, $K \leq L$ denotes the fact that there is a strictly increasing map from K into L.

choose a linear ordering $<_u$ of its immediate successors. Let $<_0$ be the lexicographical ordering generated by $<_u$ $(u \in T)$ and let $<_1$ be the lexicographical ordering generated by $<_u$ $(u \in K_0)$ and $>_u$ $(u \notin K_0)$. By SC, there is an uncountable set $X \subseteq T$ such that the cartesian square of $(X, <_0)$ can be decomposed into countably many chains. Applying SC again, we get an uncountable $Y \subseteq X$ so that the square of $(Y, <_1)$ can also be decomposed into countably many chains. Let $\mathcal{P}$ be the poset of all finite subsets of Y on which $<_0$ and $<_1$ agree. If $\mathcal{P}$ satisfies the countable chain condition, then an application of (*) would give us an uncountable $Z \subseteq Y$ on which $<_0$ and $<_1$ agree, which translates into the fact that $x \wedge y \in K_0$ for all $x, y \in Z$, $x \neq y$. If $\mathcal{P}$ fails to satisfy the countable chain condition, then using the properties of $<_0$ and $<_1$ on Y, one can easily produce an uncountable $Z \subseteq Y$ on which $<_0$ and $<_1$ are reverse of each other. This in turn would translate into the fact that $x \wedge y \notin K_0$ for all $x, y \in Z$, $x \neq y$. □

8.8 Corollary (*). *The Lipschitz-map conjecture is equivalent to Shelah's conjecture asserting that every uncountable linear ordering contains either an uncountable well-ordered subset, an uncountable conversely well-ordered subset, an uncountable separable ordering, or an uncountable ordering whose cartesian square is the union of countably many chains.*

Using a deep result of Baumgartner[2] one can say a bit more:

8.9 Proposition (*). *The Lipschitz-map conjecture implies that for any set of reals B of size $\aleph_1$, the family $\omega_1, \omega_1^*, B, C(\rho_3), C(\rho_3)^*$ forms a basis for the class of all uncountable linear orderings.*[22]

Proof. The fact that there is a strictly increasing map from B into any other uncountable separable linear ordering is the result of Baumgartner (see [2]). So let us consider an uncountable linear ordering L which contains neither an uncountable well-ordered nor conversely well-ordered subset, nor an uncountable separable subordering. By the proof of the implication from CAT to SC, we know that there is an uncountable set $X \subseteq C(\rho_3)$ such that $X \leq L$ or $X^* \leq L$. By Lemma 8.6, we have that $C(\rho_3) \leq X$. This finishes the proof. □

Added in proof. Building on this work J. Moore[7] has recently verified Shelah's conjecture by deducing SC from a standard set-theoretic postulate that is stronger than (*). It follows that the additional structure theory of the classes $\mathcal{C}$, $\mathcal{A}_{irr}$, and $\mathcal{A}$ described above in Propositions

[22]I.e., every uncountable linear ordering contains an isomorphic copy of one of these.

8.1 and 8.2 using LMC is a consistent extension of the structure theory developed above in the first seven sections of this paper.

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