Contents

I. Coherent Sequences

Stevo Todorcevic

A transfinite sequence $C_\xi \subseteq \xi$ $(\xi < \theta)$ of sets may have a number of 'coherence properties' and the purpose of this chapter is to study some of them, as well as some of their uses. Here, 'coherence' usually means that the C_ξ's are chosen in some canonical way, beyond the natural requirement that C_ξ be closed and unbounded in ξ for all ξ. For example, choosing a canonical 'fundamental sequence' of sets $C_\xi \subseteq \xi$ for $\xi < \varepsilon_0$ relying on the specific properties of the Cantor normal form for ordinals below the first ordinal satisfying the equation $x = x^x$ is a basis for a number of important results in proof theory. In set theory, one is interested in longer sequences as well and usually has a different perspective in applications, so one is naturally led to use some other tools beside the Cantor normal form. It turns out that the sets C_ξ can not only be used as 'ladders' for climbing up in recursive constructions but also as tools for 'walking' from an ordinal to a smaller one. This notion of a 'walk' and the corresponding 'distance functions' constitute the main body of study in this chapter. We show that the resulting 'metric theory of ordinals' not only provides a unified approach to a number of classical problems in set theory but also has its own intrinsic interest. For example, from this theory one learns that the triangle inequality of an ultrametric

$$e(\alpha, \gamma) \leq \max\{e(\alpha, \beta), e(\beta, \gamma)\}$$

has three versions, depending on the natural ordering between the ordinals α, β and γ, that are of a quite different character and are occurring in quite different places and constructions in set theory. For example, the most frequent occurrence is the case $\alpha < \beta < \gamma$ when the triangle inequality becomes something that one can call 'transitivity' of e. Considerably more subtle is the case $\alpha < \gamma < \beta$ of this inequality. It is this case of the inequality that captures most of the coherence properties found in this chapter. Another thing one learns from this theory is the special role of the first uncountable ordinal in this theory. Any natural coherence requirement on the sets C_ξ $(\xi < \theta)$ that one finds in this theory is satisfiable in the case $\theta = \omega_1$. The first uncountable cardinal is the only cardinal on which the theory can be carried out without relying on additional axioms of set theory. The first uncountable cardinal is the place where the theory has its deepest applications as well as its most important open problems. This special role can perhaps be explained by the fact that many set-theoretical problems, especially those coming from other fields of mathematics, are usually concerned only about the duality between the countable and the uncountable rather than some intricate relationship between two or more uncountable cardinalities. This is of course not to say that an intricate relationship between two or more uncountable cardinalities may not be a profitable detour in the course of solving such a problem. In fact, this is one of the reasons for our attempt to develop the metric theory of ordinals without restricting ourselves only to the realm of countable ordinals.

The chapter is organized as a discussion of five basic distance functions on ordinals, $\rho, \rho_0, \rho_1, \rho_2$ and ρ_3, and the reader may choose to follow the analysis of any of these functions in various contexts. The distance functions will naturally lead us to many other derived objects, most prominent of which is the 'square-bracket operation' that gives us a way to transfer the quantifier 'for every unbounded set' to the quantifier 'for every closed and unbounded set'. This reduction of quantifiers has proven to be quite useful in constructions of various mathematical structures, some of which have been mentioned or reproduced here.

I wish to thank Bernhard König, Piotr Koszmider, Justin Moore and Christine Härtl for help in the preparation of the manuscript.

1. The Space of Countable Ordinals

This is by far the most interesting space considered in this chapter. There are many mathematical problems whose combinatorial essence can be reformulated as a problem about ω_1, the smallest uncountable structure. What we mean by 'structure' is ω_1 together with a system C_α $(\alpha < \omega_1)$ of fundamental sequences, i.e. a system with the following two properties:

(a) $C_{\alpha+1} = \{\alpha\}$,

(b) C_α is an unbounded subset of α of order-type ω, whenever α is a countable limit ordinal > 0.

Despite its simplicity, this structure can be used to derive virtually all other known structures that have been defined so far on ω_1. There is a natural recursive way of picking up the fundamental sequences C_α, a recursion that refers to the Cantor normal form which works well for, say, ordinals $< \varepsilon_0$.[1] For longer fundamental sequences one typically relies on some other

[1] One is tempted to believe that the recursion can be stretched all the way up to ω_1 and this is probably the way P.S. Alexandroff found his famous Pressing Down Lemma (see [1] and [2, appendix]).

principles of recursive definition and one typically works with fundamental sequences with as few extra properties as possible. We shall see that the following assumption is what is frequently needed and will therefore be implicitly assumed whenever necessary:

(c) If α is a limit ordinal, then C_α does not contain limit ordinals.

1.1 Definition. A *step* from a countable ordinal β towards a smaller ordinal α is the minimal point of C_β that is $\geq \alpha$. The cardinality of the set $C_\beta \cap \alpha$, or better to say the order-type of this set, is the *weight* of the step.

1.2 Definition. A *walk* (or a *minimal walk*) from a countable ordinal β to a smaller ordinal α is the sequence $\beta = \beta_0 > \beta_1 > \ldots > \beta_n = \alpha$ such that for each $i < n$, the ordinal β_{i+1} is the step from β_i towards α.

Analysis of this notion leads to several two-place functions on ω_1 that give a rich structure with many applications. So let us describe some of these functions.

1.3 Definition. The *full code* of the walk is the function $\rho_0 : [\omega_1]^2 \longrightarrow \omega^{<\omega}$ defined recursively by

$$\rho_0(\alpha, \beta) = \langle |C_\beta \cap \alpha| \rangle^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha)),$$

where $\rho_0(\alpha, \alpha) = 0$,[2] and the symbol $^\frown$ refers to the sequence obtained by concatenating the one term sequence $\langle |C_\beta \cap \alpha| \rangle$ with the already known finite sequence $\rho_0(\alpha, \min(C_\beta \setminus \alpha)$ of integers. Clearly, knowing $\rho_0(\alpha, \beta)$ and the ordinal β one can reconstruct the *upper trace*

$$\mathrm{Tr}(\alpha, \beta) = \{\beta_0, \ldots, \beta_n\},$$

remembering that $\beta = \beta_0 > \beta_1 > \ldots > \beta_n = \alpha$, of the walk from β to α. The *lower trace* is defined to be

$$\mathrm{L}(\alpha, \beta) = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\},$$

where $\lambda_i = \max(\bigcup_{j=0}^{i} C_{\beta_j} \cap \alpha)$ for $i < n$ and so $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$.

1.4 Definition. The *full lower trace* of the minimal walk is the function $\mathrm{F} : [\omega_1]^2 \longrightarrow [\omega_1]^{<\omega}$ defined recursively by

$$\mathrm{F}(\alpha, \beta) = \mathrm{F}(\alpha, \min(C_\beta \setminus \alpha)) \cup \bigcup_{\xi \in C_\beta \cap \alpha} \mathrm{F}(\xi, \alpha),$$

where $\mathrm{F}(\gamma, \gamma) = \{\gamma\}$ for all γ.

[2] Technically speaking, ρ_0 operates on $[\omega_1]^2$ so $\rho_0(\alpha, \alpha) = 0$ makes no sense. What we mean is that whenever the formal recursive definition of $\rho_0(\alpha, \beta)$ involves the term $\rho_0(\alpha, \alpha)$ we take it to be equal to 0. This will be applied frequently in this chapter, not always in explicit form.

Clearly, $\mathrm{F}(\alpha,\beta) \supseteq \mathrm{L}(\alpha,\beta)$ but $\mathrm{F}(\alpha,\beta)$ is considerably larger than $\mathrm{L}(\alpha,\beta)$ as it includes also the traces of walks between any two ordinals $\leq \alpha$ that have ever been referred to during the walk from β to α. The following two properties of the full lower trace are straightforward to check (see [76]).

1.5 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $\mathrm{F}(\alpha,\gamma) \subseteq \mathrm{F}(\alpha,\beta) \cup \mathrm{F}(\beta,\gamma)$,

(b) $\mathrm{F}(\alpha,\beta) \subseteq \mathrm{F}(\alpha,\gamma) \cup \mathrm{F}(\beta,\gamma)$. ⊣

1.6 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $\rho_0(\alpha,\beta) = \rho_0(\min(\mathrm{F}(\beta,\gamma) \setminus \alpha),\beta)^\frown\rho_0(\alpha,\min(\mathrm{F}(\beta,\gamma) \setminus \alpha))$,

(b) $\rho_0(\alpha,\gamma) = \rho_0(\min(\mathrm{F}(\beta,\gamma) \setminus \alpha),\gamma)^\frown\rho_0(\alpha,\min(\mathrm{F}(\beta,\gamma) \setminus \alpha))$. ⊣

1.7 Definition. The ordering $<_c$ on ω_1 is defined as follows:

$$\alpha <_c \beta \quad \text{iff} \quad \rho_0(\xi,\alpha) <_r \rho_0(\xi,\beta),$$

where $\xi = \Delta(\alpha,\beta) = \min\{\eta \leq \min\{\alpha,\beta\} : \rho_0(\eta,\alpha) \neq \rho_0(\eta,\beta)\}$. Here $<_r$ refers to the *right lexicographical ordering* on $\omega^{<\omega}$ defined by letting $s <_r t$ iff s is an end-extension of t or $s(i) < t(i)$ for $i = \min\{j : s(j) \neq t(j)\}$.

1.8 Lemma. *The Cartesian square of the total ordering $<_c$ of ω_1 is the union of countably many chains.*

Proof. It suffices to decompose the set of all pairs (α,β) where $\alpha < \beta$. To each such pair we associate a hereditarily finite set $p(\alpha,\beta)$ which codes the finite structure obtained from $\mathrm{F}(\alpha,\beta) \cup \{\beta\}$ by adding relations that describe the way ρ_0 acts on it. To show that this parametrization works, suppose we are given two pairs (α,β) and (γ,δ) such that

$$p(\alpha,\beta) = p(\gamma,\delta) = p, \text{ and } \alpha <_c \gamma.$$

We must show that $\beta \leq_c \delta$. Let

$$\xi_{\alpha\beta} = \min(\mathrm{F}(\alpha,\beta) \setminus \Delta(\alpha,\gamma)), \text{ and}$$
$$\xi_{\gamma\delta} = \min(\mathrm{F}(\gamma,\delta) \setminus \Delta(\alpha,\gamma)).$$

Note that $\mathrm{F}(\alpha,\beta) \cap \Delta(\alpha,\gamma) = \mathrm{F}(\gamma,\delta) \cap \Delta(\alpha,\gamma)$ so $\xi_{\alpha\beta}$ and $\xi_{\gamma\delta}$ correspond to each other in the isomorphism of the (α,β) and (γ,δ) structures. It follows that:

$$\rho_0(\xi_{\alpha\beta},\alpha) = \rho_0(\xi_{\gamma\delta},\gamma)(= t_{\alpha,\gamma}),$$
$$\rho_0(\xi_{\alpha\beta},\beta) = \rho_0(\xi_{\gamma\delta},\delta)(= t_{\beta,\delta}).$$

Applying Lemma 1.6 we get:

$$\rho_0(\Delta(\alpha,\gamma),\alpha) = t_{\alpha\gamma}{}^\frown\rho_0(\Delta(\alpha,\gamma),\xi_{\alpha\beta}),$$
$$\rho_0(\Delta(\alpha,\gamma),\gamma) = t_{\alpha\gamma}{}^\frown\rho_0(\Delta(\alpha,\gamma),\xi_{\gamma\delta}).$$

It follows that $\rho_0(\Delta(\alpha,\gamma),\xi_{\alpha\beta}) \neq \rho_0(\Delta(\alpha,\gamma),\xi_{\gamma\delta})$. Applying Lemma 1.6 for β and δ and the ordinal $\Delta(\alpha,\gamma)$ we get:

$$\rho_0(\Delta(\alpha,\gamma),\beta) = t_{\beta\delta}{}^\frown\rho_0(\Delta(\alpha,\gamma),\xi_{\alpha\beta}), \tag{I.1}$$

$$\rho_0(\Delta(\alpha,\gamma),\delta) = t_{\beta\delta}{}^\frown\rho_0(\Delta(\alpha,\gamma),\xi_{\gamma\delta}). \tag{I.2}$$

It follows that $\rho_0(\Delta(\alpha,\gamma),\beta) \neq \rho_0(\Delta(\alpha,\gamma),\delta)$. This shows $\Delta(\alpha,\gamma) \geq \Delta(\beta,\delta)$. A symmetrical argument shows the other inequality $\Delta(\beta,\delta) \geq \Delta(\alpha,\gamma)$. It follows that

$$\Delta(\alpha,\gamma) = \Delta(\beta,\delta)(=\bar{\xi}).$$

Our assumption is that $\rho_0(\bar{\xi},\alpha) <_r \rho_0(\bar{\xi},\gamma)$ and since these two sequences have $t_{\alpha\gamma}$ as common initial part, this reduces to

$$\rho_0(\bar{\xi},\xi_{\alpha\beta}) <_r \rho_0(\bar{\xi},\xi_{\gamma\delta}). \tag{I.3}$$

On the other hand $t_{\beta\delta}$ is a common initial part of $\rho_0(\bar{\xi},\beta)$ and $\rho_0(\bar{\xi},\delta)$, so their lexicographical relationship depends on their tails which by (I.1) and (I.2) are equal to $\rho_0(\bar{\xi},\xi_{\alpha\beta})$ and $\rho_0(\bar{\xi},\xi_{\gamma\delta})$ respectively. Referring to (I.3) we conclude that indeed $\rho_0(\bar{\xi},\beta) <_r \rho_0(\bar{\xi},\delta)$, i.e. $\beta <_c \delta$. ⊣

1.9 Notation. *Well-ordered sets of rationals.* The set $\omega^{<\omega}$ ordered by the right lexicographical ordering $<_r$ is a particular copy of the rationals of the interval $(0,1]$ which we are going to denote by $\mathbb{Q}_r$ or simply by $\mathbb{Q}$. The next lemma shows that for a fixed α, $\rho_0(\xi,\alpha)$ is a strictly increasing function of ξ from α into $\mathbb{Q}_r$. Let $(\rho_0)_\alpha$ denote this function which we identify with its range, i.e. view as a member of the tree $\sigma\mathbb{Q}_r$ of all well-ordered subsets of $\mathbb{Q}_r$, ordered by end-extension.

1.10 Lemma. *$\rho_0(\alpha,\gamma) <_r \rho_0(\beta,\gamma)$ whenever $\alpha < \beta < \gamma$.* ⊣

At this point we recall several standard concepts for trees of height ω_1, concepts that generally figure in what follows: A tree of height ω_1 is an *Aronszajn tree* if all of its levels and chains are countable. A tree of height ω_1 is a *special Aronszajn tree* if it is an Aronszajn tree that admits a decomposition into countably many antichains or, equivalently, admits a strictly increasing map into the rationals. Finally, a tree of height ω_1 is a *Souslin tree* if all of its chains and antichains countable.

The sequence $(\rho_0)_\alpha$ $(\alpha < \omega_1)$ of members of $\sigma\mathbb{Q}_r$ naturally determines the subtree

$$T(\rho_0) = \{(\rho_0)_\beta \restriction \alpha : \alpha \leq \beta < \omega_1\}.$$

Note that for a fixed α, the restriction $(\rho_0)_\beta\restriction\alpha$ is determined by the way $(\rho_0)_\beta$ acts on the finite set $\mathrm{F}(\alpha,\beta)$. This is the content of Lemma 1.6. Hence all levels of $T(\rho_0)$ are countable, and therefore $T(\rho_0)$ is a particular example of an Aronszajn tree. We shall now see that $T(\rho_0)$ is in fact a special Aronszajn tree. The proof of this will depend on the following straightforward fact.

1.11 Lemma. $\{\xi < \beta : \rho_0(\xi,\beta) = \rho_0(\xi,\gamma)\}$ *is a closed subset of* β *whenever* $\beta < \gamma$. ⊣

It follows that $T(\rho_0)$ does not branch at limit levels. From this we can conclude that $T(\rho_0)$ is a special subtree of $\sigma\mathbb{Q}$ since this is easily seen to be so for any subtree of $\sigma\mathbb{Q}$ which is finitely branching at limit nodes.

1.12 Definition. Identifying the power set of $\mathbb{Q}$ with the particular copy $2^{\mathbb{Q}}$ of the Cantor set, define for every countable ordinal α,

$$G_\alpha = \{x \in 2^{\mathbb{Q}} : x \text{ end-extends no } (\rho_0)_\beta\restriction\alpha \text{ for } \beta \geq \alpha\}.$$

1.13 Lemma. G_α $(\alpha < \omega_1)$ *is an increasing sequence of proper* G_δ*-subsets of the Cantor set whose union is equal to the Cantor set.* ⊣

1.14 Lemma. *The set* $X = \{(\rho_0)_\beta : \beta < \omega_1\}$ *considered as a subset of the Cantor set* $2^{\mathbb{Q}}$ *has universal measure zero.*

Proof. Let μ be a given non-atomic Borel measure on $2^{\mathbb{Q}}$. For $t \in T(\rho_0)$, set

$$P_t = \{x \in 2^{\mathbb{Q}} : x \text{ end-extends } t\}.$$

Note that each P_t is a perfect subset of $2^{\mathbb{Q}}$ and therefore is μ-measurable. Let

$$S = \{t \in T(\rho_0) : \mu(P_t) > 0\}.$$

Then S is a downward closed subtree of $\sigma\mathbb{Q}$ with no uncountable antichains. By an old result of Kurepa (see [59]), no Souslin tree admits a strictly increasing map into the reals (as for example $\sigma\mathbb{Q}$ does). It follows that S must be countable and so we are done. ⊣

1.15 Definition. The *maximal weight* of the walk is the two-place function $\rho_1 : [\omega_1]^2 \longrightarrow \omega$ defined recursively by

$$\rho_1(\alpha,\beta) = \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\},$$

where we stipulate that $\rho_1(\alpha,\alpha) = 0$ for all $\alpha < \omega_1$.[3] Thus $\rho_1(\alpha,\beta)$ is simply the maximal integer appearing in the sequence $\rho_0(\alpha,\beta)$.

[3] This is another use of the convention $\rho_1(\alpha,\alpha) = 0$ that is necessary for the recursive definition to work.

1.16 Lemma. *For all $\alpha < \beta < \omega_1$ and $n < \omega$,*

(a) $\{\xi \leq \alpha : \rho_1(\xi, \alpha) \leq n\}$ *is finite,*

(b) $\{\xi \leq \alpha : \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\}$ *is finite.*

Proof. The proof is by induction. To prove (a) it suffices to show that for every $n < \omega$ and every $A \subseteq \alpha$ of order-type ω there is a $\xi \in A$ such that $\rho_1(\xi, \alpha) > n$. Let $\eta = \sup(A)$. If $\eta = \alpha$ one chooses arbitrary $\xi \in A$ with the property that $|C_\alpha \cap \xi| > n$, so let us consider the case $\eta < \alpha$. Let $\alpha_1 = \min(C_\alpha \setminus \eta)$. By the inductive hypothesis there is a $\xi \in A$ such that:

$$\xi > \max(C_\alpha \cap \eta) \text{ and } \rho_1(\xi, \alpha_1) > n.$$

Note that $\rho_0(\xi, \alpha) = \langle |C_\alpha \cap \eta| \rangle^\frown \rho_0(\xi, \alpha_1)$, and therefore

$$\rho_1(\xi, \alpha) \geq \rho_1(\xi, \alpha_1) > n.$$

To prove (b) we show by induction that for every $A \subseteq \alpha$ of order-type ω there exists a $\xi \in A$ such that $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta)$. Let $\eta = \sup(A)$ and let $\beta_1 = \min(C_\beta \setminus \eta)$. Let $n = |C_\beta \cap \eta|$ and let

$$B = \{\xi \in A : \xi > \max(C_\beta \cap \eta) \text{ and } \rho_1(\xi, \beta_1) > n\}.$$

Then B is infinite, so by the induction hypothesis we can find $\xi \in B$ such that $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta_1)$. Then

$$\rho_1(\xi, \beta) = \max\{n, \rho_1(\xi, \beta_1)\} = \rho_1(\xi, \beta_1),$$

so we are done. $\dashv$

1.17 Remark. Define $(\rho_1)_\alpha$ from ρ_1 just as $(\rho_0)_\alpha$ was defined from ρ_0 above. It follows that the sequence

$$(\rho_1)_\alpha : \alpha \longrightarrow \omega \ (\alpha < \omega_1)$$

of finite-to-one functions is *coherent* in the sense that $(\rho_1)_\alpha =^* (\rho_1)_\beta \restriction \alpha$ whenever $\alpha \leq \beta$. (Here $=^*$ denotes the fact that the functions agree on all but finitely many arguments.) The corresponding tree

$$T(\rho_1) = \{t : \alpha \longrightarrow \omega : \alpha < \omega_1 \text{ and } t =^* (\rho_1)_\alpha\}$$

is a homogeneous, special Aronszajn tree with many other interesting properties, some of which we are going to describe here. For example, we have the following fact whose proof is quite analogous to that of Lemma 1.8.

1.18 Lemma. *The Cartesian square of $T(\rho_1)$ ordered lexicographically is the union of countably many chains.* $\dashv$

1.19 Definition. Consider the following extension of $T(\rho_1)$:

$$\widetilde{T}(\rho_1) = \{t : \alpha \longrightarrow \omega : \alpha < \omega_1 \text{ and } t{\restriction}\xi \in T(\rho_1) \text{ for all } \xi < \alpha\}.$$

If we order $\widetilde{T}(\rho_1)$ by the right lexicographical ordering $<_r$ we get a complete linearly ordered set. It is not continuous, as it contains jumps of the form

$$[t^\frown\langle m\rangle, t^\frown\langle m+1\rangle^\frown\vec{0}],$$

where $t \in T(\rho_1)$ and $m < \omega$. Removing the right-hand points from all the jumps we get a linearly ordered continuum which we denote by $\widetilde{A}(\rho_1)$.

1.20 Lemma. *$\widetilde{A}(\rho_1)$ is a homogeneous nonreversible ordered continuum that can be represented as the union of an increasing ω_1-sequence of Cantor sets.* ⊣

1.21 Definition. The set $\widetilde{T}(\rho_1)$ has another natural structure, a topology generated by the family of sets of the form

$$\widetilde{V}_t = \{u \in \widetilde{T}(\rho_1) : t \subseteq u\},$$

for t a node of $T(\rho_1)$ of successor length as a clopen subbase. Let $T^0(\rho_1)$ denote the set of all nodes of $T(\rho_1)$ of successor length. Then $\widetilde{T}(\rho_1)$ can be regarded as the set of all downward closed chains of the tree $T^0(\rho_1)$ and the topology on $\widetilde{T}(\rho_1)$ is simply the topology one obtains from identifying the power set of $T^0(\rho_1)$ with the cube $\{0,1\}^{T^0(\rho_1)}$ with its Tychonoff topology.[4] $\widetilde{T}(\rho_1)$ being a closed subset of the cube is compact. In fact $\widetilde{T}(\rho_1)$ has some very strong topological properties such as the property that closed subsets of $\widetilde{T}(\rho_1)$ are its retracts. A *compactum* is a metrizable compact space, and a compactum X is *Eberlein* if its function space $\mathcal{C}(X)$ can be generated by a subset which is compact in the weak topology.

1.22 Lemma. *$\widetilde{T}(\rho_1)$ is a homogeneous Eberlein compactum.*

Proof. The proof that $\widetilde{T}(\rho_1)$ is homogeneous is quite similar to the corresponding part of the proof of the Lemma 1.20. To see that $\widetilde{T}(\rho_1)$ is an Eberlein compactum, i.e. that the function space $\mathcal{C}(\widetilde{T}(\rho_1))$ is weak compactly generated, let $\{X_n\}$ be a countable antichain decomposition of $T(\rho_1)$ and consider the set $K = \{2^{-n}\chi_{\widetilde{V}_t} : n < \omega, t \in X_n\} \cup \{\chi_\emptyset\}$. Note that K is a weakly compact subset of $\mathcal{C}(\widetilde{T}(\rho_1))$ which separates the points of $\widetilde{T}(\rho_1)$. ⊣

The coherent sequence $(\rho_1)_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$ of finite-to-one maps can easily be turned into a coherent sequence of maps that are actually one-to-one. For example, one way to achieve this is via the following formula:

$$\bar{\rho}_1(\alpha,\beta) = 2^{\rho_1(\alpha,\beta)} \cdot (2 \cdot |\{\xi \leq \alpha : \rho_1(\xi,\beta) = \rho_1(\alpha,\beta)\}| + 1).$$

[4] This is done by identifying a subset V of $T^0(\rho_1)$ with its characteristic function $\chi_V : T^0(\rho_1) \longrightarrow 2$.

Define $(\bar{\rho}_1)_\alpha$ from $\bar{\rho}_1$ just as $(\rho_1)_\alpha$ was defined from ρ_1; then the $(\bar{\rho}_1)\alpha$'s are one-to-one. From ρ_1 one has a natural sequence r_α $(\alpha < \omega_1)$ of elements of ω^ω defined as follows:

$$r_\alpha(n) = |\{\xi \leq \alpha : \rho_1(\xi, \alpha) \leq n\}|.$$

Note that r_β eventually dominates r_α whenever $\alpha + \omega < \beta$.

1.23 Definition. The sequences $e_\alpha = (\bar{\rho}_1)_\alpha$ $(\alpha < \omega_1)$ and r_α $(\alpha < \omega_1)$ can be used in describing a functor $G \longmapsto G^*$, which to every graph G on ω_1 associates another graph G^* on ω_1 as follows:

$$\{\alpha, \beta\} \in G^* \text{ iff } \{e_\alpha^{-1}(l), e_\beta^{-1}(l)\} \in G$$

for all $l < \Delta(r_\alpha, r_\beta)$ for which these preimages are both defined and different.

The proof of the following lemma can be found in [76].

1.24 Lemma. *Suppose that every uncountable family $\mathcal{F}$ of pairwise disjoint finite subsets of ω_1 contains two sets A and B such that $A \otimes B \subseteq G$.*[5] *Then the same is true about G^* provided the uncountable family $\mathcal{F}$ consists of finite cliques*[6] *of G^*.* $\dashv$

1.25 Lemma. *If there is an uncountable $\Gamma \subseteq \omega_1$ such that $[\Gamma]^2 \subseteq G^*$ then ω_1 can be decomposed into countably many sets Σ such that $[\Sigma]^2 \subseteq G$.*

Proof. Fix an uncountable $\Gamma \subseteq \omega_1$ such that $[\Gamma]^2 \subseteq G^*$. For a finite binary sequence s of length equal to some $l + 1$, set

$$\Gamma_s = \{\xi < \omega_1 : e(\xi, \alpha) = l \text{ for some } \alpha \text{ in } \Gamma \text{ with } s \subseteq r_\alpha\}.$$

Then the sets Γ_s cover ω_1 and $[\Gamma_s]^2 \subseteq G$ for all s. $\dashv$

1.26 Remark. Let G be the comparability graph of some Souslin tree T. Then for every uncountable family $\mathcal{F}$ of pairwise disjoint cliques of G (finite chains of T) there exist $A \neq B$ in $\mathcal{F}$ such that $A \cup B$ is a clique of G (a chain of T). However, it is not hard to see that G^* fails to have this property (i.e. the conclusion of Lemma 1.24). This shows that some assumption on the graph G in Lemma 1.24 is necessary. There are indeed many graphs that satisfy the hypothesis of Lemma 1.24. Many examples appear when one is trying to apply Martin's Axiom to some Ramsey-theoretic problems. Note that the conclusion of Lemma 1.24 is simply saying that the poset of all finite cliques of G^* is ccc while its hypothesis is a bit stronger than the fact that the poset of all finite cliques of G is ccc in all of its finite powers. Applying 1.25 to the case when G is the incomparability graph of some Aronszajn tree, we see that the statement saying that all Aronszajn trees are special is a purely Ramsey-theoretic statement in the same way Souslin's Hypothesis, that there are no Souslin trees, is.

---

[5] Here, $A \otimes B = \{\{\alpha, \beta\} : \alpha \in A, \beta \in B, \alpha \neq \beta\}$.

[6] A *clique* of G^* is a subset C of ω_1 with the property that $[C]^2 \subseteq G^*$.

2. Subadditive Functions

In this section we describe a metric feature of the space ω_1 of countable ordinals. One first encounters this feature by analyzing properties of the following function.

2.1 Definition. The *rho-function* $\rho : [\omega_1]^2 \longrightarrow \omega$ is defined recursion follows:

$$\rho(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, \rho(\alpha, \min(C_\beta \setminus \alpha)), \rho(\xi, \alpha) : \xi \in C_\beta \cap \alpha\},$$

where we stipulate that $\rho(\alpha, \alpha) = 0$ for all $\alpha < \omega_1$.

2.2 Lemma. *For all $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$,*

(a) $\{\xi \leq \alpha : \rho(\xi, \alpha) \leq n\}$ *is finite,*

(b) $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$,

(c) $\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$.

Proof. Note that $\rho(\alpha, \beta) \geq \rho_1(\alpha, \beta)$, so (a) follows from the corresponding property of ρ_1. The proof of (b) and (c) is simultaneous by induction on α, β and γ:

To prove (b), consider $n = \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$, and let

$$\xi_\alpha = \min(C_\gamma \setminus \alpha) \text{ and } \xi_\beta = \min(C_\gamma \setminus \beta).$$

We have to show that $\rho(\alpha, \gamma) \leq n$.

Case 1^b: $\xi_\alpha = \xi_\beta$. Then by the inductive hypothesis,

$$\rho(\alpha, \xi_\alpha) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \xi_\beta)\}.$$

From the definition of $\rho(\beta, \gamma) \leq n$ we get that $\rho(\beta, \xi_\beta) \leq \rho(\beta, \gamma)$, so replacing $\rho(\beta, \xi_\beta)$ by $\rho(\beta, \gamma)$ in the above inequality we get $\rho(\alpha, \xi_\alpha) \leq n$. Consider a $\xi \in C_\gamma \cap \alpha = C_\gamma \cap \beta$. By the inductive hypothesis

$$\rho(\xi, \alpha) \leq \max\{\rho(\xi, \beta), \rho(\alpha, \beta)\}.$$

From the definition of $\rho(\beta, \gamma)$ we see that $\rho(\xi, \beta) \leq \rho(\beta, \gamma)$, so replacing $\rho(\xi, \beta)$ with $\rho(\beta, \gamma)$ in the last inequality we get that $\rho(\xi, \alpha) \leq n$. Since $|C_\gamma \cap \alpha| = |C_\gamma \cap \beta| \leq \rho(\beta, \gamma) \leq n$, referring to the definition of $\rho(\alpha, \gamma)$ we conclude that $\rho(\alpha, \gamma) \leq n$.

Case 2^b: $\xi_\alpha < \xi_\beta$. Then $\xi_\alpha \in C_\gamma \cap \beta$, so

$$\rho(\xi_\alpha, \beta) \leq \rho(\beta, \gamma) \leq n.$$

By the inductive hypothesis

$$\rho(\alpha, \xi_\alpha) \leq \max\{\rho(\alpha, \beta), \rho(\xi_\alpha, \beta)\} \leq n.$$

Similarly, for every $\xi \in C_\gamma \cap \alpha \subseteq C_\gamma \cap \beta$,

$$\rho(\xi, \alpha) \leq \max\{\rho(\xi, \beta), \rho(\alpha, \beta)\} \leq n.$$

Finally $|C_\gamma \cap \alpha| \leq |C_\gamma \cap \beta| \leq \rho(\beta, \gamma) \leq n$. Combining these inequalities we get the desired conclusion $\rho(\alpha, \gamma) \leq n$.

To prove (c), consider now $n = \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$. We have to show that $\rho(\alpha, \beta) \leq n$. Let ξ_α and ξ_β be as above and let us consider the same two cases as above.

Case 1^c: $\xi_\alpha = \xi_\beta = \bar{\xi}$. Then by the inductive hypothesis

$$\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \bar{\xi}), \rho(\beta, \bar{\xi})\}.$$

This gives the desired bound $\rho(\alpha, \beta) \leq n$, since $\rho(\alpha, \xi_\alpha) \leq \rho(\alpha, \gamma) \leq n$ and $\rho(\beta, \xi_\beta) \leq \rho(\beta, \gamma) \leq n$.

Case 2^c: $\xi_\alpha < \xi_\beta$. Applying the inductive hypothesis again we get

$$\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \xi_\alpha), \rho(\xi_\alpha, \beta)\} \leq n.$$

This completes the proof. ⊣

The following simple consequence shows that the function ρ has a considerably finer coherence property than ρ_1.

2.3 Lemma. *If $\alpha < \beta < \gamma$ and $\rho(\alpha, \beta) > \rho(\beta, \gamma)$, then $\rho(\alpha, \gamma) = \rho(\alpha, \beta)$.* ⊣

2.4 Definition. Define $\bar{\rho} : [\omega_1]^2 \longrightarrow \omega$ as follows

$$\bar{\rho}(\alpha, \beta) = 2^{\rho(\alpha, \beta)} \cdot (2 \cdot |\{\xi \leq \alpha : \rho(\xi, \alpha) \leq \rho(\alpha, \beta)\}| + 1).$$

Using the properties of ρ one easily checks the following facts about its stretching $\bar{\rho}$.

2.5 Lemma. *For all $\alpha < \beta < \gamma < \omega_1$,*

(a) $\bar{\rho}(\alpha, \gamma) \neq \bar{\rho}(\beta, \gamma)$,

(b) $\bar{\rho}(\alpha, \gamma) \leq \max\{\bar{\rho}(\alpha, \beta), \bar{\rho}(\beta, \gamma)\}$,

(c) $\bar{\rho}(\alpha, \beta) \leq \max\{\bar{\rho}(\alpha, \gamma), \bar{\rho}(\beta, \gamma)\}$. ⊣

The following property of $\bar{\rho}$ is also sometimes useful (see [76]).

2.6 Lemma. *Suppose $\eta_\alpha \neq \eta_\beta < \min\{\alpha, \beta\}$ and $\bar{\rho}(\eta_\alpha, \alpha) = \bar{\rho}(\eta_\beta, \beta) = n$. Then $\bar{\rho}(\eta_\alpha, \beta), \bar{\rho}(\eta_\beta, \alpha) > n$.* ⊣

2.7 Definition. For $p \in \omega^{<\omega}$ define a binary relation $<_p$ on ω_1 by letting $\alpha <_p \beta$ iff $\alpha < \beta$, $\bar{\rho}(\alpha, \beta) \in |p|$, and

$$p(\bar{\rho}(\xi, \alpha)) = p(\bar{\rho}(\xi, \beta))$$

for any $\xi < \alpha$ such that $\bar{\rho}(\xi, \alpha) < |p|$.

2.8 Lemma.

(a) *$<_p$ is a tree ordering on ω_1 of height $\leq |p| + 1$,*

(b) *$p \subseteq q$ implies $<_p \subseteq <_q$.*

Proof. This follows immediately from Lemma 2.5. ⊣

2.9 Definition. For $x \in \omega^\omega$, set

$$<_x = \bigcup \{<_{x \restriction n} : n < \omega\}.$$

The proof of the following lemma can also be found in [76].

2.10 Lemma. *For every $p \in \omega^{<\omega}$ there is a partition of ω_1 into finitely many pieces such that if $\alpha < \beta$ belong to the same piece then there is a $q \supseteq p$ in $\omega^{<\omega}$ such that $\alpha <_q \beta$.* ⊣

2.11 Theorem. *For every infinite subset $\Gamma \subseteq \omega_1$, the set*

$$G_\Gamma = \{x \in \omega^\omega : \alpha <_x \beta \text{ for some } \alpha, \beta \in \Gamma\}$$

is a dense open subset of the Baire space.

Proof. This is an immediate consequence of Lemma 2.10. ⊣

2.12 Definition. For $\alpha < \beta < \omega_1$, let $\alpha <_{\bar{\rho}} \beta$ denote the fact that $\bar{\rho}(\xi, \alpha) = \bar{\rho}(\xi, \beta)$ for all $\xi < \alpha$. Then $<_{\bar{\rho}}$ is a tree ordering on ω_1 obtained by identifying α with the member $\bar{\rho}(\cdot, \alpha)$ of the tree $T(\bar{\rho})$. Note that $<_{\bar{\rho}} \subseteq <_x$ for all $x \in \omega^\omega$ and that there exists an $x \in \omega^\omega$ such that $<_x = <_{\bar{\rho}}$ (e.g., one such x is the identity map $\mathrm{id} : \omega \longrightarrow \omega$).

The following result is an analogue of Lemma 2.10 for the incomparability relation, though its proof is considerably simpler.

2.13 Lemma. *If Γ is an infinite $<_{\bar{\rho}}$-antichain, the set*

$$H_\Gamma = \{x \in \omega^\omega : \alpha \not<_x \beta \text{ for some } \alpha < \beta \text{ in } \Gamma\}$$

is a dense open subset of the Baire space. ⊣

2.14 Definition. For a family $\mathcal{F}$ of infinite $<_{\bar\rho}$-antichains, we say that a real $x \in \omega^\omega$ is $\mathcal{F}$*-Cohen* if $x \in G_\Gamma \cap H_\Gamma$ for all $\Gamma \in \mathcal{F}$. We say that x is $\mathcal{F}$*-Souslin* if no member of $\mathcal{F}$ is a $<_x$-chain or a $<_x$-antichain. We say that a real $x \in \omega^\omega$ is *Souslin* if the tree ordering $<_x$ on ω_1 has no uncountable chains nor antichains, i.e. when x is $\mathcal{F}$-Souslin for $\mathcal{F}$ equal to the family of all uncountable subsets of ω_1.

Note that since every uncountable subset of ω_1 contains an uncountable $<_{\bar\rho}$-antichain, if a family $\mathcal{F}$ refines the family of all uncountable $<_{\bar\rho}$-antichains, then every $\mathcal{F}$-Souslin real is Souslin. The following fact summarizes Theorems 2.11 and 2.13 and connects the two kinds of reals.

2.15 Theorem. *If $\mathcal{F}$ is a family of infinite $<_{\bar\rho}$-antichains, then every $\mathcal{F}$-Cohen real is $\mathcal{F}$-Souslin.* $\dashv$

2.16 Corollary. *If the density of the family of all uncountable subsets of ω_1 is smaller than the number of nowhere dense sets needed to cover the real line, then there is a Souslin tree.* $\dashv$

2.17 Remark. Recall that the *density* of a family $\mathcal{F}$ of infinite subsets of some set S is the minimal size of a family $\mathcal{F}_0$ of infinite subsets of S with the property that every member of $\mathcal{F}$ is refined by a member of $\mathcal{F}_0$. A special case of Corollary 2.16, when the density of the family of all uncountable subsets of ω_1 is equal to $\aleph_1$, was first observed by T. Miyamoto (unpublished).

2.18 Corollary. *Every Cohen real is Souslin.*

Proof. Every uncountable subset of ω_1 in the Cohen extension contains an uncountable subset from the ground model. So it suffices to consider the family $\mathcal{F}$ of all infinite $<_{\bar\rho}$-antichains from the ground model. $\dashv$

If ω_1 is a successor cardinal in the constructible subuniverse, then $\bar\rho$ can be chosen to be coanalytic and so the transformation $x \longmapsto <_x$ will transfer combinatorial notions of Souslin, Aronszajn or special Aronszajn trees into the corresponding classes of reals that lie in the third level of the projective hierarchy. This transformation has been explored on several places in the literature (see, e.g. [3], [25]).

2.19 Remark. We have just seen how the combination of the subadditivity properties (2.5(b),(c)) of the coherent sequence $\bar\rho_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$ of one-to-one mappings can be used in controlling the finite disagreement between them. It turns out that in many contexts the coherence and the subadditivities are essentially equivalent restrictions on a given sequence $e_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$. For example, the following construction shows that this is so for any sequence of finite-to-one mappings $e_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$.

2.20 Definition. Given a coherent sequence $e_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$ of finite-to-one mappings, define $\tau_e : [\omega_1]^2 \longrightarrow \omega$ as follows

$$\tau_e(\alpha,\beta) = \max\{\max\{e(\xi,\alpha), e(\xi,\beta)\} : \xi \leq \alpha \text{ and } e(\xi,\alpha) \neq e(\xi,\beta)\}.$$

2.21 Lemma. *For every* $\alpha < \beta < \gamma < \omega_1$,

(a) $\tau_e(\alpha,\beta) \geq e(\alpha,\beta)$,

(b) $\tau_e(\alpha,\gamma) \leq \max\{\tau_e(\alpha,\beta), \tau_e(\beta,\gamma)\}$,

(c) $\tau_e(\alpha,\beta) \leq \max\{\tau_e(\alpha,\gamma), \tau_e(\beta,\gamma)\}$.

Proof. Since (a) is true if $e(\alpha,\beta) = 0$, let us assume $e(\alpha,\beta) > 0$. By our convention, $e(\alpha,\alpha) = 0$ and so $e(\alpha,\alpha) \neq e(\alpha,\beta) = 0$. It follows that $\tau_e(\alpha,\beta) \geq \max\{\max\{e(\alpha,\alpha), e(\alpha,\beta)\} = e(\alpha,\beta)$. This shows (a).

To show (b), let $n = \max\{\tau_e(\alpha,\beta), \tau_e(\beta,\gamma)\}$. Suppose $\tau_e(\alpha,\gamma) > n$. Then we can choose $\xi \leq \alpha$ such that $e(\xi,\alpha) > n$ or $e(\xi,\gamma) > n$. If $e(\xi,\alpha) > n$ then $e(\xi,\beta) = e(\xi,\alpha) > n$ and so $e(\xi,\beta) \neq e(\xi,\gamma)$. It follows that $\tau_e(\beta,\gamma) \geq e(\xi,\beta) > n$, a contradiction. If $e(\xi,\gamma) > n$ then $e(\xi,\beta) = e(\xi,\gamma) > n$. In particular, $e(\xi,\alpha) \neq e(\xi,\beta)$. It follows that $\tau_e(\beta,\gamma) \geq e(\xi,\beta) > n$, a contradiction.

The proof of (c) is similar. ⊣

2.22 Definition. A mapping $a : [\omega_1]^2 \longrightarrow \omega$ is *transitive* if

$$a(\alpha,\gamma) \leq \max\{a(\alpha,\beta), a(\beta,\gamma)\}$$

for all $\alpha < \beta < \gamma < \omega_1$.

Transitive maps occur quite frequently in set-theoretic constructions. For example, given a sequence A_α $(\alpha < \omega_1)$ of subsets of ω that increases relative to the ordering $\subseteq^*$ of inclusion modulo a finite set, the mapping $a : [\omega_1]^2 \longrightarrow \omega$ defined by

$$a(\alpha,\beta) = \min\{n : A_\alpha \setminus n \subseteq A_\beta\}$$

is a transitive map. The transitivity condition by itself is not nearly as useful as its combination with the other subadditivity property (2.5(c)). Fortunately, there is a general procedure that produces a subadditive dominant to every transitive map.

2.23 Definition. For a transitive $a : [\omega_1]^2 \longrightarrow \omega$ define $\rho_a : [\omega_1]^2 \longrightarrow \omega$ recursively as follows:

$$\rho_a(\alpha,\beta) = \max\{|C_\beta \cap \alpha|, a(\min(C_\beta \setminus \alpha), \beta), \\ \rho_a(\alpha, \min(C_\beta \setminus \alpha)), \rho_a(\xi,\alpha) : \xi \in C_\beta \cap \alpha\}.$$

2.24 Lemma. *For all $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$,*

(a) $\{\xi \leq \alpha : \rho_a(\xi, \alpha) \leq n\}$ *is finite,*

(b) $\rho_a(\alpha, \gamma) \leq \max\{\rho_a(\alpha, \beta), \rho_a(\beta, \gamma)\}$,

(c) $\rho_a(\alpha, \beta) \leq \max\{\rho_a(\alpha, \gamma), \rho_a(\beta, \gamma)\}$,

(d) $\rho_a(\alpha, \beta) \geq a(\alpha, \beta)$.

Proof. The proof of (a),(b),(c) is quite similar to the corresponding part of the proof of Lemma 2.2. This comes of course from the fact that the definition of ρ and ρ_a are closely related. The occurrence of the factor $a(\min(C_\beta \setminus \alpha), \beta)$ complicates a bit the proof that ρ_a is subadditive, and the fact that a is transitive is quite helpful in getting rid of the additional difficulty. The details are left to the interested reader. Given $\alpha < \beta$, for every step $\beta_n \to \beta_{n+1}$ of the minimal walk $\beta = \beta_0 > \beta_1 > \ldots > \beta_k = \alpha$, we have $\rho_a(\alpha, \beta) \geq \rho_a(\beta_n, \beta_{n+1}) \geq a(\beta_n, \beta_{n+1})$ by the very definition of ρ_a. Applying the transitivity of a to this path of inequalities we get the conclusion (d). ⊣

2.25 Lemma. $\rho_a(\alpha, \beta) \geq \rho_a(\alpha+1, \beta)$ *whenever* $0 < \alpha < \beta$ *and* α *is a limit ordinal.*

Proof. Recall the assumption (c) about the fixed C-sequence C_ξ $(\xi < \omega_1)$ on which all our definitions are based: if ξ is a limit ordinal > 0, then no point of C_ξ is a limit ordinal. It follows that if $0 < \alpha < \beta$ and α is a limit ordinal, then the minimal walk $\beta \to \alpha$ must pass through $\alpha + 1$ and therefore $\rho_a(\alpha, \beta) \geq \rho_a(\alpha + 1, \beta)$. ⊣

Let us now give an application of ρ_a to a classical phenomenon of occurrence of gaps in the quotient algebra $\mathcal{P}(\omega)/\text{fin}$.

2.26 Definition. A *Hausdorff gap* in $\mathcal{P}(\omega)/\text{fin}$ is a pair of sequences A_α $(\alpha < \omega_1)$ and B_α $(\alpha < \omega_1)$ such that

(a) $A_\alpha \subseteq^* A_\beta \subseteq^* B_\beta \subseteq^* B_\alpha$ whenever $\alpha < \beta$, but

(b) there is no C such that $A_\alpha \subseteq^* C \subseteq^* B_\alpha$ for all α.

The following straightforward reformulation shows that a Hausdorff gap is just another instance of a nontrivial coherent sequence

$$f_\alpha : A_\alpha \longrightarrow 2 \ \ (\alpha < \omega_1),$$

where the domain A_α of f_α is not the ordinal α itself but a subset of ω and that the corresponding sequence of domains A_α $(\alpha < \omega_1)$ is a realization of ω_1 inside $\mathcal{P}(\omega)/\text{fin}$ in the sense that $A_\alpha \subset^* A_\beta$ whenever $\alpha < \beta$.

2.27 Lemma. *A pair of ω_1-sequences A_α $(\alpha < \omega_1)$ and B_α $(\alpha < \omega_1)$ form a Hausdorff gap iff the pair of ω_1-sequences $\bar{A}_\alpha = A_\alpha \cup (\omega \setminus B_\alpha)$ $(\alpha < \omega_1)$ and $\bar{B}_\alpha = \omega \setminus B_\alpha$ $(\alpha < \omega_1)$ has the following three properties:*

(a) $\bar{A}_\alpha \subseteq^* \bar{A}_\beta$ *whenever* $\alpha < \beta$,

(b) $\bar{B}_\alpha =^* \bar{B}_\beta \cap \bar{A}_\alpha$ *whenever* $\alpha < \beta$,

(c) *there is no B such that $\bar{B}_\alpha =^* B \cap \bar{A}_\alpha$ for all α.* ⊣

From now on we fix a strictly $\subseteq^*$-increasing chain A_α $(\alpha < \omega_1)$ of infinite subsets of ω and let $a : [\omega_1]^2 \longrightarrow \omega$ be defined by

$$a(\alpha, \beta) = \min\{n : A_\alpha \setminus n \subseteq A_\beta\}.$$

Let $\rho_a : [\omega_1]^2 \longrightarrow \omega$ be the corresponding subadditive dominant of a defined above. For $\alpha < \omega_1$, set

$$D_\alpha = A_{\alpha+1} \setminus A_\alpha.$$

2.28 Lemma. *The sets $D_\alpha \setminus \rho_a(\alpha, \gamma)$ and $D_\beta \setminus \rho_a(\beta, \gamma)$ are disjoint whenever $0 < \alpha < \beta < \gamma$ and α and β are limit ordinals.*

Proof. This follows immediately from Lemmas 2.24 and 2.25. ⊣

We are in a position to define a partial mapping $m : [\omega_1]^2 \longrightarrow \omega$ by

$$m(\alpha, \beta) = \min(D_\alpha \setminus \rho_a(\alpha, \beta)),$$

whenever $\alpha < \beta$ and α is a limit ordinal.

2.29 Lemma. *The mapping m is coherent, i.e., $m(\alpha, \beta) = m(\alpha, \gamma)$ for all but finitely many limit ordinals $\alpha < \min\{\beta, \gamma\}$.*

Proof. This is by the coherence of ρ_a and the fact that $\rho_a(\alpha, \beta) = \rho_a(\alpha, \gamma)$ already implies $m(\alpha, \beta) = m(\alpha, \gamma)$. ⊣

2.30 Lemma. *$m(\alpha, \gamma) \neq m(\beta, \gamma)$ whenever $\alpha \neq \beta < \gamma$ and α, β are limit ordinals.*

Proof. This follows from Lemma 2.28. ⊣

For $\beta < \omega_1$, set

$$B_\beta = \{m(\alpha, \beta) : \alpha < \beta \text{ and } \alpha \text{ limit}\}.$$

2.31 Lemma. *$B_\beta =^* B_\gamma \cap A_\beta$ whenever $\beta < \gamma$.*

Proof. By the coherence of m. ⊣

Note the following immediate consequence of Lemma 2.28 and the definition of m.

2.32 Lemma. $m(\alpha,\beta) = \max(B_\beta \cap D_\alpha)$ *whenever* $\alpha < \beta$ *and* α *is a limit ordinal.* ⊣

2.33 Lemma. *There is no* $B \subseteq \omega$ *such that* $B \cap A_\beta =^* B_\beta$ *for all* β.

Proof. Suppose that such a B exists and for a limit ordinal α let us define $g(\alpha) = \max(B \cap D_\alpha)$. Then by Lemma 2.32, $g(\alpha) = m(\alpha,\beta)$ for all $\beta < \omega_1$ and all but finitely many limit ordinals $\alpha < \beta$. By Lemma 2.30, it follows that g is a finite-to-one map, a contradiction. ⊣

2.34 Theorem. *For every strictly* $\subset^*$*-increasing chain* A_α $(\alpha < \omega_1)$ *of subsets of* ω*, there is a sequence* B_α $(\alpha < \omega_1)$ *of subsets of* ω *such that:*

(a) $B_\alpha =^* B_\beta \cap A_\alpha$ *whenever* $\alpha < \beta$,

(b) *there is no* B *such that* $B_\alpha =^* B \cap A_\alpha$ *for all* α. ⊣

3. Steps and Coherence

3.1 Definition. The *number of steps* of the minimal walk is the function $\rho_2 : [\omega_1]^2 \longrightarrow \omega$ defined recursively by

$$\rho_2(\alpha,\beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1,$$

with the convention that $\rho_2(\gamma,\gamma) = 0$ for all γ.

This is an interesting mapping which is particularly useful on higher cardinalities and especially in situations where the more informative mappings ρ_0, ρ_1 and ρ lack their usual coherence properties. Here is a typical property of this mapping which will be explained in a much more general term in later sections of this chapter.

3.2 Lemma. $\sup_{\xi<\alpha} |\rho_2(\xi,\alpha) - \rho_2(\xi,\beta)| < \infty$ *for all* $\alpha < \beta < \omega_1$. ⊣

In this section we use ρ_2 only to succinctly express the following mapping.

3.3 Definition. The *last step function* of the minimal walk is the map $\rho_3 : [\omega_1]^2 \longrightarrow 2$ defined by letting

$$\rho_3(\alpha,\beta) = 1 \text{ iff } \rho_0(\alpha,\beta)(\rho_2(\alpha,\beta) - 1) = \rho_1(\alpha,\beta).$$

In other words, we let $\rho_3(\alpha,\beta) = 1$ just in case the last step of the walk $\beta \to \alpha$ comes with the maximal weight.

3.4 Lemma. $\{\xi < \alpha : \rho_3(\xi,\alpha) \neq \rho_3(\xi,\beta)\}$ *is finite for all* $\alpha < \beta < \omega_1$.

Proof. It suffices to show that for every infinite $\Gamma \subseteq \alpha$ there exists a $\xi \in \Gamma$ such that $\rho_3(\xi,\alpha) = \rho_3(\xi,\beta)$. Shrinking Γ we may assume that for some fixed $\bar{\alpha} \in F(\alpha,\beta)$ and all $\xi \in \Gamma$:

(1) $\bar{\alpha} = \min(F(\alpha, \beta) \setminus \xi)$,

(2) $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta)$,

(3) $\rho_1(\xi, \alpha), \rho_1(\xi, \beta) > \max\{\rho_1(\bar{\alpha}, \alpha), \rho_1(\bar{\alpha}, \beta)\}$.

It follows (see 1.6) that for every $\xi \in \Gamma$:

$$\rho_0(\xi, \alpha) = \rho_0(\bar{\alpha}, \alpha)^\frown \rho_0(\xi, \bar{\alpha}),$$
$$\rho_0(\xi, \beta) = \rho_0(\bar{\alpha}, \beta)^\frown \rho_0(\xi, \bar{\alpha}).$$

So for any $\xi \in \Gamma$, $\rho_3(\xi, \alpha) = 1$ iff the last term of $\rho_0(\xi, \bar{\alpha})$ is its maximal term iff $\rho_3(\xi, \beta) = 1$. ⊣

The sequence $(\rho_3)_\alpha : \alpha \longrightarrow 2$ $(\alpha < \omega_1)$[7] is therefore coherent in the sense that $(\rho_3)_\alpha =^* (\rho_3)_\beta \restriction \alpha$ whenever $\alpha < \beta$. We need to show that the sequence is not trivial, i.e. that it cannot be uniformized by a single total map from ω_1 into 2. In other words, we need to show that ρ_3 still contains enough information about the C-sequence C_α $(\alpha < \omega_1)$ from which it is defined. For this it will be convenient to assume that C_α $(\alpha < \omega_1)$ satisfies the following natural condition:

(d) If α is a limit ordinal > 0 and if ξ occupies the nth place in the increasing enumeration of C_α (that starts with $\min(C_\alpha)$ on its 0th place), then $\xi = \lambda + n + 1$ for some limit ordinal λ (possibly 0).

3.5 Definition. Let Λ denote the set of all countable limit ordinals and for an integer $n \in \omega$, let $\Lambda + n = \{\lambda + n : \lambda \in \Lambda\}$.

The assumption (d) about the C-sequence is behind the following property of ρ_3.

3.6 Lemma. *$\rho_3(\lambda + n, \beta) = 1$ for all but finitely many n with $\lambda + n < \beta$.* ⊣

3.7 Lemma. *For all $\beta < \omega_1$, $n < \omega$, the set $\{\lambda \in \Lambda : \lambda + n < \beta$ and $\rho_3(\lambda + n, \beta) = 1\}$ is finite.*

Proof. Given an infinite subset Γ of $(\Lambda + n) \cap \beta$ we need to find a $\lambda + n \in \Gamma$ such that $\rho_3(\lambda + n, \beta) = 0$. Shrinking Γ if necessary assume that $\rho_1(\lambda + n, \beta) > n + 2$ for all $\lambda + n \in \Gamma$. So if $\rho_3(\lambda + n, \beta) = 1$ for some $\lambda + n \in \Gamma$ then the last step of $\beta \to \lambda + n$ would have to be of weight$> n + 2$ which is impossible by our assumption (d) about C_α $(\alpha < \omega_1)$. ⊣

The meaning of these properties of ρ_3 perhaps is easier to comprehend if we reformulate them in a way that resembles the original formulation of the existence of Hausdorff gaps.

[7]Recall the way one always defines the fiber-functions from a two-variable function applied to the context of ρ_3: $(\rho_3)_\alpha(\xi) = \rho_3(\xi, \alpha)$.

3.8 Lemma. *Let* $B_\alpha = \{\xi < \alpha : \rho_3(\xi, \alpha) = 1\}$ *for* $\alpha < \omega_1$. *Then:*

(1) $B_\alpha =^* B_\beta \cap \alpha$ *for* $\alpha < \beta$,

(2) $(\Lambda + n) \cap B_\beta$ *is finite for all* $n < \omega$ *and* $\beta < \omega_1$,

(3) $\{\lambda + n : n < \omega\} \subseteq^* B_\beta$ *whenever* $\lambda + \omega \leq \beta$. ⊣

In particular, there is no uncountable $\Gamma \subseteq \omega_1$ such that $\Gamma \cap \beta \subseteq^* B_\beta$ for all $\beta < \omega_1$. On the other hand, the P-ideal[8] $\mathfrak{I}$ generated by B_β $(\beta < \omega_1)$ is large as it contains all intervals of the form $[\lambda, \lambda + \omega)$. The following general dichotomy about P-ideals shows that here indeed we have quite a canonical example of a P-ideal on ω_1.

3.9 Definition.

The P-ideal dichotomy. For every P-ideal $\mathfrak{I}$ of countable subsets of some set S either:

(1) there is an uncountable $X \subseteq S$ such that $[X]^\omega \subseteq \mathfrak{I}$, or

(2) S can be decomposed into countably many sets orthogonal to $\mathfrak{I}$.

3.10 Remark. It is known that the P-ideal dichotomy is a consequence of the Proper Forcing Axiom and moreover that it does not contradict the Continuum Hypothesis (see [74]). This is an interesting dichotomy which will be used in this article for testing various notions of coherence as we encounter them. For example, let us consider the following notion of coherence, already encountered above at several places, and see how it is influenced by the P-ideal dichotomy.

3.11 Definition. A mapping $a : [\omega_1]^2 \longrightarrow \omega$ is *coherent* if for every $\alpha < \beta < \omega_1$ there exist only finitely many $\xi < \alpha$ such that $a(\xi, \alpha) \neq a(\xi, \beta)$, or in other words, $a_\alpha =^* a_\beta \restriction \alpha$.[9] We say that a is *nontrivial* if there is no $h : \omega_1 \longrightarrow \omega$ such that $h \restriction \alpha =^* a_\alpha$ for all $\alpha < \omega_1$.

Note that the existence of a coherent and nontrivial $a : [\omega_1]^2 \longrightarrow 2$ (such as, for example, the function ρ_3 defined above) is something that corresponds to the notion of a Hausdorff gap (cf. the previous lemma) in this context. Notice moreover, that this notion is also closely related to the notion of an Aronszajn tree since

$$T(a) = \{t : \alpha \longrightarrow \omega : \alpha < \omega_1 \text{ and } t =^* a_\alpha\}$$

[8] Recall that an ideal $\mathfrak{I}$ of subsets of some set S is a *P-ideal* if for every sequence A_n $(n < \omega)$ of elements of $\mathfrak{I}$ there is B in $\mathfrak{I}$ such that $A_n \setminus B$ is finite for all $n < \omega$. A set X is *orthogonal* to $\mathfrak{I}$ if $X \cap A$ is finite for all A in $\mathfrak{I}$.

[9] A mapping $a : [\omega_1]^2 \longrightarrow \omega$ is naturally identified with a sequence a_α $(\alpha < \omega_1)$, where $a_\alpha : \alpha \longrightarrow \omega$ is defined by $a_\alpha(\xi) = a(\xi, \alpha)$.

is such an Aronszajn tree whenever $a : [\omega_1]^2 \longrightarrow \omega$ is coherent and nontrivial.[10] In fact, we shall call an arbitrary Aronszajn tree T *coherent* if T is isomorphic to $T(a)$ for some coherent and nontrivial $a : [\omega_1]^2 \longrightarrow \omega$. In case the range of the map a is actually smaller than ω, e.g. equal to some integer k, then it is natural to let $T(a)$ be the collection of all $t : \alpha \longrightarrow k$ such that $\alpha < \omega_1$ and $t =^* a_\alpha$. This way, we have coherent binary, ternary, etc. Aronszajn trees rather than only ω-ary coherent Aronszajn trees.

3.12 Definition. The *support* of a map $a : [\omega_1]^2 \longrightarrow \omega$ is the sequence $\mathrm{supp}(a_\alpha) = \{\xi < \alpha : a(\xi,\alpha) \neq 0\}$ $(\alpha < \omega_1)$ of subsets of ω_1. A set Γ is *orthogonal* to a if $\mathrm{supp}(a_\alpha) \cap \Gamma$ is finite for all $\alpha < \omega_1$. We say that $a : [\omega_1]^2 \longrightarrow \omega$ is *nowhere dense* if there is no uncountable $\Gamma \subseteq \omega_1$ such that $\Gamma \cap \alpha \subseteq^* \mathrm{supp}(a_\alpha)$ for all $\alpha < \omega_1$.

Note that ρ_3 is an example of a nowhere dense coherent map for the simple reason that ω_1 can be covered by countably many sets $\Lambda + n$ $(n < \omega)$ that are orthogonal to ρ_3. The following immediate fact shows that ρ_3 is indeed a prototype of a nowhere dense and coherent map $a : [\omega_1]^2 \longrightarrow \omega$.

3.13 Proposition. *Under the P-ideal dichotomy, for every nowhere dense and coherent map $a : [\omega_1]^2 \longrightarrow \omega$ the domain ω_1 can be decomposed into countably many sets orthogonal to a.* $\dashv$

3.14 Notation. To every $a : [\omega_1]^2 \longrightarrow \omega$ associate the corresponding Δ-function $\Delta_a : [\omega_1]^2 \longrightarrow \omega$ as follows:

$$\Delta_a(\alpha,\beta) = \min\{\xi < \alpha : a(\xi,\alpha) \neq a(\xi,\beta)\}$$

with the convention that $\Delta_a(\alpha,\beta) = \alpha$ whenever $a(\xi,\alpha) = a(\xi,\beta)$ for all $\xi < \alpha$. Given this notation, it is natural to let

$$\Delta_a(\Gamma) = \{\Delta_a(\alpha,\beta) : \alpha,\beta \in \Gamma, \alpha < \beta\}$$

for an arbitrary set $\Gamma \subseteq \omega_1$.

The following simple fact reveals a crucial property of coherent trees.

3.15 Lemma. *Suppose that $a : [\omega_1]^2 \longrightarrow \omega$ is nontrivial and coherent and that every uncountable subset of $T(a)$ contains an uncountable antichain. Then for every pair Σ,Ω of uncountable subsets of ω_1 there exists an uncountable subset Γ of ω_1 such that $\Delta_a(\Gamma) \subseteq \Delta_a(\Sigma) \cap \Delta_a(\Omega)$.* $\dashv$

3.16 Notation. For $a : [\omega_1]^2 \longrightarrow \omega$, set

$$\mathcal{U}(a) = \{A \subseteq \omega_1 : A \supseteq \Delta_a(\Gamma) \text{ for some uncountable } \Gamma \subseteq \omega_1\}.$$

[10] Similarities between the notion of a Hausdorff gap and the notion of an Aronszajn tree have been further explained recently in the two papers of Talayco ([57], [58]), where it is shown that they naturally correspond to first cohomology groups over a pair of very similar spaces.

By Lemma 3.15, $\mathcal{U}(a)$ is a uniform filter on ω_1 for every nontrivial coherent $a : [\omega_1]^2 \longrightarrow \omega$ for which $T(a)$ contains no Souslin subtrees. It turns out that under some very mild assumption, $\mathcal{U}(a)$ is in fact a uniform ultrafilter on ω_1. The proof of this can be found in [76].

3.17 Theorem. *Under* MA_{ω_1}*, the filter* $\mathcal{U}(a)$ *is an ultrafilter for every nontrivial and coherent* $a : [\omega_1]^2 \longrightarrow \omega$*.* $\dashv$

3.18 Remark. One may find Theorem 3.17 a bit surprising in view of the fact that it gives us an ultrafilter $\mathcal{U}(a)$ on ω_1 that is Σ_1-definable over the structure $(H_{\omega_2}, \in)$. It is well-known that there is no ultrafilter on ω that is Σ_1-definable over the structure $(H_{\omega_1}, \in)$.

It turns out that the transformation $a \longmapsto \mathcal{U}(a)$ captures some of the essential properties of the corresponding and more obvious transformation $a \longmapsto T(a)$. To state this we need some standard definitions.

3.19 Definition. For two trees S and T, by $S \leq T$ we denote the fact that there is a strictly increasing map $f : S \longrightarrow T$. Let $S < T$ whenever $S \leq T$ and $T \nleq S$ and let $S \equiv T$ whenever $S \leq T$ and $T \leq S$. In general, the equivalence relation $\equiv$ on trees is very far from the finer relation $\cong$, the isomorphism relation. However, the following fact shows that in the realm of trees $T(a)$, these two relations may coincide and moreover, that the mapping $T(a) \longmapsto \mathcal{U}(a)$ reduces $\equiv$ and $\cong$ to the equality relation among ultrafilters on ω_1 (see [75]).

The following fact reveals in particular that the class of coherent trees has the Schroeder-Bernstein property. Its proof can again be found in [76].

3.20 Theorem. *Assuming* MA_{ω_1}*, for every pair of coherent and nontrivial mappings* $a : [\omega_1]^2 \longrightarrow \omega$ *and* $b : [\omega_1]^2 \longrightarrow \omega$*, the trees* $T(a)$ *and* $T(b)$ *are isomorphic iff* $T(a) \equiv T(b)$ *iff* $\mathcal{U}(a) = \mathcal{U}(b)$*.* $\dashv$

3.21 Definition. The *shift* of $a : [\omega_1]^2 \longrightarrow \omega$ is defined to be the mapping $a^{(1)} : [\omega_1]^2 \longrightarrow \omega$ determined by the equation $a^{(1)}(\alpha, \beta) = a(\alpha + 1, \hat{\beta})$, where $\hat{\beta} = \min\{\lambda \in \Lambda : \lambda \geq \beta\}$. The n-fold iteration of the shift operation is defined recursively by the formula $a^{(n+1)} = (a^{(n)})^{(1)}$.

The following fact, whose proof can be found in [76], shows that Aronszajn trees are not well-quasi-ordered under the quasi-ordering $\leq$ (see also [75]).

3.22 Theorem. *If* a *is nontrivial, coherent, and orthogonal to* Λ*, then* $T(a) > T(a^{(1)})$*.* $\dashv$

3.23 Corollary. *If* a *is nontrivial, coherent and orthogonal to* $\Lambda + n$ *for all* $n < \omega$*, then* $T(a^{(n)}) > T(a^{(m)})$ *whenever* $n < m < \omega$*.*

Proof. Note that if a is orthogonal to $\Lambda + n$ for all $n < \omega$, then so is every of its finite shifts $a^{(m)}$. $\dashv$

3.24 Corollary. $T(\rho_3^{(n)}) > T(\rho_3^{(m)})$ *whenever* $n < m < \omega$. $\dashv$

Somewhat unexpectedly, with very little extra assumptions we can say much more about $\leq$ in the domain of coherent Aronszajn trees (for proofs see [75] and [76]).

3.25 Theorem. *Under* MA_{ω_1}*, the family of coherent Aronszajn trees is totally ordered under* $\leq$. $\dashv$

3.26 Remark. While under MA_{ω_1}, the class of coherent Aronszajn trees is totally ordered by $\leq$, Corollary 3.24 gives us that this chain of trees is not well-ordered. This should be compared with an old result of Ohkuma [43] that the class of all scattered trees is well-ordered by $\leq$ (see also [36]). It turns out that the class of all Aronszajn trees is not totally ordered under $\leq$, i.e. there exist Aronszajn trees S and T such that $S \nleq T$ and $T \nleq S$. The reader is referred to [75] and [76] for more information on this and other related results that we chose not to reproduce here.

4. The Trace and the Square-Bracket Operation

Recall the notion of a *minimal walk* from a countable ordinal β to a smaller ordinal α along the fixed C-sequence C_ξ $(\xi < \omega_1)$: $\beta = \beta_0 > \beta_1 > \ldots > \beta_n = \alpha$ where $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$. Recall also the notion of a *trace*

$$\mathrm{Tr}(\alpha, \beta) = \{\beta_0, \beta_1, \ldots, \beta_n\},$$

the finite set of places visited in the minimal walk from β to α. The following simple fact about the trace lies at the heart of all known definitions of square-bracket operations not only on ω_1 but also at higher cardinalities.

4.1 Lemma. *For every uncountable subset* Γ *of* ω_1 *the union of* $\mathrm{Tr}(\alpha, \beta)$ *for* $\alpha < \beta$ *in* Γ *contains a closed and unbounded subset of* ω_1.

Proof. It suffices to show that the union of traces contains every countable limit ordinal δ such that $\sup(\Gamma \cap \delta) = \delta$. Pick an arbitrary $\beta \in \Gamma \setminus \delta$ and let

$$\beta = \beta_0 > \beta_1 > \cdots > \beta_k = \delta$$

be the minimal walk from β to δ. Let $\gamma < \delta$ be an upper bound of all sets of the form $C_{\beta_i} \cap \delta$ for $i < k$. By the choice of δ there is an $\alpha \in \Gamma \cap \delta$ above γ. Then the minimal walk from β to α starts as $\beta_0 > \beta_1 > \cdots > \beta_k$, so in particular δ belongs to $\mathrm{Tr}(\alpha, \beta)$. $\dashv$

We shall now see that it is possible to pick a single place $[\alpha\beta]$ in $\mathrm{Tr}(\alpha,\beta)$ so that Lemma 4.1 remains valid with $[\alpha\beta]$ in place of $\mathrm{Tr}(\alpha,\beta)$. Recall that by Lemma 1.11,

$$\Delta(\alpha,\beta) = \min\{\xi \leq \alpha : \rho_0(\xi,\alpha) \neq \rho_0(\xi,\beta)\}$$

is a successor ordinal. We shall be interested in its predecessor,

4.2 Definition. $\sigma(\alpha,\beta) = \Delta(\alpha,\beta) - 1$.

Thus, if $\xi = \sigma(\alpha,\beta)$, then $\rho_0(\xi,\alpha) = \rho_0(\xi,\beta)$ and so there is a natural isomorphism between $\mathrm{Tr}(\xi,\alpha)$ and $\mathrm{Tr}(\xi,\beta)$. We shall define $[\alpha\beta]$ by comparing the three sets $\mathrm{Tr}(\alpha,\beta), \mathrm{Tr}(\xi,\alpha)$ and $\mathrm{Tr}(\xi,\beta)$.

4.3 Definition. The *square-bracket operation* on ω_1 is defined as follows:

$$[\alpha\beta] = \min(\mathrm{Tr}(\alpha,\beta) \cap \mathrm{Tr}(\sigma(\alpha,\beta),\beta)) = \min(\mathrm{Tr}(\sigma(\alpha,\beta),\beta) \setminus \alpha).$$

Next, recall the function $\rho_0 : [\omega_1]^2 \to \omega^{<\omega}$ defined from the C-sequence C_ξ $(\xi < \omega_1)$ and the corresponding tree $T(\rho_0)$. For $\gamma < \omega_1$ let $(\rho_0)_\gamma$ be the fiber-mapping $: \gamma \to \omega^{<\omega}$ defined by $(\rho_0)_\gamma(\alpha) = \rho_0(\alpha,\gamma)$.

4.4 Lemma. *For every uncountable subset Γ of ω_1 the set of all ordinals of the form $[\alpha\beta]$ for some $\alpha < \beta$ in Γ contains a closed and unbounded subset of ω_1.*

Proof. For $t \in T(\rho_0)$ let $\Gamma_t = \{\gamma \in \Gamma : (\rho_0)_\gamma$ end-extends $t\}$. Let S be the collection of all $t \in T(\rho_0)$ for which Γ_t is uncountable. Clearly, S is a downward closed uncountable subtree of T. The lemma is established once we prove that every countable limit ordinal $\delta > 0$ with the following two properties can be represented as $[\alpha\beta]$ for some $\alpha < \beta$ in Γ:

(1) $\sup(\Gamma_t \cap \delta) = \delta$ for every $t \in S$ of length $< \delta$,

(2) every $t \in S$ of length $< \delta$ has two incomparable successors in S both of length $< \delta$.

Fix such a δ and choose $\beta \in \Gamma \setminus \delta$ such that $(\rho_0)_\beta \restriction \delta \in S$ and consider the minimal walk from β to δ:

$$\beta = \beta_0 > \beta_1 > \cdots > \beta_k = \delta.$$

Let $\gamma < \delta$ be an upper bound of all sets of the form $C_{\beta_i} \cap \delta$ for $i < k$. Since the restriction $t = (\rho_0)_\beta \restriction \gamma$ belongs to S, by (2) we can find one of its end-extensions s in S which is incomparable with $(\rho_0)_\beta$. It follows that for $\alpha \in \Gamma_s$, the ordinal $\sigma(\alpha,\beta)$ has the fixed value

$$\xi = \min\{\xi < |s| : s(\xi) \neq \rho_0(\xi,\beta)\} - 1.$$

Note that $\xi \geq \gamma$, so the walk $\beta \to \delta$ is a common initial part of walks $\beta \to \xi$ and $\beta \to \alpha$ for every $\alpha \in \Gamma_s \cap \delta$. Hence if we choose $\alpha \in \Gamma_s \cap \delta$ above $\min(C_\delta \setminus \xi)$ (which we can by (1)), we get that the walks $\beta \to \xi$ and $\beta \to \alpha$ never meet after δ. In other words for any such α, the ordinal δ is the minimum of $\mathrm{Tr}(\alpha, \beta) \cap \mathrm{Tr}(\xi, \beta)$, or equivalently δ is the minimum of $\mathrm{Tr}(\xi, \beta) \setminus \alpha$. ⊣

It should be clear that the above argument can easily be adjusted to give us the following slightly more general fact about the square-bracket operation.

4.5 Lemma. *For every uncountable family A of pairwise disjoint finite subsets of ω_1, all of the same size n, the set of all ordinals of the form $[a(1)b(1)] = [a(2)b(2)] = \cdots = [a(n)b(n)]$ for some $a \neq b$ in A contains a closed and unbounded subset of ω_1.*[11] ⊣

It turns out that the square-bracket operation can be used in constructions of various mathematical objects of complex behavior where all known previous constructions needed the Continuum Hypothesis or stronger enumeration principles. The usefulness of $[\cdot\cdot]$ in these constructions is based on the fact that $[\cdot\cdot]$ reduces the quantification over uncountable subsets of ω_1 to the quantification over closed unbounded subsets of ω_1. For example composing $[\cdot\cdot]$ with a unary operation $* : \omega_1 \longrightarrow \omega_1$ which takes each of the values stationary many times one gets the following fact about the mapping $c(\alpha, \beta) = [\alpha\beta]^*$.

4.6 Theorem. *There is a mapping $c : [\omega_1]^2 \longrightarrow \omega_1$ which takes all the values from ω_1 on any square $[\Gamma]^2$ of some uncountable subset Γ of ω_1.* ⊣

Note that the basic C-sequence C_α $(\alpha < \omega_1)$ which we have fixed at the beginning of this chapter can be used to actually define a unary operation $* : \omega_1 \longrightarrow \omega_1$ which takes each of the ordinals from ω_1 stationarily many times. So the projection $[\alpha\beta]^*$ can actually be defined in our basic structure $(\omega_1, \omega, \vec{C})$. We are now at the point to see that our basic structure is actually rigid.

4.7 Lemma. *The algebraic structure $(\omega_1, [\cdot\cdot], *)$ has no nontrivial automorphisms.*

Proof. Let h be a given automorphism of $(\omega_1, [\cdot\cdot], *)$. If the set Γ of fixed points of h is uncountable, h must be the identity map. To see this, consider a $\xi < \omega_1$. By the property of the map $c(\alpha, \beta) = [\alpha\beta]^*$ stated in Theorem 4.6

[11] For a finite set x of ordinals of size n we use the notation $x(1), x(2), \ldots, x(n)$ or $x(0), x(1), \ldots, x(n-1)$, depending on the context, for the enumeration of x according to the natural ordering on the ordinals.

there exists a $\gamma < \delta$ in Γ such that $[\gamma\delta]^* = \xi$. Applying h to this equation we get

$$h(\xi) = h([\gamma\delta]^*) = (h([\gamma\delta]))^* = [h(\gamma)h(\delta)]^* = [\gamma\delta]^* = \xi.$$

It follows that $\Delta = \{\delta < \omega_1 : h(\delta) \neq \delta\}$ is in particular uncountable. Shrinking Δ and replacing h by h^{-1}, if necessary, we may safely assume that $h(\delta) > \delta$ for all $\delta \in \Delta$. Consider a $\xi < \omega_1$ and let S_ξ be the set of all $\alpha < \omega_1$ such that $\alpha^* = \xi$. By our choice of $*$ the set S_ξ is stationary. By Lemma 4.5 applied to the family $A = \{\{\delta, h(\delta)\} : \delta \in \Delta\}$ we can find $\gamma < \delta$ in Δ such that $[\gamma\delta] = [h(\gamma)h(\delta)]$ belongs to S_ξ, or in other words,

$$[\gamma\delta]^* = [h(\gamma)h(\delta)]^* = \xi.$$

Since $[h(\gamma)h(\delta)]^* = h([\gamma\delta]^*)$ we conclude that $h(\xi) = \xi$. Since ξ was an arbitrary countable ordinal, this shows that h is the identity map. ⊣

We give now an application of this rigidity result to a problem in model theory about the quantifier Qx = 'there exist uncountably many x' and its higher dimensional analogues $Q^n x_1 \cdots x_n$ = 'there exist an uncountable n-cube many $x_1, \cdots, x_n$'. By a result of Ebbinghaus and Flum [16] (see also [44]) every model of every sentence of $L(Q)$ has nontrivial automorphisms. However we shall now see that this is no longer true about the quantifier Q^2.

4.8 Example. *A sentence of $L(Q^2)$ with only rigid models.* The sentence ϕ will talk about one unary relation N, one binary relation $<$ and two binary functional symbols C and E. It is the conjunction of the following seven sentences

(ϕ_1) $Qx\ x = x$,

(ϕ_2) $\neg Qx\ N(x)$,

(ϕ_3) $<$ is a total ordering,

(ϕ_4) E is a symmetric binary operation,

(ϕ_5) $\forall x < y\ N(E(x, y))$,

(ϕ_6) $\forall x < y < z\ E(x, z) \neq E(y, z)$,

(ϕ_7) $\forall x \forall n \{N(n) \rightarrow \neg Q^2 uv[\exists u' < u \exists v' < v(u' \neq v' \wedge E(u', u) = E(v', v) = n) \wedge \forall u' < u\ \forall v' < v(E(u', u) = E(v', v) = n \rightarrow (C(u', v') \neq x \vee C(u, v) \neq x))]\}$.

The model of ϕ that we have in mind is the model $(\omega_1, \omega, <, c, e)$ where $c(\alpha, \beta) = [\alpha\beta]^*$ and $e : [\omega_1]^2 \longrightarrow \omega$ is any mapping such that $e(\alpha, \gamma) \neq e(\beta, \gamma)$ whenever $\alpha < \beta < \gamma$ (e.g. we can take $e = \bar{\rho}_1$ or $e = \bar{\rho}$). The sentence ϕ_7 is simply saying that for every $\xi < \omega_1$ and every uncountable family A of pairwise disjoint unordered pairs of countable ordinals there exist $a \neq b$ in A such that

$$c(\min a, \min b) = c(\max a, \max b) = \xi.$$

This is a consequence of Lemma 4.5 and the fact that $S_\xi = \{\alpha : \alpha^* = \xi\}$ is a stationary subset of ω_1. These are the properties of $[\cdot\cdot]$ and $*$ which we have used in the proof of Lemma 4.7 in order to prove that $(\omega_1, [\cdot\cdot], *)$ is a rigid structure. So a quite analogous proof will show that any model $(M, N, <, C, E)$ of ϕ must be rigid. ⊣

The crucial property of $[\cdot\cdot]$ stated in Lemma 4.5 can also be used to provide a negative answer to the basis problem for uncountable graphs by constructing a large family of pairwise orthogonal uncountable graphs.

4.9 Definition. For a subset Γ of ω_1, let $\mathcal{G}_\Gamma$ be the graph whose vertex-set is ω_1 and whose edge-set is equal to $\{\{\alpha, \beta\} : [\alpha\beta] \in \Gamma\}$.

4.10 Lemma. *If the symmetric difference between Γ and Δ is a stationary subset of ω_1, then the corresponding graphs $\mathcal{G}_\Gamma$ and $\mathcal{G}_\Delta$ are orthogonal to each other, i.e. they do not contain uncountable isomorphic subgraphs.* ⊣

We have seen above that comparing $[\cdot\cdot]$ with a map $\pi : \omega_1 \longrightarrow I$ where I is some set of mathematical objects/requirements in such a way that each object/requirement is given a stationary preimage, gives us a way to meet each of these objects/requirements in the square of any uncountable subset of ω_1. This observation is the basis of all known applications of the square-bracket operation. A careful choice of I and $\pi : \omega_1 \longrightarrow I$ gives us a projection of the square-bracket operation that can be quite useful. So let us illustrate this on yet another example.

4.11 Definition. Let $\mathcal{H}$ be the collection of all maps $h : 2^n \longrightarrow \omega_1$ where n is a positive integer denoted by $n(h)$. Choose a mapping $\pi : \omega_1 \longrightarrow \mathcal{H}$ which takes each value from $\mathcal{H}$ stationarily many times. Choose also a one-to-one sequence r_α $(\alpha < \omega_1)$ of elements of the Cantor set 2^ω. Note that both these objects can actually be defined in our basic structure $(\omega_1, \omega, \vec{C})$. Consider the following projection of the square-bracket operation:

$$[\![\alpha\beta]\!] = \pi([\alpha\beta])(r_\alpha \restriction n(\pi([\alpha\beta]))).$$

It is easily checked that the property of $[\cdot\cdot]$ stated in Lemma 4.5 corresponds to the following property of the projection $[\![\alpha\beta]\!]$:

4.12 Lemma. *For every uncountable family A of pairwise disjoint finite subsets of ω_1, all of the same size n, and for every n-sequence $\xi_1, \ldots, \xi_n$ of countable ordinals there exist a and b in A such that $[\![a(i)b(i)]\!] = \xi_i$ for $i = 1, \ldots, n$.* $\dashv$

This projection of $[\cdot\cdot]$ leads to an interesting example of a Banach space with 'few' operators, which we will now describe.

4.13 Theorem. *There is a nonseparable reflexive Banach space E with the property that every bounded linear operator $T : E \longrightarrow E$ can be expressed as $T = \lambda I + S$ where λ is a scalar, I the identity operator of E, and S an operator with separable range.*

Proof. Let $I = 3 \times [\omega_1]^{<\omega}$ and let us identify the index-set I with ω_1, i.e. pretend that $[\![\cdot\cdot]\!]$ takes its values in I rather than ω_1. Let $[\![\cdot\cdot]\!]_0$ and $[\![\cdot\cdot]\!]_1$ be the two projections of $[\![\cdot\cdot]\!]$.

$$\mathcal{G} = \{G \in [\omega_1]^{<\omega} : [\![\alpha\beta]\!]_0 = 0 \text{ for all } \{\alpha, \beta\} \in [G]^2\},$$

$$\mathcal{H} = \{H \in [\omega_1]^{<\omega} : [\![\alpha\beta]\!]_0 = 1 \text{ for all } \{\alpha, \beta\} \in [H]^2\}.$$

Let $\mathcal{K}$ be the collection of all finite sets $\{\{\alpha_i, \beta_i\} : i < k\}$ of pairs of countable ordinals such that for all $i < j < k$:

(i) $\max\{\alpha_i, \beta_i\} < \min\{\alpha_j, \beta_j\}$,

(ii) $[\![\alpha_i\alpha_j]\!]_0 = [\![\beta_i\beta_j]\!]_0 = 2$,

(iii) $[\![\alpha_i\alpha_j]\!]_1 = [\![\beta_i\beta_j]\!]_1 = \{\alpha_l : l < i\} \cup \{\beta_l : l < i\}$.

The following properties of $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ should be clear:

(1) $\mathcal{G}$ and $\mathcal{H}$ contain all the singletons, are closed under subsets and they are 1-orthogonal to each other in the sense that $\mathcal{G} \cap \mathcal{H}$ contains no doubleton.

(2) $\mathcal{G}$ and $\mathcal{H}$ are both 2-orthogonal to the family of the unions of members of $\mathcal{K}$.

(3) If K and L are two distinct members of $\mathcal{K}$, then there are no more than 5 ordinals α such that $\{\alpha, \beta\} \in K$ and $\{\alpha, \gamma\} \in L$ for some $\beta \neq \gamma$.

(4) For every sequence $\{\alpha_\xi, \beta_\xi\}$ ($\xi < \omega_1$) of pairwise disjoint pairs of countable ordinals there exist arbitrarily large finite sets $\Gamma, \Delta \subseteq \omega_1$ such that $\{\alpha_\xi : \xi \in \Gamma\} \in \mathcal{G}$, $\{\beta_\xi : \xi \in \Gamma\} \in \mathcal{H}$ and $\{\{\alpha_\xi, \beta_\xi\} : \xi \in \Delta\} \in \mathcal{K}$.

For a function x from ω_1 into $\mathbb{R}$, set

$$\|x\|_{\mathcal{H},2} = \sup\{(\sum_{\alpha\in H} x(\alpha)^2)^{\frac{1}{2}} : H \in \mathcal{H}\},$$
$$\|x\|_{\mathcal{K},2} = \sup\{(\sum_{\{\alpha,\beta\}\in K} (x(\alpha) - x(\beta))^2)^{\frac{1}{2s}} : K \in \mathcal{K}\}.$$

Let $\|\cdot\| = \max\{\|\cdot\|_\infty, \|\cdot\|_{\mathcal{H},2}, \|\cdot\|_{\mathcal{K},2}\}$ and define $\bar{E}_2 = \{x : \|x\| < \infty\}$. Let 1_α be the characteristic function of $\{\alpha\}$. Finally, let E_2 be the closure of the linear span of $\{1_\alpha : \alpha \in \omega_1\}$ inside $(\bar{E}_2, \|\cdot\|)$. The following facts about the norm $\|\cdot\|$ are easy to establish using the properties of the families $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ listed above.

(i) If x is supported by some $G \in \mathcal{G}$, then $\|x\| \leq 2 \cdot \|x\|_\infty$.

(ii) If x is supported by $\bigcup K$ for some K in $\mathcal{K}$, then $\|x\| \leq 10 \cdot \|x\|_\infty$.

The role of the seminorm $\|\cdot\|_{\mathcal{H},2}$ is to ensure that every bounded operator $T : E_2 \longrightarrow E_2$ can be expressed as $D + S$, where D is a diagonal operator relative to the basis[12] 1_α $(\alpha < \omega_1)$ and where S has separable range.

Note that $\|x\| \leq 2\|x\|_2$ for all $x \in \ell_2(\omega_1)$. It follows that $\ell_2(\omega_1) \subseteq E_2$ and the inclusion is a bounded linear operator. Note also that $\ell_2(\omega_1)$ is a dense subset of E_2. Therefore E_2 is a weak compactly generated space. For example, $W = \{x \in \ell_2(\omega_1) : \|x\|_2 \leq 1\}$ is a weakly compact subset of E_2 and its linear span is dense in E_2. To get a reflexive example out of E_2 one uses an interpolation method of Davis, Figiel, Johnson and Pelczynski [9] as follows. Let p_n be the Minkowski functional of the set $2^n W + 2^{-n}\mathrm{Ball}(E_2)$.[13] Let

$$E = \{x \in E_2 : \|x\|_E = (\sum_{n=0}^{\infty} p_n(x)^2)^{\frac{1}{2}} < \infty\}.$$

By [9, Lemma 1], E is a reflexive Banach space and $\ell_2(\omega_1) \subseteq E \subseteq E_2$ are continuous inclusions. Note that $p_n(x) < r$ iff $x = y + z$ for some $y \in E_2$ and $z \in \ell_2(\omega_1)$ such that $\|y\| < 2^{-n}r$ and $\|z\|_2 < 2^n r$. Then the reflexive version of the space also has the property that every bounded operator $T : E \longrightarrow E$ has the form $\lambda I + S$. ⊣

4.14 Remark. The above example is reproduced from Wark [80] who based his example on a previous construction due to Shelah and Steprans [54]. The reader is referred to these sources and to [76] for more information.

We only mention yet another interesting application of the square-bracket operation, given recently by Erdős, Jackson and Mauldin [18]:

4.15 Theorem. *For every positive integer n there exist collections $\mathcal{H}$ and X of hyperplanes and points of $\mathbb{R}^n$, respectively, and a coloring $P : \mathcal{H} \longrightarrow \omega$ such that:*

[12] Indeed it can be shown that 1_α $(\alpha < \omega_1)$ is a 'transfinite basis' of E_2 in the sense of [55]. So every vector x of E_2 has a unique representation as $\Sigma_{\alpha<\omega_1} x(\alpha)1_\alpha$ and the projection operators $P_\beta : E_2 \to E_2 \restriction \beta$ $(\beta < \omega_1)$ are uniformly bounded.

[13] I.e. $p_n(x) = \inf\{\lambda > 0 : x \in \lambda B\}$, where B denotes this set.

(1) *any n hyperplanes of distinct colors meet in at most one point,*

(2) *there is no coloring $Q : X \longrightarrow \omega$ such that for every $H \in \mathcal{H}$ there exists at most $n-1$ points x in $X \cap H$ such that $Q(x) = P(H)$.* ⊣

Let us now introduce yet another projection of the square-bracket operation which has some universality properties.

4.16 Definition. Let $\mathcal{H}$ now be the collection of all maps $h : 2^n \times 2^n \longrightarrow \omega_1$ where $n = n(h) < \omega$ and let π be a mapping from ω_1 onto $\mathcal{H}$ that takes each of the values stationarily many times. Define a new operation on ω_1 by

$$|\alpha\beta| = \pi([\alpha\beta])(r_\alpha \restriction n(\pi([\alpha\beta])), r_\beta \restriction n(\pi([\alpha\beta]))).$$

4.17 Lemma. *For every positive integer n, every uncountable subset Γ of ω_1 and every symmetric $n \times n$-matrix M of countable ordinals there is a one-to-one $\phi : n \longrightarrow \Gamma$ such that $|\phi(i)\phi(j)| = M(i,j)$ for $i, j < n$.* ⊣

5. A Square-Bracket Operation on a Tree

In this section we try to show that the basic idea of the square-bracket operation on ω_1 can perhaps be more easily grasped by working on an arbitrary special Aronszajn tree rather than $T(\rho_0)$. So let $T = \langle T, <_T \rangle$ be a fixed special Aronszajn tree and let $a : T \longrightarrow \omega$ be a fixed map witnessing this, i.e. a mapping with the property that $a^{-1}(\{n\})$ is an antichain of T for all $n < \omega$. We shall assume that for every $s, t \in T$ the greatest lower bound $s \wedge t$ exists in T. For $t \in T$ and $n < \omega$, set

$$F_n(t) = \{s \leq_T t : s = t \text{ or } a(s) \leq n\}.$$

Finally, for $s, t \in T$ with $\mathrm{ht}(s) \leq \mathrm{ht}(t)$, let

$$[st]_T = \min\{v \in F_{a(s \wedge t)}(t) : \mathrm{ht}(v) \geq \mathrm{ht}(s)\}.$$

(If $\mathrm{ht}(s) \geq |ht(t)$ we let $[st]_T = [ts]_T$.)

The following fact corresponds to Lemma 4.4 when $T = T(\rho_0)$.

5.1 Lemma. *If X is an uncountable subset of T, the set of nodes of T of the form $[st]_T$ for some $s, t \in X$ intersects a closed and unbounded set of levels of T.* ⊣

We do not give a proof of this fact as it is almost identical to the proof of Lemma 4.4 which deals with the special case $T = T(\rho_0)$. But one can go further and show that $[\cdot\cdot]_T$ shares all the other properties of the square-bracket operation $[\cdot\cdot]$ described in the previous section. Some of these properties, however, are easier to visualize and prove in the general context. For example, consider the following fact which in the case $T = T(\rho_0)$ is the essence of Lemma 4.

5.2 Lemma. *Suppose $A \subseteq T$ is an uncountable antichain and that for each $t \in A$ be given a finite set F_t of its successors. Then for every stationary set $\Gamma \subseteq \omega_1$ there exists an arbitrarily large finite set $B \subseteq A$ such that the height of $[xy]_T$ belongs to Γ whenever $x \in F_s$ and $y \in F_t$ for some $s \neq t$ in B.* ⊣

Let us now examine in more detail the collection of graphs $\mathcal{G}_\Gamma(\Gamma \subseteq \omega_1)$ of 4.9 but in the present more general context.

5.3 Definition. For $\Gamma \subseteq \omega_1$, let $K_\Gamma = \{\{s,t\} \in [T]^2 : \mathrm{ht}([st]_T) \in \Gamma\}$.

Working as in 4.10 one shows that (T, K_Γ) and (T, K_Δ) have no isomorphic uncountable subgraph whenever the symmetric difference between Γ and Δ is a stationary subset of ω_1, i.e. whenever they represent different members of the quotient algebra $\mathcal{P}(\omega_1)/NS$. In particular, K_Γ contains no square $[X]^2$ of an uncountable set $X \subseteq T$ whenever Γ contains no closed and unbounded subset of ω_1. The following fact is a sort of converse to this. Its proof can be found in [76].

5.4 Lemma. *If Γ contains a closed and unbounded subset of ω_1 then there is a proper forcing notion introducing an uncountable set $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$.* ⊣

5.5 Corollary. *The graph K_Γ contains the square of some uncountable subset of T in some ω_1-preserving forcing extension if and only if Γ is a stationary subset of ω_1.*

Proof. If Γ is disjoint from a closed and unbounded subset then in any ω_1-preserving forcing extension its complement $\Delta = \omega_1 \setminus \Gamma$ will be a stationary subset of ω_1. So by the basic property 5.1 of the square-bracket operation no such a forcing extension will contain an uncountable set $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$. On the other hand, if Γ is a stationary subset of ω_1, going first to some standard ω_1-preserving forcing extension in which Γ contains a closed and unbounded subset of ω_1 and then applying Lemma 5.4, we get an ω_1-preserving forcing extension having an uncountable set $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$. ⊣

5.6 Remark. Corollary 5.5 gives us a further indication of the extreme complexity of the class of graphs on the vertex-set ω_1. It also bears some relevance to the recent work of Woodin [84] who, working in his $\mathbb{P}_{\max}$-forcing extension, was able to associate a stationary subset of ω_1 to any partition of $[\omega_1]^2$ into two pieces. So one may view Corollary 5.5 as some sort of converse to this since in the $\mathbb{P}_{\max}$-extension one is able to get a sufficiently generic filter to the forcing notion $\mathbb{P} = \mathbb{P}_\Gamma$ of Lemma 5.4 that would give us an uncountable $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$. In other words, under a bit of PFA or Woodin's axiom (*), a set $\Gamma \subseteq \omega_1$ contains a closed and unbounded subset of ω_1 if and only if K_Γ contains $[X]^2$ for some uncountable $X \subseteq T$.

6. Special Trees and Mahlo Cardinals

One of the most basic questions frequently asked about set-theoretical trees is the question whether they contain any *cofinal branch*, a branch that intersects each level of the tree. The fundamental importance of this question has already been realized in the work of Kurepa [35] and then later in the works of Erdős and Tarski in their respective attempts to develop the theory of partition calculus and large cardinals (see [20]). A tree T of height equal to some regular cardinal θ may not have a cofinal branch for a very special reason as the following definition indicates.

6.1 Definition. For a tree $T = \langle T, <_T\rangle$, a function $f : T \to T$ is *regressive* if $f(t) <_T t$ for every $t \in T$ that is not a minimal node of T. A tree T of height θ is *special* if there is a regressive map $f : T \longrightarrow T$ with the property that the f-preimage of every point of T can be written as the union of $< \theta$ antichains of T.

This definition in case $\theta = \omega_1$ reduces indeed to the old definition of special tree, a tree that can be decomposed into countably many antichains. More generally we have the following:

6.2 Lemma. *If θ is a successor cardinal then a tree T of height θ is special if and only if T is the union of $< \theta$ antichains.* ⊣

The new definition, however, seems to be the right notion of speciality as it makes sense even if θ is a limit cardinal.

6.3 Definition. A tree T of height θ is *Aronszajn* if T has no cofinal branches and if every level of T has size $< \theta$.

Recall the well-known characterization of weakly compact cardinals due to Tarski and his collaborators: a strongly inaccessible cardinal θ is weakly compact if and only if there are no Aronszajn trees of height θ. We supplement this with the following:

6.4 Theorem. *The following are equivalent for a strongly inaccessible cardinal θ:*

(1) *θ is Mahlo.*

(2) *there are no special Aronszajn trees of height θ.*

Proof. Suppose θ is a Mahlo cardinal and let T be a given tree of height θ all of whose levels have size $< \theta$. To show that T is not special let $f : T \longrightarrow T$ be a given regressive mapping. By our assumption of θ there is an elementary submodel M of some large enough structure H_κ such that $T, f \in M$ and $\lambda = M \cap \theta$ is a regular cardinal $< \theta$. Note that $T \restriction \lambda$ is a subset of M and since this tree of height λ is clearly not special, there is an

$t \in T\restriction\lambda$ such that the preimage $f^{-1}(\{t\})$ is not the union of $< \lambda$ antichains. Using the elementarity of M we conclude that $f^{-1}(\{t\})$ is actually not the union of $< \theta$ antichains.

The proof that (2) implies (1) uses the method of minimal walks in a rather crucial way. So suppose to the contrary that our cardinal contains a closed and unbounded subset C consisting of singular strong limit cardinals. Using C, we choose a C-sequence C_α $(\alpha < \theta)$ such that: $C_{\alpha+1} = \{\alpha\}$, $C_\alpha = (\bar{\alpha}, \alpha)$ for α limit such that $\bar{\alpha} = \sup(C \cap \alpha) < \alpha$ but if $\alpha = \sup(C \cap \alpha)$ then take C_α such that:

(a) $\mathrm{tp}(C_\alpha) = \mathrm{cf}(\alpha) < \min(C_\alpha)$,

(b) $\xi = \sup(C_\alpha \cap \xi)$ implies $\xi \in C$,

(c) $\xi \in C_\alpha$ and $\xi > \sup(C_\alpha \cap \xi)$ imply that $\xi = \eta + 1$ for some $\eta \in C$.

Given the C-sequence C_α $(\alpha < \theta)$ we have the notion of minimal walk along the sequence and various distance functions defined above. In this proof we are particularly interested in the function ρ_0 from $[\theta]^2$ into the set $\mathbb{Q}_\theta$ of all finite sequences of ordinals from θ:

$$\rho_0(\alpha, \beta) = \langle \mathrm{tp}(C_\beta \cap \alpha) \rangle ^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha))$$

where we stipulate that $\rho_0(\gamma, \gamma) = 0$ for all $\gamma < \theta$. We would like to show that the tree

$$T(\rho_0) = \{(\rho_0)_\beta\restriction\alpha : \alpha \leq \beta < \theta\}$$

is a special Aronszajn tree of height θ. Note that the size of the αth level $(T(\rho_0))_\alpha$ of $T(\rho_0)$ is controlled in the following way:

$$|(T(\rho_0))_\alpha| \leq |\{C_\beta \cap \alpha : \alpha \leq \beta < \theta\}| + |\alpha + \omega|. \tag{I.4}$$

So under the present assumption that θ is a strongly inaccessible cardinal, all levels of $T(\rho_0)$ do indeed have size $< \theta$. It remains to define the regressive map

$$f : T(\rho_0) \longrightarrow T(\rho_0)$$

that will witness speciality of $T(\rho_0)$. Note that it really suffices defining f on all levels whose index belong to our club C of singular cardinals. So let $t = (\rho_0)_\beta\restriction\alpha$ be a given node of T such that $\alpha \in C$ and $\alpha \leq \beta < \theta$. Note that by our choice of the C-sequence every term of the finite sequence of ordinals $\rho_0(\alpha, \beta)$ is strictly smaller than α. So, if we let $f(t) = t\restriction\ulcorner\rho_0(\alpha, \beta)\urcorner$, where $\ulcorner\cdot\urcorner$ is a standard coding of finite sequences of ordinals by ordinals, we get a regressive map. To show that f is one-to-one on chains of $T(\rho_0)$, which would be more than sufficient, suppose $t_i = (\rho_0)_{\beta_i}\restriction\alpha_i$ $(i < 2)$ are two nodes such that $t_0 \subsetneqq t_1$. Our choice of the C-sequence allows us to deduce the following general fact about the corresponding ρ_0-function as in the case $\theta = \omega_1$ dealt with above in Lemma 1.11.

> If $\alpha \leq \beta \leq \gamma$, α is a limit ordinal, and if $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$ for all $\xi < \alpha$, then $\rho_0(\alpha, \beta) = \rho_0(\alpha, \gamma)$.

Applying this to the triple of ordinals α_0, β_0 and β_1 we conclude that $\rho_0(\alpha_0, \beta_0) = \rho_0(\alpha_0, \beta_1)$. Now observe another fact about the ρ_0-function whose proof is identical to that of $\theta = \omega_1$ dealt with above in Lemma 1.10.

$$\text{If } \alpha \leq \beta \leq \gamma \text{ then } \rho_0(\alpha, \gamma) <_r \rho_0(\beta, \gamma).$$

Applying this to the triple $\alpha_0 < \alpha_1 \leq \beta_1$ we in particular have that $\rho_0(\alpha_0, \beta_1) \neq \rho_0(\alpha_1, \beta_1)$. Combining this with the above equality gives us that $\rho_0(\alpha_0, \beta_0) \neq \rho_0(\alpha_1, \beta_1)$ and therefore that $f(t_0) \neq f(t_1)$. ⊣

A similar argument gives us the following characterization of Mahlo cardinals due to Hajnal, Kanamori and Shelah [23] which improves a bit an earlier characterization of this sort due to Schmerl [50]. Its proof can also be found in [76].

6.5 Theorem. *A cardinal θ is a Mahlo cardinal if and only if every regressive map f defined on a cube $[C]^3$ of a closed and unbounded subset of θ has an infinite min-homogeneous set $X \subseteq C$.*[14]

Starting from the case $n = 1$ one can now easily deduce the following characterization also due to Hajnal, Kanamori and Shelah [23].

6.6 Theorem. *The following are equivalent for an uncountable cardinal θ and a positive integer n:*

(1) *θ is n-Mahlo.*

(2) *Every regressive map defined on $[C]^{n+2}$ for some closed and unbounded subset C of θ has an infinite min-homogeneous subset.* ⊣

The proof of Theorem 6.4 gives us the following well-known fact, first established by Silver (see [40]) when θ is a successor of a regular cardinal, which we are going to reprove now.

6.7 Theorem. *If θ is a regular uncountable cardinal which is not Mahlo in the constructible universe, then there is a constructible special Aronszajn tree of height θ.*

Proof. Working in L we choose a closed and unbounded subset C of θ consisting of singular ordinals and a C-sequence C_α $(\alpha < \theta)$ such that $C_{\alpha+1} = \{\alpha\}$, $C_\alpha = (\bar{\alpha}, \alpha)$ when $\bar{\alpha} = \sup(C \cap \alpha) < \alpha$, while if α is a limit point of C we take C_α to have the following properties:

[14] Recall, that X is min-homogeneous for f if $f(\alpha, \beta, \gamma) = f(\alpha', \beta', \gamma')$ for every pair $\alpha < \beta < \gamma$ and $< \alpha' < \beta' < \gamma'$ of triples of elements of X such that $\alpha = \alpha'$.

(i) $\xi = \sup(C_\alpha \cap \xi)$ implies $\xi \in C$,

(ii) $\xi > \sup(C_\alpha \cap \xi)$ implies $\xi = \eta + 1$ for some $\eta \in C$.

We choose the C-sequence to also have the following crucial property:

(iii) $|\{C_\alpha \cap \xi : \xi \leq \alpha < \theta\}| \leq |\xi| + \aleph_0$ for all $\xi < \theta$.

It is clear then that the tree $T(\rho_0)$, where ρ_0 is the ρ_0-function of C_α $(\alpha < \theta)$, is a constructible special Aronszajn tree of height θ. ⊣

We are also in a position to deduce the following well-known fact.

6.8 Theorem. *The following are equivalent for a successor cardinal θ:*

(a) *There is a special Aronszajn tree of height θ.*

(b) *There is a C-sequence C_α $(\alpha < \theta)$ such that* $\text{tp}(C_\alpha) \leq \theta^-$ *for all α and such that $\{C_\alpha \cap \xi : \alpha < \theta\}$ has size $\leq \theta^-$ for all $\xi < \theta$.*

Proof. If C_α $(\alpha < \theta)$ is a C-sequence satisfying (b) and if ρ_0 is the associated ρ_0-function then $T(\rho_0)$ is a special Aronszajn tree of height θ. Suppose $<_T$ is a special Aronszajn tree ordering on θ such that $[\theta^- \cdot \alpha, \theta^- \cdot (\alpha+1))$ is its αth level. Let C be the club of ordinals $< \theta$ divisible by θ^-. Let $f : \theta \longrightarrow \theta^-$ be such that the f-preimage of every ordinal $< \theta^-$ is an antichain of the tree $(\theta, <_T)$. We choose a C-sequence C_α $(\alpha < \theta)$ such that $C_{\alpha+1} = \{\alpha\}$, $C_\alpha = (\bar{\alpha}, \alpha)$ for α limit with the property that $\bar{\alpha} = \sup(C \cap \alpha) < \alpha$, but if α is a limit point of C we take C_α more carefully as follows: $C_\alpha = \{\alpha_\xi : \xi < \eta\}$ where

$$
\begin{aligned}
\alpha_\lambda &= \sup\{\alpha_\xi : \xi < \lambda\} \text{ for } \lambda \text{ limit } < \eta,\\
\alpha_0 &= \text{the } <_T\text{-predecessor of } \alpha \text{ with minimal } f\text{-image},\\
\alpha_{\xi+1} &= \text{the } <_T\text{-predecessor of } \alpha \text{ with minimal } f\text{-image subject}\\
&\quad \text{to the requirement that } f(\alpha_{\xi+1}) > f(\alpha_{\zeta+1}) \text{ for all } \zeta < \xi,\\
\eta &= \text{the limit ordinal} \leq \theta^- \text{ where the process stops, i.e.}\\
&\quad \sup\{f(\alpha_{\xi+1}) : \xi < \eta\} = \theta^-.
\end{aligned}
$$

Note that if α and β are two limit points of C and if $\gamma <_T \alpha, \beta$ then $C_\alpha \cap \gamma = C_\beta \cap \gamma$. From this one concludes that the C-sequence is locally small, i.e. that $\{C_\alpha \cap \gamma : \gamma \leq \alpha < \theta\}$ has size $\leq \theta^-$ for all $\gamma < \theta$. ⊣

6.9 Corollary. *If $\theta^{<\theta} = \theta$ then there exists a special Aronszajn tree of height θ^+.* ⊣

6.10 Corollary. *In the constructible universe, special Aronszajn trees of any regular uncountable non-Mahlo height exist.* ⊣

6.11 Remark. In a large portion of the literature on this subject the notion of a special Aronszajn tree of height equal to some successor cardinal θ^+ is somewhat weaker, equivalent to the fact that the tree can be embedded inside the tree $\{f : \alpha \longrightarrow \theta : \alpha < \theta \;\&\; f \text{ is } 1-1\}$. One would get our notion of speciality by restricting the tree on successor ordinals losing thus the frequently useful property of a tree that different nodes of the same limit height have different sets of predecessors. The result 6.9 in this weaker form is due to Specker [56], while the result 6.10 is essentially due to Jensen [27].

7. The Weight Function on Successor Cardinals

In this section we assume that $\theta = \kappa^+$ and we fix a C-sequence C_α $(\alpha < \kappa^+)$ such that

$$\mathrm{tp}(C_\alpha) \leq \kappa \text{ for all } \alpha < \kappa^+.$$

Let $\rho_1 : [\kappa^+]^2 \longrightarrow \kappa$ be defined recursively by

$$\rho_1(\alpha, \beta) = \max\{\mathrm{tp}(C_\beta \cap \alpha), \rho_1(\alpha, \min(C_\beta \setminus \alpha))\},$$

where we stipulate that $\rho_1(\gamma, \gamma) = 0$ for all $\gamma < \kappa^+$.

7.1 Lemma. $|\{\xi \leq \alpha : \rho_1(\xi, \alpha) \leq \nu\}| \leq |\nu| + \aleph_0$ *for all* $\alpha < \kappa^+$ *and* $\nu < \kappa$.

Proof. Let ν^+ be the first infinite cardinal above the ordinal ν. The proof of the conclusion is by induction on α. So let $\Gamma \subseteq \alpha$ be a given set of order-type ν^+. We need to find $\xi \in \Gamma$ such that $\rho_1(\xi, \alpha) > \nu$. This will clearly be true if there is an $\xi \in \Gamma$ such that $\mathrm{tp}(C_\alpha \cap \xi) > \nu$. So, we may assume that $\mathrm{tp}(C_\alpha \cap \xi) \leq \nu$ for all $\xi \in \Gamma$. Then there must be an ordinal $\alpha_1 \in C_\alpha$ such that $\Gamma_1 = \{\xi \in \Gamma : \alpha_1 = \min(C_\alpha \setminus \xi)\}$ has size ν^+. By the inductive hypothesis there is an $\xi \in \Gamma_1$ such that, $\rho_1(\xi, \alpha_1) > \nu \geq \mathrm{tp}(C_\alpha \cap \xi)$. It follows that

$$\rho_1(\xi, \alpha) = \max\{\mathrm{tp}(C_\alpha \cap \xi), \rho_1(\xi, \alpha_1)\} = \rho_1(\xi, \alpha_1) > \nu.$$

This finishes the proof. ⊣

7.2 Lemma. *If* κ *is regular, then* $\{\xi \leq \alpha : \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\}$ *has size* $< \kappa$ *for all* $\alpha < \beta < \kappa^+$.

Proof. The proof is by induction on α and β. Let $\Gamma \subseteq \alpha$ be a given set of order-type κ. We need to find $\xi \in \Gamma$ such that $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta)$. Let $\gamma = \sup(\Gamma)$, $\gamma_0 = \max(C_\beta \cap \gamma)$, and $\beta_0 = \min(C_\beta \setminus \gamma)$. Note that by our assumption on κ and the C-sequence, these two ordinals are well-defined and

$$\gamma_0 < \gamma \leq \beta_0 < \beta.$$

By Lemma 7.1 and the inductive hypothesis there is an ξ in $\Gamma \cap (\gamma_0, \gamma)$ such that

$$\rho_1(\xi, \alpha) = \rho_1(\xi, \beta_0) > \mathrm{tp}(C_\beta \cap \gamma).$$

It follows that $C_\beta \cap \gamma = C_\beta \cap \xi$ and $\beta_0 = \min(C_\beta \setminus \xi)$, and so

$$\rho_1(\xi, \beta) = \max\{\mathrm{tp}(C_\beta \cap \xi), \rho_1(\xi, \beta_0)\} = \rho_1(\xi, \beta_0) = \rho_1(\xi, \alpha).$$

⊣

7.3 Remark. The assumption about the regularity of κ in Lemma 7.2 is essential. For example, it can be seen (see [5, p.72]) that the conclusion of this lemma fails if κ is a singular limit of supercompact cardinals.

7.4 Definition. For κ regular, define $\bar{\rho}_1 : [\kappa^+]^2 \longrightarrow \kappa$ by

$$\bar{\rho}_1(\alpha, \beta) = 2^{\rho_1(\alpha,\beta)} \cdot (2 \cdot \mathrm{tp}\{\xi \leq \alpha : \rho_1(\xi, \beta) = \rho_1(\alpha, \beta)\} + 1).$$

7.5 Lemma. *If κ is a regular cardinal then*

(a) $\bar{\rho}_1(\alpha, \gamma) \neq \bar{\rho}_1(\beta, \gamma)$ *whenever* $\alpha < \beta < \gamma < \kappa^+$,

(b) $|\{\xi \leq \alpha : \bar{\rho}_1(\xi, \alpha) \neq \bar{\rho}_1(\xi, \beta)\}| < \kappa$ *whenever* $\alpha < \beta < \kappa^+$. ⊣

7.6 Remark. Note that Lemma 7.5 gives an alternative proof of Corollary 6.9 since under the assumption $\kappa^{<\kappa} = \kappa$ the tree $T(\bar{\rho}_1)$ will have levels of size at most κ. It should be noted that the coherent sequence $(\bar{\rho})_\alpha$ $(\alpha < \kappa^+)$ of one-to-one mappings is an object of independent interest which can be particularly useful in stepping-up combinatorial properties of κ to κ^+. It is also an object that has interpretations in such areas as the theory of Čech-Stone compactifications of discrete spaces (see, e.g. [81], [8], [47], [34], [14]). We have already noted that if κ is singular then we may no longer have the coherence property of Lemma 7.2. To get this property, one needs to make some additional assumption on the C-sequence C_α $(\alpha < \kappa^+)$, an assumption about the coherence of the C-sequence. This will be subject of some of the following chapters where we will concentrate on the finer function ρ rather than ρ_1.

8. The Number of Steps

The purpose of this section is to isolate a condition on C-sequences C_α $(\alpha < \theta)$ on regular uncountable cardinals θ as weak as possible subject to a requirement that the corresponding function

$$\rho_2(\alpha, \beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1$$

is in some sense nontrivial, and in particular, far from being constant. Without doubt the C-sequence $C_\alpha = \alpha$ $(\alpha < \theta)$ is the most trivial choice and the corresponding ρ_0-function gives no information about the cardinal θ. The following notion of the triviality of a C-sequence on θ seems to be only marginally different.

8.1 Definition. A C-sequence C_α $(\alpha < \theta)$ on a regular uncountable cardinal θ is *trivial* if there is a closed and unbounded set $C \subseteq \theta$ such that for every $\alpha < \theta$ there is a $\beta \geq \alpha$ with $C \cap \alpha \subseteq C_\beta$.

The proof of the following fact can be found in [76].

8.2 Theorem. *The following are equivalent for any C-sequence C_α $(\alpha < \theta)$ on a regular uncountable cardinal θ and the corresponding function ρ_2:*

(i) *C_α $(\alpha < \theta)$ is nontrivial.*

(ii) *For every family A of θ pairwise disjoint finite subsets of θ and every integer n there is a subfamily B of A of size θ such that $\rho_2(\alpha, \beta) > n$ for all $\alpha \in a$, $\beta \in b$ and $a \neq b$ in B.* ⊣

8.3 Corollary. *Suppose that C_α $(\alpha < \theta)$ is a nontrivial C-sequence and let $T(\rho_0)$ be the corresponding tree (see Section 6 above). Then every subset of $T(\rho_0)$ of size θ contains an antichain of size θ.*

Proof. Consider a subset K of $[\theta]^2$ of size θ which gives us a subset of $T(\rho_0)$ of size θ as follows: $\{(\rho_0)_\beta \restriction (\alpha + 1) \colon \{\alpha, \beta\} \in K\}$. Here, we are assuming without loss of generality that the set consists of successor nodes of $T(\rho_0)$. Clearly, we may also assume that the set takes at most one point from a given level of $T(\rho_0)$. Shrinking K further, we obtain that ρ_2 is constant on K. Let n be the constant value of $\rho_2 \restriction K$. Applying 8.2(ii) to K and n, we get $K_0 \subseteq K$ of size θ such that $\rho_2(\alpha, \delta) > n$ for all $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ from K_0 with properties $\alpha < \beta$, $\gamma < \delta$ and $\alpha < \gamma$. Then $\{(\rho_0)_\beta \restriction (\alpha + 1) \colon \{\alpha, \beta\} \in K_0\}$ is an antichain in $T(\rho_0)$. ⊣

8.4 Remark. It should be clear that nontrivial C-sequences exist on any successor cardinal. Indeed, with very little extra work one can show that nontrivial C-sequences exist for some inaccessible cardinals quite high in the Mahlo-hierarchy. To show how close this is to the notion of weak compactness, we will give the following characterization of it which is of independent interest.[15]

Similar arguments would prove the following result which gives us an interesting characterization of weakly compact cardinals (see [76]).

8.5 Theorem. *The following are equivalent for an inaccessible cardinal θ:*

[15] It turns out that every C-sequence on θ being trivial is not quite as strong as the weak compactness of θ. As pointed out to us by Donder and König, one can show this using a model of Kunen [33, §3].

(i) *θ is weakly compact.*

(ii) *For every C-sequence C_α $(\alpha < \theta)$ there is a closed and unbounded set $C \subseteq \theta$ such that for all $\alpha < \theta$ there is a $\beta \geq \alpha$ with $C_\beta \cap \alpha = C \cap \alpha$.* ⊣

We have already remarked that every successor cardinal $\theta = \kappa^+$ admits a nontrivial C-sequence C_α $(\alpha < \theta)$. It suffices to take the C_α's to be all of order-type $\leq \kappa$. It turns out that for such a C-sequence the corresponding ρ_2-function has a property that is considerably stronger than 8.2(ii). The proof of this can again be found in [76].

8.6 Theorem. *For every infinite cardinal κ there is a C-sequence on κ^+ such that the corresponding ρ_2-function has the following unboundedness property: for every family A of κ^+ pairwise disjoint subsets of κ^+, all of size $< \kappa$, and for every $n < \omega$ there exists a $B \subseteq A$ of size κ^+ such that $\rho_2(\alpha, \beta) > n$ whenever $\alpha \in a$ and $\beta \in b$ for some $a \neq b$ in B.* ⊣

Theorems 8.2 and 8.6 admit the following variation.

8.7 Theorem. *Suppose that a regular uncountable cardinal θ supports a nontrivial C-sequence and let ρ_2 be the associated function. Then for every integer n and every pair of θ-sized families A_0 and A_1, where the members of A_0 are pairwise disjoint bounded subsets of θ and the members of A_1 are pairwise disjoint finite subsets of θ, there exist $B_0 \subseteq A_0$ and $B_1 \subseteq A_1$ of size θ such that $\rho_2(\alpha, \beta) > n$ whenever $\alpha \in a$ and $\beta \in b$ for some $a \in B_0$ and $b \in B_1$ such that* $\sup(a) < \min(b)$. ⊣

9. Square Sequences

9.1 Definition. A C-sequence C_α $(\alpha < \theta)$ is a *square-sequence* if and only if it is *coherent*, i.e. it has the property that $C_\alpha = C_\beta \cap \alpha$ whenever α is a limit point of C_β.

Note that the nontriviality conditions appearing in Definition 8.1 and Theorem 8.5 coincide in the realm of square-sequences:

9.2 Lemma. *A square-sequence C_α $(\alpha < \theta)$ is trivial if and only if there is a closed and unbounded subset C of θ such that $C_\alpha = C \cap \alpha$ whenever α is a limit point of C.* ⊣

To a given square-sequence C_α $(\alpha < \theta)$ one naturally associates a tree ordering $<^2$ on θ as follows by letting $\alpha <^2 \beta$ if and only if α is a limit point of C_β. The triviality of C_α $(\alpha < \theta)$ is then equivalent to the statement that the tree $(\theta, <^2)$ has a chain of size θ. In fact, one can characterize the tree orderings $<_T$ on θ for which there exists a square sequence C_α $(\alpha < \theta)$ such that for all $\alpha < \beta < \theta$,

$$\alpha <_T \beta \text{ if and only if } \alpha \text{ is a limit point of } C_\beta. \tag{I.5}$$

9.3 Lemma. *A tree ordering $<_T$ on θ admits a square sequence C_α $(\alpha < \theta)$ satisfying* (I.5) *if and only if*

(i) *$\alpha <_T \beta$ can hold only for limit ordinals α and β such that $\alpha < \beta$,*

(ii) *$P_\beta = \{\alpha : \alpha <_T \beta\}$ is a closed subset of β, which is unbounded in β whenever* $\mathrm{cf}(\beta) > \omega$ *and*

(iii) *minimal as well as successor nodes of the tree $<_T$ on θ are ordinals of cofinality ω.*

Proof. For each ordinal $\alpha < \theta$ of countable cofinality we fix a subset $S_\alpha \subseteq \alpha$ of order-type ω cofinal with α. Given a tree ordering $<_T$ on θ with properties (i)-(iii), for a limit ordinal $\beta < \theta$ let P_β^+ be the set of all successor nodes from $P_\beta \cup \{\beta\}$ including the minimal one. For $\alpha \in P_\beta^+$ let α^- be its immediate predecessor in P_β. Finally, set

$$C_\beta = P_\beta \cup \bigcup\{S_\alpha \cap [\alpha^-, \alpha) : \alpha \in P_\beta^+\}.$$

It is easily checked that this defines a square-sequence C_β $(\beta < \theta)$ with the property that $\alpha <_T \beta$ holds if and only if α is a limit point of C_β. ⊣

9.4 Remark. It should be clear that the proof of Lemma 9.3 shows that the exact analogue of this result is true for any cofinality $\kappa < \theta$ rather than just for the cofinality ω.

The most important result about square sequences is of course the following well-known result of Jensen [27].

9.5 Theorem. *If a regular uncountable cardinal θ is not weakly compact in the constructible subuniverse then there is a nontrivial square sequence on θ which is moreover constructible.* ⊣

9.6 Corollary. *If a regular uncountable cardinal θ is not weakly compact in the constructible subuniverse then there is a constructible Aronszajn tree on θ.*

Proof. Let C_α $(\alpha < \theta)$ be a fixed nontrivial square sequence which is constructible. Changing the C_α's a bit, we may assume that if β is a limit ordinal with $\alpha = \min C_\beta$ or if $\alpha \in C_\beta$ but $\sup(C_\beta \cap \alpha) < \alpha$ then α must be a successor ordinal in θ. Consider the corresponding function $\rho_0 : [\theta]^2 \longrightarrow \mathbb{Q}_\theta$

$$\rho_0(\alpha, \beta) = \mathrm{tp}(C_\beta \cap \alpha)^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha)),$$

where $\rho_0(\gamma, \gamma) = \emptyset$ for all $\gamma < \theta$. Consider the tree

$$T(\rho_0) = \{(\rho_0)_\beta \restriction \alpha : \alpha \leq \beta < \theta\}.$$

Clearly $T(\rho_0)$ is constructible. By (I.4) the αth level of $T(\rho_0)$ is bounded by the size of the set $\{C_\beta \cap \alpha : \beta \geq \alpha\}$. Since the intersection of the form $C_\beta \cap \alpha$ is determined by its maximal limit point modulo a finite subset of α, we conclude that the αth level of $T(\rho_0)$ has size $\leq |\alpha| + \aleph_0$. Since the sequence C_α $(\alpha < \theta)$ is nontrivial, the proof of Theorem 8.5 shows that $T(\rho_0)$ has no cofinal branches. $\dashv$

9.7 Lemma. *Suppose C_α $(\alpha < \theta)$ is a square sequence on θ, $<^2$ the associated tree ordering on θ and $T(\rho_0) = \{(\rho_0)_\beta \restriction \alpha : \alpha \leq \beta < \theta\}$ where $\rho_0 : [\theta]^2 \longrightarrow \mathbb{Q}_\theta$ is the associated ρ_0-function. Then $\alpha \longmapsto (\rho_0)_\alpha$ is a strictly increasing map from the tree $(\theta, <^2)$ into the tree $T(\rho_0)$.*

Proof. If α is a limit point of C_β then $C_\alpha = C_\beta \cap \alpha$ so the walks $\alpha \to \xi$ and $\beta \to \xi$ for $\xi < \alpha$ get the same code $\rho_0(\xi, \alpha) = \rho_0(\xi, \beta)$. $\dashv$

The purpose of this section, however, is to analyze a family of ρ-functions associated with a square sequence C_α $(\alpha < \theta)$ on some regular uncountable cardinal θ, both fixed from now on. Recall that an ordinal α *divides* an ordinal γ if there is a β such that $\gamma = \alpha{\cdot}\beta$, i.e. γ can be written as the union of an increasing β-sequence of intervals of type α. Let $\kappa \leq \theta$ be a fixed infinite regular cardinal. Let $\Lambda_\kappa : [\theta]^2 \longrightarrow \theta$ be defined by

$$\Lambda_\kappa(\alpha, \beta) = \max\{\xi \in C_\beta \cap (\alpha + 1) : \kappa \text{ divides } \mathrm{tp}(C_\beta \cap \xi)\}.$$

Finally, we are ready to define the main object of study in this section:

$$\rho_\kappa : [\theta]^2 \longrightarrow \kappa$$

defined recursively by

$$\begin{aligned}\rho_\kappa(\alpha, \beta) = \sup\{&\mathrm{tp}(C_\beta \cap [\Lambda_\kappa(\alpha, \beta), \alpha)), \rho_\kappa(\alpha, \min(C_\beta \setminus \alpha)),\\ &\rho_\kappa(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda_\kappa(\alpha, \beta), \alpha)\},\end{aligned}$$

where we stipulate that $\rho_\kappa(\gamma, \gamma) = 0$ for all γ.

The following consequence of the coherence property of C_α $(\alpha < \theta)$ will be quite useful.

9.8 Lemma. *If α is a limit point of C_β then $\rho_\kappa(\xi, \alpha) = \rho_\kappa(\xi, \beta)$ for all $\xi < \alpha$.* $\dashv$

Note that ρ_κ is something that corresponds to the function $\rho : [\omega_1]^2 \longrightarrow \omega$ considered in Definition 2.1 (see also Section 11) and that the ρ_κ's are simply various *local versions* of the key definition. It turs out that they all have the crucial subadditive properties (see [76]).

9.9 Lemma. *If $\alpha < \beta < \gamma < \theta$ then*

(a) $\rho_\kappa(\alpha, \gamma) \leq \max\{\rho_\kappa(\alpha, \beta), \rho_\kappa(\beta, \gamma)\}$,

(b) $\rho_\kappa(\alpha,\beta) \leq \max\{\rho_\kappa(\alpha,\gamma), \rho_\kappa(\beta,\gamma)\}$. ⊣

The following is an immediate consequence of the fact that the definition of ρ_κ is closely tied to the notion of a walk along the fixed square sequence.

9.10 Lemma. *$\rho_\kappa(\alpha,\gamma) \geq \rho_\kappa(\alpha,\beta)$ whenever $\alpha \leq \beta \leq \gamma$ and β belongs to the trace of the walk from γ to α.* ⊣

9.11 Lemma. *Suppose $\beta \leq \gamma < \theta$ and that β is a limit ordinal > 0. Then $\rho_\kappa(\alpha,\gamma) \geq \rho_\kappa(\alpha,\beta)$ for coboundedly many $\alpha < \beta$.*

Proof. Let $\gamma = \gamma_0 > \gamma_1 > \ldots > \gamma_{n-1} > \gamma_n = \beta$ be the trace of the walk from γ to β. Let $\bar{\gamma} = \gamma_{n-1}$ if β is a limit point of $C_{\gamma_{n-1}}$, otherwise let $\bar{\gamma} = \beta$. Note that by Lemma 9.8, in any case we have that

$$\rho_\kappa(\alpha,\beta) = \rho_\kappa(\alpha,\bar{\gamma}) \text{ for all } \alpha < \beta. \tag{I.6}$$

Let $\bar{\beta} < \beta$ be an upper bound of all $C_{\gamma_i} \cap \beta$ $(i < n)$ which are bounded in β. Then $\bar{\gamma}$ is a member of the trace of any walk from γ to some ordinal α in the interval $[\bar{\beta},\beta)$. Applying Lemma 9.10 to this fact gives us

$$\rho_\kappa(\alpha,\gamma) \geq \rho_\kappa(\alpha,\bar{\gamma}) \text{ for all } \alpha \in [\bar{\beta},\beta).$$

Since $\rho_\kappa(\alpha,\bar{\gamma}) = \rho_\kappa(\alpha,\beta)$ for all $\alpha < \beta$ (see (I.6)), this gives us the conclusion of the lemma. ⊣

The proof of the following lemma can be found in [76].

9.12 Lemma. *The set $P^\kappa_\nu(\beta) = \{\xi < \beta : \rho_\kappa(\xi,\beta) \leq \nu\}$ is a closed subset of β for every $\beta < \theta$ and $\nu < \kappa$.* ⊣

For $\alpha < \beta < \theta$ and $\nu < \kappa$ set

$$\alpha <^\kappa_\nu \beta \text{ if and only if } \rho_\kappa(\alpha,\beta) \leq \nu.$$

9.13 Lemma.

(1) *$<^\kappa_\nu$ is a tree ordering on θ,*

(2) *$<^\kappa_\nu \subseteq <^\kappa_\mu$ whenever $\nu < \mu < \kappa$,*

(3) $\in\restriction(\theta \times \theta) = \bigcup_{\nu<\kappa} <^\kappa_\nu$.

Proof. This follows immediately from Lemma 9.9. ⊣

Recall the notion of a special tree of height θ from Section 6, a tree T for which one can find a T-regressive map $f : T \longrightarrow T$ with the property that the preimage of any point is the union of $< \theta$ antichains. By a *tree on θ* we mean a tree of the form $(\theta, <_T)$ with the property that $\alpha <_T \beta$ implies $\alpha < \beta$.

9.14 Lemma. *If a tree T naturally placed on θ is special, then there is an ordinal-regressive map $f : \theta \longrightarrow \theta$ and a closed and unbounded set $C \subseteq \theta$ such that f is one-to-one on all chains separated by C.*

Proof. Let $g : \theta \longrightarrow \theta$ be a T-regressive map such that for each $\xi < \theta$ the preimage $g^{-1}(\{\xi\})$ can be written as a union of a sequence $A_\delta(\xi)$ $(\delta < \lambda_\xi)$ of antichains, where $\lambda_\xi < \theta$. Let C be the collection of all limit $\alpha < \theta$ with the property that $\lambda_\xi < \alpha$ for all $\xi < \alpha$. Choose an ordinal-regressive $f : \theta \longrightarrow \theta$ as follows. If there is a $\delta \in C$ such that $g(\alpha) < \delta \leq \alpha$, then $f(\alpha)$ is smaller than the minimal member of C above $g(\alpha)$, $f(\alpha)$ codes in some standard way the ordinal $g(\alpha)$ as well as the index δ of the antichain $A_\delta(g(\alpha))$ to which α belongs, and $f(\alpha) \notin C$. If no member of C separates $g(\alpha)$ and α, let $f(\alpha)$ be the maximal member of C that is smaller than α. ⊣

By Lemma 9.13 we have a sequence $<^\kappa_\nu$ $(\nu < \kappa)$ of tree orderings on θ. The following lemma tells us that they are frequently quite large orderings.

9.15 Lemma. *If $\theta > \kappa$ is not a successor of a cardinal of cofinality κ then there must be $\nu < \kappa$ such that $(\theta, <^\kappa_\nu)$ is a nonspecial tree on θ.*

Proof. Suppose to the contrary that all trees are special. By Lemma 9.14 we may choose ordinal-regressive maps $f_\nu : \theta \longrightarrow \theta$ for all $\nu < \kappa$ and a single closed and unbounded set $C \subseteq \theta$ such that each of the maps f_ν is one-to-one on $<^\kappa_\nu$-chains separated by C. Using the Pressing Down Lemma we find a stationary set Γ of cofinality κ^+ ordinals $< \theta$ and $\lambda < \theta$ such that $f_\nu(\gamma) < \lambda$ for all $\gamma \in \Gamma$ and $\nu < \kappa$. If $|\lambda|^+ < \theta$, let $\Delta = \lambda$, $\Gamma = \Gamma_0$ and if $|\lambda|^+ = \theta$, represent λ as the increasing union of a sequence Δ_ξ $(\xi < \mathrm{cf}(|\lambda|))$ of sets of size $< |\lambda|$. Since $\kappa \neq \mathrm{cf}(|\lambda|)$ there is a $\bar{\xi} < \mathrm{cf}(|\lambda|)$ and a stationary $\Gamma_0 \subseteq \Gamma$ such that for all $\gamma \in \Gamma_0$, $f_\nu(\gamma) \in \Delta_{\bar{\xi}}$ for κ many $\nu < \kappa$. Let $\Delta = \Delta_{\bar{\xi}}$. This gives us subsets Δ and Γ_0 of θ such that

(1) $|\Delta|^+ < \theta$ and Γ_0 is stationary in θ,

(2) $\Sigma_\gamma = \{\nu < \kappa : f_\nu(\gamma) \in \Delta\}$ is unbounded in κ for all $\gamma \in \Gamma_0$.

Let $\bar{\theta} = \kappa^+ \cdot |\Delta|^+$. Then $\bar{\theta} < \theta$ and so we can find $\beta \in \Gamma_0$ such that $\Gamma_0 \cap C \cap \beta$ has size $\bar{\theta}$. Then there will be $\nu_0 < \kappa$ and $\Gamma_1 \subseteq \Gamma_0 \cap C \cap \beta$ of size $\bar{\theta}$ such that $\rho_\kappa(\alpha, \beta) \leq \nu_0$ for all $\alpha \in \Gamma_1$. By (2) we can find $\Gamma_2 \subseteq \Gamma_1$ of size $\bar{\theta}$ and $\nu_1 \geq \nu_0$ such that $f_{\nu_1}(\alpha) \in \Delta$ for all $\alpha \in \Gamma_2$. Note that Γ_2 is a $<^\kappa_{\nu_1}$-chain separated by C, so f_{ν_1} is one-to-one on Γ_2. However, this gives us the desired contradiction since the set Δ, in which f_{ν_1} embeds Γ_2 has size smaller than the size of Γ_2. This finishes the proof. ⊣

It is now natural to ask the following question: under which assumption on the square sequence C_α $(\alpha < \theta)$ can we conclude that neither of the trees $(\theta, <^\kappa_\nu)$ will have a branch of size θ?

9.16 Lemma. *If the set* $\Gamma_\kappa = \{\alpha < \theta : \text{tp } C_\alpha = \kappa\}$ *is stationary in* θ*, then none of the trees* $(\theta, <^\kappa_\nu)$ *has a branch of size* θ.

Proof. Assume that B is a $<^\kappa_\nu$-branch of size θ. By Lemma 9.12, B is a closed and unbounded subset of θ. Pick a limit point β of B which belongs to Γ_κ. Pick $\alpha \in B \cap \beta$ such that $\text{tp}(C_\beta \cap \alpha) > \nu$. By definition of $\rho_\kappa(\alpha, \beta)$ we have that $\rho_\kappa(\alpha, \beta) \geq \text{tp}(C_\beta \cap \alpha) > \nu$ since clearly $\Lambda_\kappa(\alpha, \beta) = 0$. This contradicts the fact that $\alpha <^\kappa_\nu \beta$ and finishes the proof. ⊣

9.17 Definition. A square sequence on θ is *special* if the corresponding tree $(\theta, <^2)$ is special, i.e. there is a $<^2$-regressive map $f : \theta \longrightarrow \theta$ with the property that the f-preimage of every $\xi < \theta$ is the union of $< \theta$ antichains of $(\theta, <^2)$.

9.18 Theorem. *Suppose* $\kappa < \theta$ *are regular cardinals such that* θ *is not a successor of a cardinal of cofinality* κ*. Then to every square sequence* C_α $(\alpha < \theta)$ *for which there exist stationarily many* α *such that* $\text{tp } C_\alpha = \kappa$*, one can associate a sequence* $C_{\alpha\nu}$ $(\alpha < \theta, \nu < \kappa)$ *such that:*

(i) $C_{\alpha\nu} \subseteq C_{\alpha\mu}$ *for all* α *and* $\nu \leq \mu$,

(ii) $\alpha = \bigcup_{\nu<\kappa} C_{\alpha\nu}$ *for all limit* α,

(iii) $C_{\alpha\nu}$ $(\alpha < \theta)$ *is a nonspecial (and nontrivial) square sequence on* θ *for all* $\nu < \kappa$.

Proof. Fix $\nu < \kappa$ and define $C_{\alpha\nu}$ by induction on $\alpha < \theta$. So suppose β is a limit ordinal $< \theta$ and that $C_{\alpha\nu}$ is defined for all $\alpha < \beta$. If $P^\kappa_\nu(\beta)$ is bounded in β, let $\bar{\beta}$ be the maximal limit point of $P^\kappa_\nu(\beta)$ ($\bar{\beta} = 0$ if the set has no limit points) and let

$$C_{\beta\nu} = C_{\bar{\beta}\nu} \cup P^\kappa_\nu(\beta) \cup (C_\beta \cap [\max(P^\kappa_\nu(\beta)), \beta)).$$

If $P^\kappa_\nu(\beta)$ is unbounded in β, let

$$C_{\beta\nu} = P^\kappa_\nu(\beta) \cup \bigcup\{C_{\alpha\nu} : \alpha \in P^\kappa_\nu(\beta) \text{ and } \alpha = \sup(P^\kappa_\nu(\beta) \cap \alpha)\}.$$

By Lemmas 9.9 and 9.12, $C_{\beta\nu}$ $(\beta < \theta)$ is well defined and it forms a square sequence on θ. The properties (i) and (ii) are also immediate. To see that for each $\nu < \kappa$ the sequence $C_{\beta\nu}$ $(\beta < \theta)$ is nontrivial, one uses Lemma 9.16 and the fact that if α is a limit point of $C_{\beta\nu}$ occupying a place in $C_{\beta\nu}$ that is divisible by κ, then $\alpha <^\kappa_\nu \beta$. By Lemma 9.15, or rather its proof, we conclude that there is a $\bar{\nu} < \kappa$ such that $C_{\beta\nu}$ $(\beta < \theta)$ is nonspecial for all $\nu \geq \bar{\nu}$. This finishes the proof. ⊣

The following facts whose proof can be found in [76] gives us a square sequence satisfying the hypothesis of Lemma 9.18.

9.19 Lemma. *For every pair of regular cardinals $\kappa < \theta$, every special square sequence C_α $(\alpha < \theta)$ can be refined to a square sequence $\bar{C}_\alpha$ $(\alpha < \theta)$ with the property that* $\operatorname{tp} \bar{C}_\alpha = \kappa$ *for stationarily many $\alpha < \theta$.* ⊣

Finally we can state the main result of this section which follows from Theorem 9.18 and Lemma 9.19.

9.20 Theorem. *A regular uncountable cardinal $\theta \neq \omega_1$ carries a nontrivial square sequence iff it also carries such a sequence which is moreover nonspecial.* ⊣

9.21 Corollary. *If a regular uncountable cardinal $\theta \neq \omega_1$ is not weakly compact in the constructible subuniverse then there is a nonspecial Aronszajn tree of height θ.*

Proof. By Theorem 9.5, θ carries a nontrivial square sequence C_α $(\alpha < \theta)$. By Theorem 9.20 we may assume that the sequence is moreover nonspecial. Let ρ_0 be the associated ρ_0-function and consider the tree $T(\rho_0)$. As in Corollary 9.6 we conclude that $T(\rho_0)$ is an Aronszajn tree of height θ. By Lemma 9.7 there is a strictly increasing map from $(\theta, <^2)$ into $T(\rho_0)$, so $T(\rho_0)$ must be nonspecial. ⊣

9.22 Remark. The assumption $\theta \neq \omega_1$ in Theorem 9.20 is essential as there is always a nontrivial square sequence on ω_1 but it is possible to have a situation where all Aronszajn trees on ω_1 are special. For example MA_{ω_1} implies this. In [37], Laver and Shelah have shown that any model with a weakly compact cardinal admits a forcing extension satisfying CH and the statement that all Aronszajn trees on ω_2 are special. A well-known open problem in this area asks whether one can have GCH rather than CH in a model where all Aronszajn trees on ω_2 are special.

10. The Full Lower Trace of a Square Sequence

In this section θ is a regular uncountable cardinal and C_α $(\alpha < \theta)$ is a nontrivial square sequence on θ. Recall the function $\Lambda = \Lambda_\omega : [\theta]^2 \longrightarrow \theta$:

$$\Lambda(\alpha, \beta) = \text{maximal limit point of } C_\beta \cap (\alpha + 1).$$

($\Lambda(\alpha, \beta) = 0$ if $C_\beta \cap (\alpha + 1)$ has no limit points.)

The purpose of this section is to study the following recursive trace formula, describing a mapping $\mathrm{F} : [\theta]^2 \longrightarrow [\theta]^{<\omega}$:

$$\mathrm{F}(\alpha, \beta) = \mathrm{F}(\alpha, \min(C_\beta \setminus \alpha)) \cup \bigcup \{\mathrm{F}(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda(\alpha, \beta), \alpha)\},$$

where $\mathrm{F}(\gamma, \gamma) = \{\gamma\}$ for all γ.

As in the case $\theta = \omega_1$, the full lower trace has the following two properties (see [76]).

10.1 Lemma. *For all $\alpha \le \beta \le \gamma$,*

(a) $\mathrm{F}(\alpha, \gamma) \subseteq \mathrm{F}(\alpha, \beta) \cup \mathrm{F}(\beta, \gamma)$,

(b) $\mathrm{F}(\alpha, \beta) \subseteq \mathrm{F}(\alpha, \gamma) \cup \mathrm{F}(\beta, \gamma)$. ⊣

10.2 Lemma. *For all $\alpha \le \beta \le \gamma$,*

(a) $\rho_0(\alpha, \beta) = \rho_0(\min(\mathrm{F}(\beta, \gamma) \setminus \alpha), \beta)^\frown \rho_0(\alpha, \min(\mathrm{F}(\beta, \gamma) \setminus \alpha))$,

(b) $\rho_0(\alpha, \gamma) = \rho_0(\min(\mathrm{F}(\beta, \gamma) \setminus \alpha), \gamma)^\frown \rho_0(\alpha, \min(\mathrm{F}(\beta, \gamma) \setminus \alpha))$. ⊣

Recall the function $\rho_2 : [\theta]^2 \longrightarrow \omega$ which counts the number of steps in the walk along the fixed C-sequence C_α ($\alpha < \theta$) which in this section is assumed to be moreover a square sequence:

$$\rho_2(\alpha, \beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1,$$

where we let $\rho_2(\gamma, \gamma) = 0$ for all γ. Thus $\rho_2(\alpha, \beta) + 1$ is simply equal to the cardinality of the trace $\mathrm{Tr}(\alpha, \beta)$ of the minimal walk from β to α.

10.3 Lemma. $\sup_{\xi<\alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| < \infty$ *for all $\alpha < \beta < \theta$.*

Proof. By Lemma 10.2, $\sup_{\xi<\alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)|$ is less than or equal than $\sup_{\xi \in F(\alpha,\beta)} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)|$. ⊣

10.4 Definition. Set $\mathcal{I}$ to be the set of all countable $\Gamma \subseteq \theta$ such that $\sup_{\xi \in \Delta} \rho_2(\xi, \alpha) = \infty$ for all $\alpha < \theta$ and infinite $\Delta \subseteq \Gamma \cap \alpha$.

10.5 Lemma. *$\mathcal{I}$ is a P-ideal of countable subsets of θ.*

Proof. Let Γ_n ($n < \omega$) be a given sequence of members of $\mathcal{I}$ and fix $\beta < \theta$ such that $\Gamma_n \subseteq \beta$ for all n. For $n < \omega$ set $\Gamma_n^* = \{\xi \in \Gamma_n : \rho_2(\xi, \beta) \ge n\}$. Since Γ_n belongs to $\mathcal{I}$, Γ_n^* is a cofinite subset of Γ_n. Let $\Gamma_\infty = \bigcup_{n<\omega} \Gamma_n^*$. Then Γ_∞ is a member of $\mathcal{I}$ such that $\Gamma_n \setminus \Gamma_\infty$ is finite for all n. ⊣

10.6 Theorem. *The P-ideal dichotomy implies that a nontrivial square sequence can exist only on $\theta = \omega_1$.*

Proof. Applying the P-ideal dichotomy on $\mathcal{I}$ from 10.4 we get the two alternatives (see 3.9):

(1) there is an uncountable $\Delta \subseteq \theta$ such that $[\Delta]^\omega \subseteq \mathcal{I}$, or

(2) there is a decomposition $\theta = \bigcup_{n<\omega} \Sigma_n$ such that $\Sigma_n \perp \mathcal{I}$ for all n.

By Lemma 10.3, if (1) holds, then $\Delta \cap \alpha$ must be countable for all $\alpha < \theta$ and so the cofinality of θ must be equal to ω_1. Since we are working only with regular uncountable cardinals, we see that (1) gives us that $\theta = \omega_1$ must hold. Suppose now (2) holds and pick $k < \omega$ such that Σ_k is unbounded in θ. Since $\Sigma_k \perp \mathcal{I}$ we have that $(\rho_2)_\alpha$ is bounded on $\Sigma_k \cap \alpha$ for all $\alpha < \theta$. So there is an unbounded set $\Gamma \subseteq \theta$ and an integer n such that for each $\alpha \in \Gamma$ the restriction of $(\rho_2)_\alpha$ on $\Sigma_k \cap \alpha$ is bounded by n. By Theorem 8.2 we conclude that the square sequence C_α $(\alpha < \theta)$ we started with must be trivial. ⊣

10.7 Definition. By S_θ we denote the *sequential fan* with θ edges, i.e. the space on $(\theta \times \omega) \cup \{*\}$ with $*$ as the only nonisolated point, while a typical neighborhood of $*$ has the form $\mathcal{U}_f = \{(\alpha, n) : n \geq f(\alpha)\} \cup \{*\}$ where $f : \theta \longrightarrow \omega$.

The *tightness* of a point x in a space X is equal to θ if θ is the minimal cardinal such that, if a set $W \subseteq X \setminus \{x\}$ accumulates to x, then there is a subset of W of size $\leq \theta$ that accumulates to x.

10.8 Theorem. *If there is a nontrivial square sequence on θ then the square of the sequential fan S_θ has tightness equal to θ.* ⊣

The proof will be given after a sequence of definitions and lemmas.

10.9 Definition. Given a square sequence C_α $(\alpha < \theta)$ and its number of steps function $\rho_2 : [\theta]^2 \longrightarrow \omega$ we define $d : [\theta]^2 \longrightarrow \omega$ by letting

$$d(\alpha, \beta) = \sup_{\xi \leq \alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)|.$$

10.10 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $\rho_2(\alpha, \beta) \leq d(\alpha, \beta)$,

(b) $d(\alpha, \gamma) \leq d(\alpha, \beta) + d(\beta, \gamma)$,

(c) $d(\alpha, \beta) \leq d(\alpha, \gamma) + d(\beta, \gamma)$.

Proof. The conclusion (a) follows from the fact that we allow $\xi = \alpha$ in the definition of $d(\alpha, \beta)$. The conclusions (b) and (c) are consequences of the triangle inequalities of the ℓ_∞-norm and the fact that in both inequalities we have that the domain of functions on the left-hand side is included in the domain of functions on the right-hand side. ⊣

10.11 Definition. For $\gamma \leq \theta$, let

$$W_\gamma = \{((\alpha, d(\alpha, \beta)), (\beta, d(\alpha, \beta))) : \alpha < \beta < \gamma\}.$$

The following lemma establishes that the tightness of the point $(*, *)$ of S_θ^2 is equal to θ, giving us the proof of Theorem 10.8.

10.12 Lemma. $(*, *) \in \bar{W}_\theta$ *but* $(*, *) \notin \bar{W}_\gamma$ *for all* $\gamma < \theta$.

Proof. To see that W_θ accumulates to $(*,*)$, let $\mathcal{U}_f^2$ be a given neighborhood of $(*,*)$. Fix an unbounded set $\Gamma \subseteq \theta$ on which f is constant. By Theorem 8.2 and Lemma 10.10(a) there exists an $\alpha < \beta$ in Γ such that $d(\alpha,\beta) \geq f(\alpha) = f(\beta)$. Then $((\alpha, d(\alpha,\beta)), (\beta, d(\alpha,\beta)))$ belongs to the intersection $W_\theta \cap \mathcal{U}_f^2$. To see that for a given $\gamma < \theta$ the set W_γ does not accumulate to $(*,*)$, choose $g : \theta \longrightarrow \omega$ such that

$$g(\alpha) = 2d(\alpha,\gamma) + 1 \text{ for } \alpha < \gamma.$$

Suppose $W_\gamma \cap \mathcal{U}_g^2$ is nonempty and choose $((\alpha, d(\alpha,\beta)), (\beta, d(\alpha,\beta)))$ from this set. Then

$$d(\alpha,\beta) \geq 2d(\alpha,\gamma) + 1 \text{ and } d(\alpha,\beta) \geq 2d(\beta,\gamma) + 1,$$

and so, $d(\alpha,\beta) \geq d(\alpha,\gamma) + d(\beta,\gamma) + 1$, contradicting Lemma 10.10(c). ⊣

Since $\theta = \omega_1$ admits a nontrivial square sequence, Theorem 10.8 leads to the following result of Gruenhage and Tanaka [22].

10.13 Corollary. *The square of the sequential fan with ω_1 edges is not countably tight.* ⊣

10.14 Question. What is the tightness of the square of the sequential fan with ω_2 edges?

10.15 Corollary. *If a regular uncountable cardinal θ is not weakly compact in the constructible subuniverse then the square of the sequential fan with θ edges has tightness equal to θ.* ⊣

11. Special Square Sequences

The following well-known result of Jensen [27] supplements the corresponding result for weakly compact cardinals listed above as Theorem 9.5.

11.1 Theorem. *If a regular uncountable cardinal θ is not Mahlo in the constructible subuniverse then there is a special square sequence on θ which is moreover constructible.* ⊣

Today we know many more inner models with sufficient amount of fine structure necessary for building special square sequences. So the existence of special square sequences, especially at successors of strong-limit singular cardinals, is tied to the existence of some other large cardinal axioms. The reader is referred to the relevant chapters of this Handbook for the specific information. In this section we give the combinatorial analysis of walks along special square sequences and the corresponding distance functions. Let us start by restating some results of Section 9.

11.2 Theorem. *Suppose $\kappa < \theta$ are regular cardinals and that θ carries a special square sequence. Then there exist $C_{\alpha\nu}$ $(\alpha < \theta, \nu < \kappa)$ such that:*

(1) $C_{\alpha\nu} \subseteq C_{\alpha\mu}$ *for all* α *and* $\nu < \mu$,

(2) $\alpha = \bigcup_{\nu<\kappa} C_\alpha$ *for all limit* α,

(3) $C_{\alpha\nu}$ $(\alpha < \theta)$ *is a nontrivial square sequence on* θ *for all* $\nu < \kappa$.

Moreover, if θ is not a successor of a cardinal of cofinality κ then each of the square sequences can be chosen to be nonspecial. $\dashv$

11.3 Theorem. *Suppose $\kappa < \theta$ are regular cardinals and that θ carries a special square sequence. Then there exist $<_\nu$ $(\nu < \kappa)$ such that:*

(i) $<_\nu$ *is a closed tree ordering of* θ *for each* $\nu < \kappa$,

(ii) $\in\restriction(\theta \times \theta) = \bigcup_{\nu<\kappa} <_\nu$,

(iii) *no tree* $(\theta, <_\nu)$ *has a chain of size* θ. $\dashv$

11.4 Lemma. *The following are equivalent when θ is a successor of some cardinal κ:*

(1) *there is a special square sequence on* θ,

(2) *there is a square sequence* C_α $(\alpha < \theta)$ *such that* $\mathrm{tp}(C_\alpha) \leq \kappa$ *for all* $\alpha < \theta$.

Proof. Let D_α $(\alpha < \kappa^+)$ be a given special square sequence. By Lemma 6.2 the corresponding tree $(\kappa^+, <^2)$ can be decomposed into κ antichains so let $f : \kappa^+ \longrightarrow \kappa$ be a fixed map such that $f^{-1}(\{\xi\})$ is a $<^2$-antichain for all $\xi < \kappa$. Let $\alpha < \kappa^+$ be a given limit ordinal. If D_α has a maximal limit point $\bar{\alpha} < \alpha$, let $C_\alpha = D_\alpha \setminus \bar{\alpha}$. Suppose now that $\{\xi : \xi <^2 \alpha\}$ is unbounded in α and define a strictly increasing continuous sequence $c_\alpha(\xi)$ $(\xi < \nu(\alpha))$ of its elements as follows. Let $c_\alpha(0) = \min\{\xi : \xi <^2 \alpha\}$, $c_\alpha(\eta) = \sup_{\xi<\eta} c_\alpha(\xi)$ for η limit, and $c_\alpha(\xi + 1)$ is the minimal $<^2$-predecessor γ of α such that $\gamma > c_\alpha(\xi)$ and has the minimal f-image among all $<^2$-predecessors that are $> c_\alpha(\xi)$. The ordinal $\nu(\alpha)$ is defined as the place where the process stops, i.e. when $\alpha = \sup_{\xi<\nu(\alpha)} c_\alpha(\xi)$. Let $C_\alpha = \{c_\alpha(\xi) : \xi < \nu(\alpha)\}$. It is easily checked that this gives a square sequence C_α $(\alpha < \kappa^+)$ with the property that $\mathrm{tp}(C_\alpha) \leq \kappa$ for all $\alpha < \kappa^+$. $\dashv$

Square sequences C_α $(\alpha < \kappa^+)$ that have the property $\mathrm{tp}(C_\alpha) \leq \kappa$ for all $\alpha < \kappa^+$ are usually called $\square_\kappa$-sequences. So let C_α $(\alpha < \kappa^+)$ be a $\square_\kappa$-sequence fixed from now on. Let

$$\Lambda(\alpha, \beta) = \text{maximal limit point of } C_\beta \cap (\alpha + 1)$$

when such a limit point exists; otherwise $\Lambda(\alpha, \beta) = 0$. The purpose of this section is to analyze the following distance function:

$$\rho : [\kappa^+]^2 \longrightarrow \kappa$$

defined recursively by

$$\rho(\alpha, \beta) = \max\{\text{tp}(C_\beta \cap \alpha), \rho(\alpha, \min(C_\beta \setminus \alpha)),\\ \rho(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda(\alpha, \beta), \alpha)\},$$

where we stipulate that $\rho(\gamma, \gamma) = 0$ for all $\gamma < \kappa^+$. Clearly $\rho(\alpha, \beta) \geq \rho_1(\alpha, \beta)$, so by Lemma 7.1 we have

11.5 Lemma. $|\{\xi \leq \alpha : \rho(\xi, \alpha) \leq \nu\}| \leq |\nu| + \aleph_0$ *for* $\alpha < \kappa^+$ *and* $\nu < \kappa$. ⊣

The following two crucial subadditive properties of ρ have proofs that are almost identical to the proofs of the corresponding properties of, say, the function ρ_ω discussed above in Section 9.

11.6 Lemma. *For all* $\alpha \leq \beta \leq \gamma$,

(a) $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$,

(b) $\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$. ⊣

The following immediate fact will also be quite useful.

11.7 Lemma. *If* α *is a limit point of* C_β*, then* $\rho(\xi, \alpha) = \rho(\xi, \beta)$ *for all* $\xi < \alpha$. ⊣

The following as well is an immediate consequence of the fact that the definition of ρ is closely tied to the notion of a minimal walk along the square sequence.

11.8 Lemma. $\rho(\alpha, \gamma) \geq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$ *whenever* $\alpha \leq \beta \leq \gamma$ *and* β *belongs to the trace of the walk from* γ *to* α. ⊣

Using Lemmas 11.7 and 11.8 one proves the following fact exactly as in the case of ρ_κ of Section 9 (the proof of Lemma 9.11).

11.9 Lemma. *If* $0 < \beta \leq \gamma$ *and* β *is a limit ordinal, then there is a* $\bar{\beta} < \beta$ *such that* $\rho(\alpha, \gamma) \geq \rho(\alpha, \beta)$ *for all* α *in the interval* $[\bar{\beta}, \beta)$. ⊣

The proof of the following fact is also completely analogous to the proof of the corresponding fact for the local version ρ_κ considered above in Section 9 (the proof of Lemma 9.12).

11.10 Lemma. $P_\nu(\gamma) = \{\beta < \gamma : \rho(\beta, \gamma) \leq \nu\}$ *is a closed subset of* γ *for all* $\gamma < \kappa^+$ *and* $\nu < \kappa$. ⊣

The discussion of $\rho : [\kappa^+]^2 \longrightarrow \kappa$ now splits naturally into two cases depending on whether κ is a regular or a singular cardinal (with the case $\text{cf}(\kappa) = \omega$ of special importance).

12. Successors of Regular Cardinals

In this section, κ is a fixed regular cardinal, C_α $(\alpha < \kappa^+)$ a fixed $\square_\kappa$-sequence and $\rho : [\kappa^+]^2 \longrightarrow \kappa$ the corresponding ρ-function. For $\nu < \kappa$ and $\alpha < \beta < \kappa^+$ set

$$\alpha <_\nu \beta \text{ if and only if } \rho(\alpha, \beta) \leq \nu.$$

The following is an immediate consequence of the analysis of ρ given in the previous section.

12.1 Lemma.

(a) $<_\nu$ *is a closed tree ordering of* κ^+ *of height* $\leq \kappa$ *for all* $\nu < \kappa$,

(b) $<_\nu \subseteq <_\mu$ *whenever* $\nu \leq \mu < \kappa$,

(c) $\in \restriction (\kappa^+ \times \kappa^+) = \bigcup_{\nu<\kappa} <_\nu$. ⊣

The following result shows that these trees have some properties of smallness not covered by statements of Lemma 12.1.

12.2 Lemma. *If* $\kappa > \omega$, *then no tree* $(\kappa^+, <_\nu)$ *has a branch of size* κ.

Proof. Suppose towards a contradiction that some tree $(\kappa^+, <_\nu)$ does have a branch of size κ and let B be one such fixed branch (maximal chain). By Lemmas 11.5 and 11.10, if $\gamma = \sup(B)$ then B is a closed and unbounded subset of γ of order-type κ. Since κ is regular and uncountable, $C_\gamma \cap B$ is unbounded in C_γ, so in particular we can find $\alpha \in C_\gamma \cap B$ such that $\text{tp}(C_\gamma \cap \alpha) > \nu$. Reading off the definition of $\rho(\alpha, \gamma)$ we conclude that $\rho(\alpha, \beta) = \text{tp}(C_\gamma \cap \alpha) > \nu$. Similarly we can find a $\beta > \alpha$ belonging to the intersection of $\lim(C_\gamma)$ and B. Then $C_\beta = C_\gamma \cap \beta$ so $\alpha \in C_\beta$ and therefore $\rho(\alpha, \beta) = \text{tp}(C_\beta \cap \alpha) > \nu$ contradicting the fact that $\alpha <_\nu \beta$. ⊣

A tree of height κ is *Souslin* if all of its chains and antichains are of cardinality less than κ.

12.3 Lemma. *If* $\kappa > \omega$, *then no tree* $(\kappa^+, <_\nu)$ *has a tree of height* κ *which is Souslin subtree.*

Proof. Forcing with subtree of $(\kappa^+, <_\nu)$ of height κ which is Souslin would produce an ordinal γ of cofinality κ and a closed and unbounded subset B of C_γ forming a chain of the tree $(\kappa^+, <_\nu)$. It is well-known that in this case B would contain a ground model subset of size κ, contradicting Lemma 12.2. ⊣

12.4 Lemma. *If* $\kappa > \omega$ *then for every* $\nu < \kappa$ *and every family* A *of* κ *pairwise disjoint finite subsets of* κ^+ *there exists an* $A_0 \subseteq A$ *of size* κ *such that for all* $a \neq b$ *in* A_0 *and all* $\alpha \in a$, $\beta \in b$ *we have* $\rho(\alpha, \beta) > \nu$.

Proof. We may assume that for some n and all $a \in A$ we have $|a| = n$. Let $a(0), \ldots, a(n-1)$ enumerate a given element a of A increasingly. By Lemma 12.3, shrinking A we may assume that $a(i)$ $(a \in A)$ is an antichain of $(\kappa^+, <_\nu)$ for all $i < n$. Going to a subfamily of A of equal size we may assume to have a well-ordering $<_w$ of A with the property that if $a <_w b$ then no node from a is above a node from b in the tree ordering $<_\nu$. Define $f : [A]^2 \longrightarrow \{0\} \cup (n \times n)$ by letting $f(a, b) = 0$ if $a \cup b$ is an $<_\nu$-antichain; otherwise, assuming $a <_w b$, let $f(a, b) = (i, j)$ where (i, j) is the minimal pair such that $a(i) <_\nu b(j)$. By the Dushnik-Miller partition theorem ([15]), either there exists an $A_0 \subseteq A$ of size κ such that f is constantly equal to 0 on $[A_0]^2$ or there exist $(i, j) \in n \times n$ and an infinite $A_1 \subseteq A$ such that f is constantly equal to (i, j) on $[A_1]^2$. The first alternative is what we want, so let us see that the second one is impossible. Otherwise, choose $a <_w b <_w c$ in A_1. Then $a(i)$ and $b(i)$ are both $<_\nu$-dominated by $c(j)$, so they must be $<_\nu$-comparable, contradicting our initial assumption about A. This completes the proof. ⊣

The unboundedness property of Lemma 12.4 can be quite useful in designing forcing notions satisfying good chain conditions. Having such applications in mind, we now state a further refiniement of this kind of unboundedness property of the ρ-function. Its tedious proof can be found for example in [76].

12.5 Lemma. *Suppose $\kappa > 0$, let $\gamma < \kappa^+$ and let $\{\alpha_\xi, \beta_\xi\}$ $(\xi < \kappa)$ be a sequence of pairwise disjoint elements of $[\kappa^+]^{\leq 2}$. Then there is an unbounded set $\Gamma \subseteq \kappa$ such that $\rho\{\alpha_\xi, \beta_\eta\} \geq \min\{\rho\{\alpha_\xi, \gamma\}, \rho\{\beta_\eta, \gamma\}\}$ for all $\xi \neq \eta$ in Γ.*[16] ⊣

This lemma allows a further refinement as follows (see [76]). A cardinal κ is *λ-inaccessible* if $\nu^\tau < \kappa$ for all $\nu < \kappa$ and $\tau < \lambda$.

12.6 Lemma. *Suppose κ is λ-inaccessible for some $\lambda < \kappa$ and that A is a family of size κ of subsets of κ^+, all of size $< \lambda$. Then for every ordinal $\nu < \kappa$ there is a subfamily B of A of size κ such that for all a and b in B:*

(a) *$\rho\{\alpha, \beta\} > \nu$ for all $\alpha \in a \setminus b$ and $\beta \in b \setminus a$.*

(b) *$\rho\{\alpha, \beta\} \geq \min\{\rho\{\alpha, \gamma\}, \rho\{\beta, \gamma\}\}$ for all $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$.* ⊣

12.7 Definition. The set-mapping $D : [\kappa^+]^2 \longrightarrow [\kappa^+]^{<\kappa}$ is defined by

$$D(\alpha, \beta) = \{\xi \leq \alpha : \rho(\xi, \alpha) \leq \rho(\alpha, \beta)\}.$$

[16] Here, and everywhere else later in this chapter, the convention is that, $\rho\{\alpha, \beta\}$ is meant to be equal to $\rho(\alpha, \beta)$ if $\alpha < \beta$, equal to $\rho(\beta, \alpha)$ if $\beta < \alpha$, and equal to 0 if $\alpha = \beta$.

(Note that $D(\alpha,\beta) = \{\xi \le \alpha : \rho(\xi,\beta) \le \rho(\alpha,\beta)\}$, so we could take the formula

$$D\{\alpha,\beta\} = \{\xi \le \min\{\alpha,\beta\} : \rho(\xi,\alpha) \le \rho\{\alpha,\beta\}\}$$

as our definition of $D\{\alpha,\beta\}$ when there is no implicit assumption about the ordering between α and β as there is whenever we write $D(\alpha,\beta)$.)

12.8 Lemma. *If κ is λ-inaccessible for some $\lambda < \kappa$, then for every family A of size κ of subsets of κ^+, all of size $< \lambda$, there exists a $B \subseteq A$ of size κ such that for all a and b in B and all $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$:*

(a) $\alpha,\beta > \gamma \implies D\{\alpha,\gamma\} \cup D\{\beta,\gamma\} \subseteq D\{\alpha,\beta\}$,
(b) $\beta > \gamma \implies D\{\alpha,\gamma\} \subseteq D\{\alpha,\beta\}$,
(c) $\alpha > \gamma \implies D\{\beta,\gamma\} \subseteq D\{\alpha,\beta\}$,
(d) $\gamma > \alpha,\beta \implies D\{\alpha,\gamma\} \subseteq D\{\alpha,\beta\}$ *or* $D\{\beta,\gamma\} \subseteq D\{\alpha,\beta\}$.

Proof. Choose $B \subseteq A$ of size κ satisfying the conclusion (b) of Lemma 12.6. Pick $a \neq b$ in B and consider $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$. By the conclusion of 12.6(b), we have

$$\rho\{\alpha,\beta\} \ge \min\{\rho\{\alpha,\gamma\}, \rho\{\beta,\gamma\}\}. \tag{I.7}$$

a. Suppose $\alpha,\beta > \gamma$. Note that in this case a single inequality $\rho(\gamma,\alpha) \le \rho\{\alpha,\beta\}$ or $\rho(\gamma,\beta) \le \rho\{\alpha,\beta\}$ given to us by (I.7) implies that we actually have both inequalities simultaneously holding. The subadditivity of ρ gives us $\rho(\xi,\alpha) \le \rho\{\alpha,\beta\}$, or equivalently $\rho(\xi,\beta) \le \rho\{\alpha,\beta\}$ for any $\xi \le \gamma$ with $\rho(\xi,\gamma) \le \rho(\gamma,\alpha)$ or $\rho(\xi,\gamma) \le \rho(\gamma,\beta)$. This is exactly the conclusion of 12.8(a).

b. Suppose that $\beta > \gamma > \alpha$. Using the subadditivity of ρ we see that in both cases given to us by (I.7) we have that $\rho(\alpha,\gamma) \le \rho\{\alpha,\beta\}$. So the inclusion $D\{\alpha,\gamma\} \subseteq D\{\alpha,\beta\}$ follows immediately.

c. Suppose that $\alpha > \gamma > \beta$. The conclusion $D\{\beta,\gamma\} \subseteq D\{\alpha,\beta\}$ follows from the previous case by symmetry.

d. Suppose that $\gamma > \alpha,\beta$. Then $\rho(\alpha,\gamma) \le \rho\{\alpha,\beta\}$ gives $D\{\alpha,\gamma\} \subseteq D\{\alpha,\beta\}$ while $\rho(\beta,\gamma) \le \rho\{\alpha,\beta\}$ gives us $D\{\beta,\gamma\} \subseteq D\{\alpha,\beta\}$.

This completes the proof. ⊣

12.9 Remark. Note that $\min\{x,y\} \in D\{x,y\}$ for every $\{x,y\} \in [\kappa^+]^2$, so the conclusion (a) of Lemma 12.8 in particular means that $\gamma < \min\{\alpha,\beta\}$ implies $\gamma \in D\{\alpha,\beta\}$. In applications, one usually needs this consequence of 12.8(a) rather than 12.8(a) itself.

12.10 Definition. The Δ-*function* of some family $\mathcal{F}$ of subsets of some ordinal κ (respectively, a family of functions with domain κ) is the function $\Delta : [\mathcal{F}]^2 \longrightarrow \kappa$ defined by $\Delta(f,g) = \min(f \bigtriangleup g)$, (respectively, $\Delta(f,g) = \min\{\xi : f(\xi) \neq g(\xi)\}$).

Note the following property of Δ:

12.11 Lemma. $\Delta(f,g) \geq \min\{\Delta(f,h), \Delta(g,h)\}$ *for all* $\{f,g,h\} \in [\mathcal{F}]^3$. $\dashv$

12.12 Remark. This property can be very useful when transferring objects that live on κ to objects on $\mathcal{F}$. This is especially interesting when $\mathcal{F}$ is of size larger than κ while all of its restrictions $\mathcal{F}\restriction\nu = \{f \cap \nu : f \in \mathcal{F}\}$ $(\nu < \kappa)$ have size $< \kappa$, i.e. when $\mathcal{F}$ is a *Kurepa family* (see for example [10]). We shall now see that it is possible to have a Kurepa family $\mathcal{F} = \{f_\alpha : \alpha < \kappa^+\}$ whose Δ-function is dominated by ρ, i.e. $\Delta(f_\alpha, f_\beta) \leq \rho(\alpha,\beta)$ for all $\alpha < \beta < \kappa^+$.

12.13 Theorem. *If* $\square_\kappa$ *holds and* κ *is* λ*-inaccessible then there is a* λ*-closed* κ*-cc forcing notion* $\mathcal{P}$ *that introduces a Kurepa family on* κ*.*

Proof. Put p in $\mathcal{P}$, if p is a one-to-one function from a subset of κ^+ of size $< \lambda$ into the family of all subsets of κ of size $< \lambda$ such that for all α and β in $\mathrm{dom}(p)$:

$$p(\alpha) \cap p(\beta) \text{ is an initial part of } p(\alpha) \text{ and of } p(\beta), \tag{I.8}$$

$$\Delta(p(\alpha), p(\beta)) \leq \rho(\alpha,\beta) \text{ provided that } \alpha \neq \beta. \tag{I.9}$$

Let $p \leq q$ whenever $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and $p(\alpha) \supseteq q(\alpha)$ for all $\alpha \in \mathrm{dom}(q)$. Clearly $\mathcal{P}$ is a λ-closed forcing notion. The proof that $\mathcal{P}$ satisfies the κ-chain condition, depends heavily on the properties of the set-mapping D and can be found in [76]. $\dashv$

Recall that a poset satisfies *property K* (for Knaster) if every uncountable subset has a further uncountable subset consisting of pairwise compatible elements.

12.14 Corollary. *If* $\square_{\omega_1}$ *holds, and so in particular if* ω_2 *is not a Mahlo cardinal in the constructible universe, then there is a property K poset, forcing the Kurepa hypothesis.* $\dashv$

12.15 Remark. This is a variation on a result of Jensen, namely that under $\square_{\omega_1}$ there is a ccc poset forcing the Kurepa hypothesis. Veličković [77] was the first to use the function ρ to reprove Jensen's result though his proof works only in case $\kappa = \omega_1$ and produces only a ccc poset rather than a property K poset. It should also be noted that Jensen also proved (see [28]) that in the Levy collapse of a Mahlo cardinal to ω_2 there is no ccc poset forcing the Kurepa hypothesis. We shall now see that ρ provides sufficient ground for another well-known forcing construction, the forcing construction of Baumgartner and Shelah [4] of a locally compact scattered topology on ω_2 all of whose Cantor-Bendixson ranks are countable.

12.16 Theorem. *If* $\square_{\omega_1}$ *holds then there is a property K forcing notion that introduces a locally compact scattered topology on* ω_2 *all of whose Cantor-Bendixson ranks are countable.*

Proof. Let $\mathcal{P}$ be the set of all $p = \langle D_p, \leq_p, M_p\rangle$ where D_p is a finite subset of ω_2, where $\leq_p$ is a partial ordering of D_p compatible with the well-ordering and $M_p : [D_p]^2 \longrightarrow [\omega_2]^{<\omega}$ has the following properties:

(i) $M_p\{\alpha,\beta\} \subseteq D\{\alpha,\beta\} \cap D_p$,

(ii) $M_p\{\alpha,\beta\} = \{\alpha\}$ if $\alpha \leq_p \beta$ and $M_p\{\alpha,\beta\} = \{\beta\}$ if $\beta \leq_p \alpha$,

(iii) $\gamma \leq_p \alpha, \beta$ for all $\gamma \in M_p\{\alpha,\beta\}$,

(iv) for every $\delta \leq_p \alpha,\beta$ there is a $\gamma \in M_p\{\alpha,\beta\}$ such that $\delta \leq_p \gamma$.

We let $p \leq q$ if and only if $D_p \supseteq D_q$, $\leq_p\restriction D_q = \leq_q$ and $M_p\restriction[D_q]^2 = M_q$. To verify that $\mathcal{P}$ satisfies property K one again relies heavily on the properties of the function D. Full details about this can be found for example in [76]. $\dashv$

12.17 Remark. A function $f : [\omega_2]^2 \longrightarrow [\omega_2]^{\leq\omega}$ has *property* Δ if for every uncountable set A of finite subsets of ω_2 there exist a and b in A such that for all $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$, $\alpha, \beta > \gamma$ implies $\gamma \in f\{\alpha,\beta\}$, if $\beta > \gamma$ implies $f\{\alpha,\gamma\} \subseteq f\{\alpha,\beta\}$, and if $\alpha > \gamma$ implies $f\{\beta,\gamma\} \subseteq f\{\alpha,\beta\}$. This definition is due to Baumgartner and Shelah [4] who used it in their forcing construction of the scattered topology on ω_2. They were also able to force a function with the property Δ using a σ-closed ω_2-cc poset. This part of their result was reproved by Veličković (see [4, p.129]) who showed that the function $D\{\alpha,\beta\} = \{\xi \leq \min\{\alpha,\beta\} : \rho(\xi,\alpha) \leq \rho\{\alpha,\beta\}\}$ has property Δ. We have seen above that D has many more properties of independent interest which are likely to be needed in similar forcing constructions. The reader is referred to papers of Koszmider [31] and Rabus [45] for further work in this area.

13. Successors of Singular Cardinals

In the previous section we saw that the function $\rho : [\kappa^+]^2 \longrightarrow \kappa$ defined from a $\square_\kappa$-sequence C_α $(\alpha < \kappa^+)$ can be quite a useful tool in stepping-up objects from κ to κ^+. In this section we analyse the stepping-up power of ρ under the assumption that κ is a singular cardinal of cofinality ω. So let κ_n $(n < \omega)$ be a strictly increasing sequence of regular cardinals converging to κ fixed from now on. This immediately gives rise to a rather striking tree decomposition $<_n$ $(n < \omega)$ of the $\in$-relation on κ^+:

$$\alpha <_n \beta \text{ if and only if } \rho(\alpha,\beta) \leq \kappa_n. \tag{I.10}$$

13.1 Lemma.

(1) $\in\restriction(\kappa^+ \times \kappa^+) = \bigcup_{n<\omega} <_n$,

(2) $<_n \subseteq <_{n+1}$,

(3) $(\kappa^+, <_n)$ *is a tree of height* $\leq \kappa_n^+$. ⊣

13.2 Definition. Let $F_n(\alpha) = \{\xi \leq \alpha : \rho(\xi, \alpha) \leq \kappa_n\}$, and let $f_\alpha(n) = \mathrm{tp}(F_n(\alpha))$ for $\alpha < \kappa^+$ and $n < \omega$. Let $L = \{f_\alpha : \alpha < \kappa^+\}$, considered as a linearly ordered set with the lexicographical ordering.

Since L is a subset of ${}^\omega\kappa$, it has an order-dense subset of size κ, so in particular it contains no well-ordered subset of size κ^+. The following result shows, however, that every subset of L of smaller size is the union of countably many well-ordered subsets.

13.3 Lemma. *For each* $\beta < \kappa^+$, $L_\beta = \{f_\alpha : \alpha < \beta\}$ *can be decomposed into countably many well-ordered subsets.*

Proof. Let $L_{\beta n} = \{f_\alpha : \alpha \in F_n(\beta)\}$ for $n < \omega$. Note that the projection $f \longmapsto f{\restriction}(n+1)$ is one-to-one on $L_{\beta n}$ so each $L_{\beta n}$ is lexicographically well-ordered. ⊣

13.4 Remark. Note that $\mathcal{K} = \{\{(n, f_\alpha(n)) : n < \omega\} : \alpha < \kappa^+\}$ is a family of countable subsets of $\omega \times \kappa$ which has the property that $\mathcal{K}{\restriction}X = \{K \cap X : K \in \mathcal{K}\}$ has size $\leq |X| + \aleph_0$ for every $X \subseteq \omega \times \kappa$ of size $< \kappa$. We shall now see that with a bit more work a considerably finer such a family can be constructed.

13.5 Definition. If a family $\mathcal{K} \subseteq [S]^\omega$ is at the same time locally countable and cofinal in $[S]^\omega$ then we call it a *cofinal Kurepa family* (*cofinal K-family* for short). Two cofinal K-families $\mathcal{H}$ and $\mathcal{K}$ are *compatible* if $H \cap K \in \mathcal{H} \cap \mathcal{K}$ for all $H \in \mathcal{H}$ and $K \in \mathcal{K}$. We say that $\mathcal{K}$ *extends* $\mathcal{H}$ if they are compatible and if $\mathcal{H} \subseteq \mathcal{K}$.

13.6 Remark. Note that the size of any cofinal K-family $\mathcal{K}$ on a set S is equal to the cofinality of $[S]^\omega$. Note also that for every $X \subseteq S$ there is a $Y \supseteq X$ of size $\mathrm{cf}([X]^\omega)$ such that $K \cap Y \in \mathcal{K}$ for all $K \in \mathcal{K}$.

13.7 Definition. Define $\mathrm{CK}(\theta)$ to be the statement that every sequence $\mathcal{K}_n$ $(n < \omega)$ of comparable cofinal K-families with domains included in θ which are closed under $\cup$, $\cap$ and $\setminus$ can be extended to a single cofinal K-family on θ, which is also closed under these three operations.

13.8 Lemma. $\mathrm{CK}(\omega_1)$ *is true and if* $\mathrm{CK}(\theta)$ *is true for some* θ *such that* $\mathrm{cf}([\theta]^\omega) = \theta$ *then* $\mathrm{CK}(\theta^+)$ *is also true.*

Proof. The easy proof of $\mathrm{CK}(\omega_1)$ is left to the reader.

Suppose $\mathrm{CK}(\theta)$ and let $\mathcal{K}_n$ $(n < \omega)$ be a given sequence of compatible cofinal K-families as in the hypothesis of $\mathrm{CK}(\theta^+)$. By Remark 13.6 there is a strictly increasing sequence δ_ξ $(\xi < \theta^+)$ of ordinals $< \theta$ such that $\mathcal{K}_n{\restriction}\delta_\xi \subseteq \mathcal{K}_n$ for all $\xi < \theta^+$ and $n < \omega$. Recursively on $\xi < \theta^+$ we construct a chain $\mathcal{H}_\xi$ ($\xi < \theta^+$) of cofinal K-families as follows. If $\xi = 0$ or $\xi = \eta + 1$ for some η, using $\mathrm{CK}(\delta_\xi)$ we can find a cofinal K-family $\mathcal{H}_\xi$ on δ_ξ extending $\mathcal{H}_{\xi-1}$ and $\mathcal{K}_n \restriction \delta_\xi$ ($n < \omega$). If ξ has uncountable cofinality then the union of $\bar{\mathcal{H}}_\xi = \bigcup_{\eta<\xi} \mathcal{H}_\eta$ is a cofinal K-family with domain included in δ_ξ, so using $\mathrm{CK}(\delta_\xi)$ we can find a cofinal K-family $\mathcal{H}_\xi$ on δ_ξ extending $\bar{\mathcal{H}}_\xi$ and $\mathcal{K}_n \restriction \delta_\xi$ ($n < \omega$). If ξ has countable cofinality, pick a sequence $\{\xi_n\}$ converging to ξ and use $\mathrm{CK}(\delta_\xi)$ to find a cofinal K-family $\mathcal{H}_\xi$ on δ_ξ extending $\mathcal{H}_{\xi_n}$ ($n < \omega$) and $\mathcal{K}_n$ ($n < \omega$). When the recursion is done, set $\mathcal{H} = \bigcup_{\xi<\theta^+} \mathcal{H}_\xi$. Then $\mathcal{H}$ is a cofinal K-family on θ^+ extending $\mathcal{K}_n$ ($n < \omega$). ⊣

13.9 Corollary. *For each $n < \omega$ there is a cofinal Kurepa family on ω_n.* ⊣

13.10 Definition. Let κ be a cardinal of cofinality ω. A *Jensen matrix* on κ^+ is a matrix $\mathrm{J}_{\alpha n}$ ($\alpha < \kappa^+, n < \omega$) of subsets of κ with the following properties, where κ_n ($n < \omega$) is some increasing sequence of cardinals converging to κ:

(1) $|\mathrm{J}_{\alpha n}| \leq \kappa_n$ for all $\alpha < \kappa^+$ and $n < \omega$,

(2) for all $\alpha < \beta$ and $n < \omega$ there is an $m < \omega$ such that $\mathrm{J}_{\alpha n} \subseteq \mathrm{J}_{\beta m}$,

(3) $\bigcup_{n<\omega} [\mathrm{J}_{\beta n}]^\omega = \bigcup_{\alpha<\beta} \bigcup_{n<\omega} [\mathrm{J}_{\alpha n}]^\omega$ whenever $\mathrm{cf}(\beta) > \omega$,

(4) $[\kappa^+]^\omega = \bigcup_{\alpha<\kappa^+} \bigcup_{n<\omega} [\mathrm{J}_{\alpha n}]^\omega$.

13.11 Remark. The notion of a Jensen matrix is the combinatorial essence behind Silver's proof of Jensen's model-theoretic two-cardinal transfer theorem in the constructible universe (see [27, appendix]), so the matrix could equally well be called 'Silver matrix'. It has been implicitly or explicitly used in several places in the literature. The reader is referred to the paper of Foreman and Magidor [21] which gives quite a complete discussion of this notion and its occurrence in the literature.

13.12 Lemma. *Suppose some cardinal κ of countable cofinality carries a Jensen matrix $\mathrm{J}_{\alpha n}$ ($\alpha < \kappa^+, n < \omega$) relative to some sequence of cardinals κ_n ($n < \omega$) that converge to κ. If $\mathrm{CK}(\kappa_n)$ holds for all $n < \omega$ then $\mathrm{CK}(\kappa^+)$ is also true.*

Proof. Let $\mathcal{K}_n$ ($n < \omega$) be a given sequence of compatible cofinal K-families with domains included in κ^+. Given $\mathrm{J}_{\alpha n}$, there is a natural continuous chain $\mathrm{J}^\xi_{\alpha n}$ ($\xi < \omega_1$) of subsets of κ^+ of size $\leq \kappa_n$ such that $\mathrm{J}^0_{\alpha n} = \mathrm{J}_{\alpha n}$ and $\mathrm{J}^{\xi+1}_{\alpha n}$ equal to the union of all $K \in \bigcup_{n<\omega} \mathcal{K}_n$ which intersect $\mathrm{J}^\xi_{\alpha n}$. Let $\mathrm{J}^*_{\alpha n} = \bigcup_{\xi<\omega_1} \mathrm{J}^\xi_{\alpha n}$. It is easily seen that $\mathrm{J}^*_{\alpha n}$ ($\alpha < \kappa^+, n < \omega$) is also a Jensen matrix. By recursion on α and n we define a sequence $\mathcal{H}_{\alpha n}$ ($\alpha < \kappa^+, n < \omega$) of compatible cofinal K-families as follows. If $\alpha = \beta + 1$ or $\alpha = 0$ and $n < \omega$ using $\mathrm{CK}(\kappa_n)$ we can find a cofinal K-family $\mathcal{H}_{\alpha n}$ with domain $\mathrm{J}^*_{\alpha n}$

compatible with $\mathcal{H}_{\alpha m}$ $(m < n)$, $\mathcal{H}_{(\alpha-1)m}$ $(m < \omega)$ and $\mathcal{K}_m \restriction \mathrm{J}^*_{\alpha n}$ $(m < \omega)$. If $\mathrm{cf}(\alpha) = \omega$ let α_n $(n < \omega)$ be an increasing sequence of ordinals converging to α. Using $\mathrm{CK}(\kappa_n)$ we can find a cofinal K-family $\mathcal{H}_{\alpha n}$ which extends $\mathcal{H}_{\alpha m}$ $(m < n)$, $\mathcal{K}_m \restriction \mathrm{J}^*_{\alpha n}$ $(m < \omega)$ and each of the families $\mathcal{H}_{\alpha_i k}$ $(i < \omega, k < \omega$ and $\mathrm{J}^*_{\alpha_i k} \subseteq \mathrm{J}^*_{\alpha n})$. Finally, suppose that $\mathrm{cf}(\alpha) > \omega$. For $n < \omega$, set

$$\mathcal{H}_{\alpha n} = [\mathrm{J}^*_{\alpha n}]^\omega \cap (\textstyle\bigcup_{\xi<\alpha} \bigcup_{m<\omega} \mathcal{H}_{\xi m}).$$

Using the properties of the Jensen matrix (especially (3)) as well as the compatibility of $\mathcal{H}_{\xi m}$ $(\xi < \alpha, m < \omega)$ one easily checks that $\mathcal{H}_{\alpha n}$ is a cofinal K-family with domain $\mathrm{J}^*_{\alpha n}$ which extends each member of $\mathcal{H}_{\alpha m}$ $(m < n)$ and $\mathcal{K}_m \restriction \mathrm{J}^*_{\alpha n}$ $(m < \omega)$ and which is compatible with all of the previously constructed families $\mathcal{H}_{\xi m}$ $(\xi < \alpha, m < \omega)$. When the recursion is done we set

$$\mathcal{H} = \textstyle\bigcup_{\alpha<\kappa^+} \bigcup_{n<\omega} \mathcal{H}_{\alpha n}.$$

Using the property (4) of $\mathrm{J}^*_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$, it follows easily that $\mathcal{H}$ is a cofinal K-family on κ^+ extending $\mathcal{K}_n$ $(n < \omega)$. ⊣

13.13 Theorem. *If a Jensen matrix exists on any successor of a cardinal of cofinality ω, then a cofinal Kurepa family exists on any domain.* ⊣

The ρ-function $\rho : [\kappa^+]^2 \longrightarrow \kappa$ associated with a $\square_\kappa$-sequence C_α $(\alpha < \kappa^+)$ for some singular cardinal κ of cofinality ω leads to the matrix

$$F_n(\alpha) = \{\xi < \alpha : \rho(\xi, \alpha) \leq n\} (\alpha < \kappa^+, n < \omega) \tag{I.11}$$

which has the properties (1)-(3) of 13.10 as well as some other properties not captured by the definition of a Jensen matrix. If one additionally has a sequence a_α $(\alpha < \kappa^+)$ of countable subsets of κ^+ that is cofinal in $[\kappa^+]^\omega$ one can extend the matrix (I.11) as follows:

$$M_{\beta n} = \textstyle\bigcup_{\alpha <_n \beta} (a_\alpha \cup \{\alpha\}) \ (\beta < \kappa^+, n < \omega).$$

(Recall that $<_n$ is the tree ordering on κ^+ defined by the formula $\alpha <_n \beta$ iff $\rho(\alpha, \beta) \leq \kappa_n$ where κ_n is a fixed increasing sequence of cardinals converging to κ.) The matrix $M_{\beta n}$ $(\beta < \kappa^+, n < \omega)$ has properties not captured by Definition 13.10 that are of independent interest.

13.14 Lemma.

(1) $\alpha <_n \beta$ *implies* $M_{\alpha n} \subseteq M_{\beta n}$,

(2) $M_{\alpha m} \subseteq M_{\alpha n}$ *whenever* $m < n$,

(3) *if* $\beta = \sup\{\alpha : \alpha <_n \beta\}$ *then* $M_{\beta n} = \bigcup_{\alpha <_n \beta} M_{\alpha n}$,

(4) *every countable subset of* κ^+ *is covered by some* $M_{\beta n}$.

(5) $\mathcal{M} = \{M_{\beta n} : \beta < \kappa^+, n < \omega\}$ *is a locally countable family if we have started with a locally countable* $\mathcal{K} = \{a_\alpha : \alpha < \kappa^+\}$. ⊣

13.15 Remark. One can think of the matrix $\mathcal{M} = \{M_{\beta n} : \beta < \kappa^+, n < \omega\}$ as a version of a 'morass' for the singular cardinal κ (see [78]). It would be interesting to see how far one can go in this analogy. We give a few applications just to illustrate the usefulness of the families we have constructed so far.

13.16 Definition. A *Bernstein decomposition* of a topological space X is a function $f : X \longrightarrow 2^{\mathbb{N}}$ with the property that f takes all the values from $2^{\mathbb{N}}$ on any subset of X homeomorphic to the Cantor set.

13.17 Remark. The classical construction of Bernstein [6] can be interpreted by saying that every space of size at most continuum admits a Bernstein decomposition. For larger spaces one must assume Hausdorff's separation axiom, a result of Nešetril and Rődl (see [42]). In this context Malykhin was able to extend Bernstein's result to all spaces of size $< \mathfrak{c}^{+\omega}$ (see [39]). To extend this to all Hausdorff spaces, some use of square sequences seems natural. In fact, the first Bernstein decompositions of an arbitrary Hausdorff space have been constructed using $\square_\kappa$ and $\kappa^\omega = \kappa^+$ for every $\kappa > \mathfrak{c}$ of cofinality ω by Weiss [82] and Wolfsdorf [83]. We shall now see that cofinal K-families are quite natural tools in constructions of Bernstein decompositions. The proof of this result can be found for example in [76].

13.18 Theorem. *Suppose every regular* $\theta > \mathfrak{c}$ *supports a cofinal Kurepa family of size* θ*. Then every Hausdorff space admits a Bernstein decomposition.* ⊣

It is interesting that various less pathological classes of spaces admit a local version of Theorem 13.18 (see [76]).

13.19 Theorem. *Every metric space that carries a cofinal Kurepa family admits a Bernstein decomposition.* ⊣

13.20 Definition. Recall the notion of a *coherent family* of partial functions indexed by some ideal $\mathfrak{I}$, a family of the form $f_a : a \longrightarrow \omega$ $(a \in \mathfrak{I})$ with the property that $\{x \in a \cap b : f_a(x) \neq f_b(x)\}$ is finite for all $a, b \in \mathfrak{I}$.

It can be seen (see [74]) that the P-ideal dichotomy (see 3.9) has a strong influence on such families provided $\mathfrak{I}$ is a P-ideal of countable subsets of some set Γ.

13.21 Theorem. *Assuming the P-ideal dichotomy, for every coherent family of functions* $f_a : a \longrightarrow \omega$ $(a \in \mathfrak{I})$ *indexed by some P-ideal* $\mathfrak{I}$ *of countable subsets of some set* Γ*, either*

(1) *There is an uncountable* $\Delta \subseteq \Gamma$ *such that* $f_a \restriction \Delta$ *is finite-to-one for all* $a \in \mathfrak{I}$, *or*

(2) *There is a* $g : \Gamma \longrightarrow \omega$ *such that* $g \restriction a =^* f_a$ *for all* $a \in \mathfrak{I}$.

Proof. Let $\mathfrak{L}$ be the family of all countable subsets b of Γ for which one can find an a in $\mathfrak{I}$ such that $b \setminus a$ is finite and f_a is finite-to-one on b. To see that $\mathfrak{L}$ is a P-ideal, let $\{b_n\}$ be a given sequence of members of $\mathfrak{L}$ and for each n fix a member a_n of $\mathfrak{I}$ such that f_{a_n} is finite-to-one on b_n. Since $\mathfrak{I}$ is a P-ideal, we can find $a \in \mathfrak{I}$ such that $a_n \setminus a$ is finite for all n. Note that for all n, $b_n \setminus a$ is finite and that f_a is finite-to-one on b_n. For $n < \omega$, let

$$b_n^* = \{\xi \in b_n \cap a : f_a(\xi) > n\}.$$

Then b_n^* is a cofinite subset of b_n for each n, so if we set b to be equal to the union of the b_n^*'s, we get a subset of a which almost includes each b_n and on which f_a is finite-to-one. It follows that b belongs to $\mathfrak{L}$. This completes the proof that $\mathfrak{L}$ is a P-ideal. Applying the P-ideal dichotomy to $\mathfrak{L}$, we get the two alternatives that translate into the alternatives (1) and (2) of the theorem. $\dashv$

This leads to the natural question whether for any set Γ one can construct a family $\{f_a : a \longrightarrow \omega\}$ of finite-to-one mappings indexed by $[\Gamma]^\omega$. This question was answered by Koszmider [30] using the notion of a Jensen matrix discussed above. We shall present Koszmider's result using the notion of a cofinal Kurepa family instead.

13.22 Theorem. *If* Γ *carries a cofinal Kurepa family then there is a coherent family* $f_a : a \longrightarrow \omega$ $(a \in [\Gamma]^\omega)$ *of finite-to-one mappings.*

Proof. Let $\mathcal{K}$ be a fixed well-founded cofinal K-family on Γ and let $<_w$ be a well-ordering of $\mathcal{K}$ compatible with $\subseteq$. It suffices to produce a coherent family of finite-to-one mappings indexed by $\mathcal{K}$. This is done by induction on $<_w$. Suppose $K \in \mathcal{K}$ and $f_H : H \longrightarrow \omega$ is determined for all $H \in \mathcal{K}$ with $H <_w K$. Let H_n $(n < \omega)$ be a sequence of elements of $\mathcal{K}$ that are $<_w K$ and have the property that for every $H \in \mathcal{K}$ with $H <_w K$ there is an $n < \omega$ such that $H \cap K =^* H_n \cap K$. So it suffices to construct a finite-to-one $f_K : K \longrightarrow \omega$ which coheres with each f_{H_n} $(n < \omega)$, a straightforward task. $\dashv$

13.23 Corollary. *For every nonnegative integer n there is a coherent family* $f_a : a \longrightarrow \omega$ $(a \in [\omega_n]^\omega)$ *of finite-to-one mappings.* $\dashv$

13.24 Remark. It is interesting that 'finite-to-one' cannot be replaced by 'one-to-one' in these results. For example, there is no coherent family of one-to-one mappings $f_a : a \longrightarrow \omega$ $(a \in [\mathfrak{c}^+]^\omega)$. We finish this section with a typical application of coherent families of finite-to-one mappings discovered by Scheepers [48].

13.25 Theorem. *If there is a coherent family $f_a : a \longrightarrow \omega$ $(a \in [\Gamma]^\omega)$ of finite-to-one mappings, then there is an $F : [[\Gamma]^\omega]^2 \longrightarrow [\Gamma]^{<\omega}$ with the property that for every strictly $\subseteq$-increasing sequence a_n $(n < \omega)$ of countable subsets of Γ, the union of $F(a_n, a_{n+1})$ $(n < \omega)$ covers the union of a_n $(n < \omega)$.*

Proof. For $a \in [\Gamma]^\omega$ let $x_a : \omega \longrightarrow \omega$ be defined by letting $x_a(n) = |\{\xi \in a : f_a(\xi) \leq n\}|$. Note that x_a is eventually dominated by x_b whenever a is a proper subset of b. Choose $\Phi : \omega^\omega \longrightarrow \omega^\omega$ with the property that $x <^* y$ implies $\Phi(y) <^* \Phi(x)$, where $<^*$ is the ordering of eventual dominance on ω^ω (i.e. $x <^* y$ if $x(n) < y(n)$ for all but finitely many n's). Define another family of functions $g_a : a \longrightarrow \omega$ $(a \in [\Gamma]^\omega)$ by letting

$$g_a(\xi) = \Phi(x_a)(f_a(\xi)).$$

Note the following interesting property of g_a $(a \in [\Gamma]^\omega)$:

$$F(a, b) = \{\xi \in a : g_b(\xi) \geq g_a(\xi)\} \text{ is finite for all } a \subsetneqq b \text{ in } [\Gamma]^\omega.$$

So if a_n $(n < \omega)$ is a strictly $\subseteq$-increasing sequence of countable subsets of Γ and $\bar{\xi}$ belongs to some $a_{\bar{n}}$ then the sequence of integers $g_{a_n}(\bar{\xi})$ $(\bar{n} \leq n < \omega)$ must have some place $n \geq \bar{n}$ with the property that $g_{a_n}(\bar{\xi}) < g_{a_{n+1}}(\bar{\xi})$, i.e. a place $n \geq \bar{n}$ such that $\xi \in F(a_n, a_{n+1})$. ⊣

13.26 Remark. Note that if κ is a singular cardinal of cofinality ω with the property that $\mathrm{cf}([\theta]^\omega) < \kappa$ for all $\theta < \kappa$, then the existence of a cofinal Kurepa family on κ^+ implies the existence of a Jensen matrix on κ^+. So these two notions appear to be quite close to each other. The three basic properties of the function $\rho : [\kappa^+] \longrightarrow \kappa$ (11.5 and 11.6(a),(b)) seem much stronger in view of the fact that the linear ordering as in 13.3 cannot exist for κ above a supercompact cardinal and the fact that Foreman and Magidor [21] have produced a model with a supercompact cardinal that carries a Jensen matrix on any successor of a singular cardinal of cofinality ω. The "Chang's conjecture" $(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega)$ is the model-theoretic transfer principle asserting that every structure of the form $(\kappa^+, \kappa, <, \ldots)$ with a countable signature has an uncountable elementary submodel B with the property that $B \cap \omega_1$ is countable. Note that $(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega)$ for some singular κ of cofinality ω implies that every locally countable family $\mathcal{K} \subseteq [\kappa]^\omega$ must have size $\leq \kappa$. So, one of the models of set theory that has no cofinal K-family on, say $\aleph_{\omega+1}$, is the model of Levinski, Magidor and Shelah [38], in which $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega)$ holds. It seems still unknown whether the conclusion of Theorem 13.25 can be proved without additional set-theoretic assumptions.

14. The Oscillation Mapping

In what follows, θ will be a fixed regular infinite cardinal.

$$\mathrm{osc} : \mathcal{P}(\theta)^2 \longrightarrow \mathrm{Card}$$

is defined by

$$\mathrm{osc}(x,y) = |x \setminus (\sup(x \cap y) + 1)/\sim|,$$

where $\sim$ is the equivalence relation on $x \setminus (\sup(x \cap y) + 1)$ defined by letting $\alpha \sim \beta$ iff the closed interval determined by α and β contains no point from y. So, if x and y are disjoint, $\mathrm{osc}(x,y)$ is simply the number of convex pieces the set x is split by the set y. The oscillation mapping has proven to be a useful device in various schemes for coding information. It usefulness in a given context depend very much of the corresponding 'oscillation theory', a set of definitions and lemmas that disclose when it is possible to achieve a given number as oscillation between two sets x and y in a given family $\mathcal{X}$. The following definition reveals the notion of largeness relevant to the oscillation theory that we develop in this section.

14.1 Definition. A family $\mathcal{X} \subseteq \mathcal{P}(\theta)$ is *unbounded* if for every closed and unbounded subset C of θ there exist $x \in \mathcal{X}$ and an increasing sequence $\{\delta_n : n < \omega\} \subseteq C$ such that $\sup(x \cap \delta_n) < \delta_n$ and $[\delta_n, \delta_{n+1}) \cap x \neq \emptyset$ for all $n < \omega$.

This notion of unboundedness has proven to be the key behind a number of results asserting the complex behaviour of the oscillation mapping on $\mathcal{X}^2$. The case $\theta = \omega$ seems to contain the deeper part of the oscillation theory known so far (see [64],[65, §1] and [73]), though in this section we shall only consider the case $\theta > \omega$. We shall also restrict ourselves to the family $\mathcal{K}(\theta)$ of all closed bounded subsets of θ rather than the whole power-set of θ. Our next lemma is the basic result about the behavior of the oscillation mapping in this context. Its proof can again be found in [76].

14.2 Lemma. *If $\mathcal{X}$ is an unbounded subfamily of $\mathcal{K}(\theta)$ then for every positive integer n there exist x and y in $\mathcal{X}$ such that* $\mathrm{osc}(x,y) = n$. ⊣

Lemma 14.2 also has a rectangular form.

14.3 Lemma. *If $\mathcal{X}$ and $\mathcal{Y}$ are two unbounded subfamilies of $\mathcal{K}(\theta)$ then for all but finitely many positive integers n there exist $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that* $\mathrm{osc}(x,y) = n$. ⊣

Recall the notion of a nontrivial C-sequence C_α $(\alpha < \theta)$ on θ from Section 8, a C-sequence with the property that for every closed and unbounded subset C of θ there is a limit point δ of C such that $C \cap \delta \nsubseteq C_\alpha$ for all $\alpha < \theta$.

14.4 Definition. For a subset D of θ let $\lim(D)$ denote the set of all $\alpha < \theta$ with the property that $\alpha = \sup(D \cap \alpha)$. A subsequence C_α $(\alpha \in \Gamma)$ of some C-sequence C_α $(\alpha < \theta)$ is *stationary* if the union of all $\lim(C_\alpha)$ $(\alpha \in \Gamma)$ is a stationary subset of θ.

14.5 Lemma. *A stationary subsequence of a nontrivial C-sequence on θ is an unbounded family of subsets of θ.*

Proof. Let C_α $(\alpha \in \Gamma)$ be a given stationary subsequence of a nontrivial C-sequence on θ. Let C be a given closed and unbounded subset of θ. Let Δ be the union of all $\lim(C_\alpha)$ $(\alpha \in \Gamma)$. Then Δ is a stationary subset of θ. For $\xi \in \Delta$ choose $\alpha_\xi \in \Gamma$ such that $\xi \in \lim(C_{\alpha_\xi})$. Applying the assumption that C_α $(\alpha \in \Gamma)$ is a nontrivial C-subsequence, we can find a $\xi \in \Delta \cap \lim(C)$ such that

$$C \cap [\eta, \xi) \nsubseteq C_{\alpha_\xi} \text{ for all } \eta < \xi. \tag{I.12}$$

If such a ξ cannot be found using the stationarity of the set $\Delta \cap \lim(C)$ we would be able to use the Pressing Down Lemma on the regressive mapping that would give us an $\eta < \xi$ violating (I.12) and get that a tail of C trivializes C_α $(\alpha \in \Gamma)$. Using (I.12) and the fact that $C_{\alpha_\xi} \cap \xi$ is unbounded in ξ we can find a strictly increasing sequence δ_n $(n < \omega)$ of elements of $(C \cap \xi) \setminus C_{\alpha_\xi}$ such that $[\delta_n, \delta_{n+1}) \cap C_{\alpha_\xi} \neq \emptyset$ for all n. So the set C_{α_ξ} satisfies the conclusion of Definition 14.1 for the given closed and unbounded set C. ⊣

Recall that $\mathbb{Q}_\theta$ denotes the set of all finite sequences of ordinals $< \theta$ and that we consider it ordered by the right lexicographical ordering. We need the following two further orderings on $\mathbb{Q}_\theta$: $s \sqsubseteq t$ if and only if s is an initial segment of t, and $s \sqsubset t$ if and only if s is a proper initial part of t.

14.6 Definition. Given a C-sequence C_α $(\alpha < \theta)$ we can define an *action* $(\alpha, t) \longmapsto \alpha_t$ of $\mathbb{Q}_\theta$ on θ recursively on the ordering $\sqsubseteq$ of $\mathbb{Q}_\theta$ as follows: $\alpha_\emptyset = \alpha$, $\alpha_{\langle \xi \rangle}$ is equal to the ξth member of C_α if $\xi < \mathrm{tp}(C_\alpha)$; otherwise $\alpha_{\langle \xi \rangle} = \alpha$, and finally, $\alpha_{t ^\frown \langle \xi \rangle} = (\alpha_t)_{\langle \xi \rangle}$.

14.7 Remark. Note that if $\rho_0(\alpha, \beta) = t$ for some $\alpha < \beta < \theta$ then $\beta_t = \alpha$. In fact, if $\beta = \beta_0 > \ldots > \beta_n = \alpha$ is the walk from β to α along the C-sequence, each member of the trace $\mathrm{Tr}(\alpha, \beta) = \{\beta_0, \beta_1, \ldots, \beta_n\}$ has the form β_s where s is the uniquely determined initial part of t. Note, however, that in general $\beta_t = \alpha$ does not imply that $\rho_0(\alpha, \beta) = t$.

14.8 Notation. Given a C-sequence C_α $(\alpha < \theta)$ on θ we shall use $\mathrm{osc}(\alpha, \beta)$ to denote $\mathrm{osc}(C_\alpha, C_\beta)$.

The proof of the following result can be found in [76].

14.9 Theorem. *If C_α $(\alpha < \theta)$ is a nontrivial C-sequence on θ, then for every unbounded set $\Gamma \subseteq \theta$ and positive integer n there exist $\alpha < \beta$ in Γ and $t \sqsubseteq \rho_0(\alpha, \beta)$ such that $\mathrm{osc}(\alpha_t, \beta_t) = n$, but $\mathrm{osc}(\alpha_s, \beta_s) = 1$ for all $s \sqsubset t$.* ⊣

14.10 Corollary. *Suppose a regular uncountable cardinal θ carries a nontrivial C-sequence. Then there is an $f : [\theta]^2 \longrightarrow \omega$ which takes all the values from ω on any set of the form $[\Gamma]^2$ for an unbounded subset of θ.*

Proof. Given $\alpha < \beta < \theta$, if there is a $t \sqsubseteq \rho_0(\alpha, \beta)$ satisfying the conclusion of 14.9, put $f(\alpha, \beta) = \mathrm{osc}(\alpha_t, \beta_t) - 2$; otherwise put $f(\alpha, \beta) = 0$. ⊣

14.11 Remark. The class of all regular cardinals θ that carry a nontrivial C-sequence is quite extensive. It includes not only all successor cardinals but also some inaccessible as well as hyperinaccessible cardinals such as for example, the first inaccessible cardinal or the first Mahlo cardinal. In view of the well-known Ramsey-theoretic characterization of weak compactness, Corollary 14.10 leads us to the following natural question.

14.12 Question. Can the weak compactness of a strong limit regular uncountable cardinal be characterized by the fact that for every $f : [\theta]^2 \longrightarrow \omega$ there exists an unbounded set $\Gamma \subseteq \theta$ such that $f``[\Gamma]^2 \neq \omega$? This is true when ω is replaced by 2, but can any other number beside 2 be used in this characterization?

15. The Square-Bracket Operation

In this section we show that the basic idea of the square-bracket operation on ω_1 introduced in Definition 4.3 extends to a general setting on an arbitrary uncountable regular cardinal θ that carries a nontrivial C-sequence C_α $(\alpha < \theta)$. The basic idea is based on the oscillation map defined in the previous section and, in particular, on the property of this map described in Theorem 14.9: for $\alpha < \beta < \theta$ we set

$$[\alpha\beta] = \beta_t, \text{ where } t \sqsubseteq \rho_0(\alpha, \beta) \text{ is such that } \mathrm{osc}(\alpha_t, \beta_t) \geq 2 \text{ but } \mathrm{osc}(\alpha_s, \beta_s) = 1 \text{ for all } s \sqsubset t; \text{ if such a } t \text{ does not exist, we let } [\alpha\beta] = \alpha. \tag{I.13}$$

Thus, $[\alpha\beta]$ is the first place visited by β on its walk to α where a nontrivial oscillation with the corresponding step of α occurs. What Theorem 14.9 is telling us is that the nontrivial oscillation indeed happens most of the time. Results that would say that the set of values $\{[\alpha\beta] : \{\alpha, \beta\} \in [\Gamma]^2\}$ is in some sense large no matter how small the unbounded set $\Gamma \subseteq \theta$ is, would correspond to the results of Lemmas 4.4-4.5 about the square-bracket operation on ω_1. It turns out that this is indeed possible and to describe it we need the following definition.

15.1 Definition. A C-sequence C_α $(\alpha < \theta)$ on θ *avoids* a given subset Δ of θ if $C_\alpha \cap \Delta = \emptyset$ for all limit ordinals $\alpha < \theta$.

The proof of the following lemma is quite similar to the proof of the corresponding fact in case $\theta = \omega_1$ considered above though its full proof can be found in [76].

15.2 Lemma. *Suppose C_α $(\alpha < \theta)$ is a given C-sequence on θ that avoids a set $\Delta \subseteq \theta$. Then for every unbounded set $\Gamma \subseteq \theta$, the set of elements of Δ not of the form $[\alpha\beta]$ for some $\alpha < \beta$ in Γ is nonstationary in θ.* $\dashv$

A similar proof gives the following more general result.

15.3 Lemma. *Suppose C_α $(\alpha < \theta)$ avoids $\Delta \subseteq \theta$ and let A be a family of size θ consisting of pairwise disjoint finite sets, all of some fixed size n. Then the set of all elements of Δ that are not of the form $[a(1)b(1)] = [a(2)b(2)] = \ldots = [a(n)b(n)]$ for some $a \neq b$ in A is nonstationary in θ.* $\dashv$

Since $[\alpha\beta]$ belongs to the trace $\mathrm{Tr}(\alpha, \beta)$ of the walk from β to α it is not surprising that $[\cdot\cdot]$ strongly depends on the behavior of Tr. The following is one of the results which brings this out.

15.4 Lemma. *The set $\Omega \setminus \{[\alpha\beta] : \{\alpha, \beta\} \in [\Gamma]^2\}$ is not stationary in θ if and only if the set $\Omega \setminus \bigcup\{\mathrm{Tr}(\alpha, \beta) : \{\alpha, \beta\} \in [\Gamma]^2\}$ is not stationary in θ.* $\dashv$

This fact suggests the following definition.

15.5 Definition. The *trace filter* of a given C-sequence C_α $(\alpha < \theta)$ is the normal filter on θ generated by sets of the form $\bigcup\{\mathrm{Tr}(\alpha, \beta) : \{\alpha, \beta\} \in [\Gamma]^2\}$ where Γ is an unbounded subset of θ.

15.6 Remark. Having a proper (i.e. $\neq \mathcal{P}(\theta)$) trace filter is a strengthening of the nontriviality requirement on a given C-sequence C_α $(\alpha < \theta)$. For example, if a C-sequence avoids a stationary set $\Omega \subseteq \theta$, then its trace filter is nontrivial and in fact no stationary subset of Γ is a member of it. Note the following analogue of Lemma 15.4: the trace filter of a given C-sequence is the normal filter generated by sets of the form $\{[\alpha\beta] : \{\alpha, \beta\} \in [\Gamma]^2\}$ where Γ is an unbounded subset of θ. So to obtain the analogues of the results of Section 4 about the square-bracket operation on ω_1 one needs a C-sequence C_α $(\alpha < \theta)$ on θ whose trace filter is not only nontrivial but also not θ-saturated, i.e. it allows a family of θ pairwise disjoint positive sets. It turns out that the hypothesis of Lemma 15.2 is sufficient for both of these conclusions.

15.7 Lemma. *If a C-sequence on θ avoids a stationary subset of θ, then there exist θ pairwise disjoint subsets of θ that are positive with respect to its trace filter.*[17]

[17] A subset A of the domain of some filter $\mathcal{F}$ is *positive* with respect to $\mathcal{F}$ if $A \cap F \neq \emptyset$ for every $F \in \mathcal{F}$.

Proof. This follows from the well-known fact (see [29]) that if there is a normal, nontrivial and θ-saturated filter on θ, then for every stationary $\Omega \subseteq \theta$ there exists a $\lambda < \theta$ such that $\Omega \cap \lambda$ is stationary in λ (and the fact that the stationary set which is avoided by the C-sequence does not reflect in this way). $\dashv$

15.8 Corollary. *If a regular cardinal θ admits a nonreflecting stationary subset then there is a $c : [\theta]^2 \longrightarrow \theta$ which takes all the values from θ on any set of the form $[\Gamma]^2$ for some unbounded set $\Gamma \subseteq \theta$.* $\dashv$

To get such a c, one composes the square-bracket operation of some C-sequence, that avoids a stationary subset of θ, with a mapping $* : \theta \longrightarrow \theta$ with the property that the $*$-preimage of each point from θ is positive with respect to the trace filter of the square sequence. In other words, c is equal to the composition of $[\cdot\cdot]$ and $*$, i.e. $c(\alpha, \beta) = [\alpha\beta]^*$. Note that, as in Section 4, the property of the square-bracket operation from Lemma 15.3 leads to the following rigidity result which corresponds to Lemma 4.7.

15.9 Lemma. *The algebraic structure $(\theta, [\cdot\cdot], *)$ has no nontrivial automorphisms.* $\dashv$

15.10 Remark. Note that every θ which is a successor of a regular cardinal κ admits a nonreflecting stationary set. For example, $\Omega = \{\delta < \theta : \mathrm{cf}(\delta) = \kappa\}$ is such a set. Thus any C-sequence on θ that avoids Ω leads to a square bracket operation which allows analogues of all the results from Section 4 about the square-bracket operation on ω_1. The reader is urged to examine these analogues.

Let us now introduce a useful projection of the square-bracket operation, the analogue of 4.11 considered above. This concerns the case when θ is the successor of some regular cardinal κ and when the square-bracket operation is based on a fixed C-sequence C_α $(\alpha < \kappa^+)$ on κ^+ such that $\mathrm{tp}(C_\alpha) \leq \kappa$ for all α, or equivalently, such that C_α $(\alpha < \kappa^+)$ avoids the set $\Omega_\kappa = \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\}$. Let $[\cdot\cdot]$ be the corresponding square-bracket operation. Let λ be the minimal cardinal such that $2^\lambda \geq \kappa^+$. Choose a sequence r_ξ $(\xi < \kappa^+)$ of distinct subsets of λ. Let $\mathcal{H}$ be the collection of all maps $h : \mathcal{P}(D(h)) \longrightarrow \kappa^+$ where $D(h)$ is a finite subset of λ. Let $\pi : \kappa^+ \longrightarrow \mathcal{H}$ be a map with the property that $\pi^{-1}(\{h\}) \cap \Omega_\kappa$ is stationary for all $h \in \mathcal{H}$. Finally, define an operation $[\![\cdot\cdot]\!]$ on κ^+ as follows:

$$[\![\alpha\beta]\!] = \pi([\alpha\beta])(r_\alpha \cap D(\pi([\alpha\beta]))).$$

The following is a simple consequence of the property 15.3 of the square-bracket operation.

15.11 Lemma. *For every family A of size κ^+ consisting of pairwise disjoint finite subsets of κ^+ all of some fixed size n and every sequence $\xi_0, \ldots, \xi_{n-1}$*

of ordinals $< \kappa^+$ *there exist* $a \neq b$ *in* A *such that* $[\![a(i)b(i)]\!] = \xi_i$ *for all* $i < n$. ⊣

For sufficiently large cardinals θ we have the following variation on the theme first encountered above in Theorem 8.2 and the reader can find its full proof in [76].

15.12 Theorem. *Suppose* θ *is bigger than the continuum and carries a* C*-sequence avoiding a stationary set* Γ *of cofinality* $> \omega$ *ordinals in* θ. *Let* A *be a family of* θ *pairwise disjoint finite subsets of* θ, *all of some fixed size* n. *Then for every stationary* $\Gamma_0 \subseteq \Gamma$ *there exist* $s, t \in \omega^n$ *and a positive integer* k *such that for every* $l < \omega$ *there exist* $a < b$ [18] *in* A *and* $\delta_0 > \delta_1 > \ldots > \delta_l$ *in* $\Gamma_0 \cap (\max(a), \min(b))$ *such that:*

(1) $\rho_2(\delta_{i+1}, \delta_i) = k$ *for all* $i < l$,

(2) $\rho_0(a(i), b(j)) = \rho_0(\delta_0, b(j))^\frown \rho_0(\delta_1, \delta_0)^\frown \ldots {}^\frown \rho_0(\delta_l, \delta_{l-1})^\frown \rho_0(a(i), \delta_l)$ *for all* $i, j < n$,

(3) $\rho_2(\delta_0, b(j)) = t_j$ *and* $\rho_2(a(i), \delta_l) = s_i$ *for all* $i, j < n$. ⊣

From now on, θ is assumed to be a fixed cardinal satisfying the hypotheses of Theorem 15.12. It turns out that Theorem 15.12 gives us a way to define another square-bracket operation which has complex behaviour not only on squares of unbounded subsets of θ but also on rectangles formed by two unbounded subsets of θ. To define this new operation we choose a mapping $h : \omega \longrightarrow \omega$ such that:

$$\text{for every } k, m, n, p < \omega \text{ and } s \in \omega^n \text{ there is an } l < \omega \text{ such that } h(m + l \cdot k + s(i)) = m + p \text{ for all } i < n. \tag{I.14}$$

15.13 Definition. $[\cdot\cdot]_h : [\theta]^2 \longrightarrow \theta$ is defined by letting $[\alpha\beta]_h = \beta_t$ where $t = \rho_0(\alpha, \beta) \restriction h(\rho_2(\alpha, \beta))$.

Thus, $[\alpha\beta]_h$ is the $h(\rho_2(\alpha, \beta))$th place that β visits on its walk to α. It is clear that Theorem 15.12 and the choice of h in (I.14) give us the following conclusion.

15.14 Lemma. *Let* A *be a family of* θ *pairwise disjoint finite subsets of* θ, *all of some fixed size* n, *and let* Ω *be an unbounded subset of* θ. *Then almost every* $\delta \in \Gamma$ *has the form* $[a(0)\beta]_h = [a(1)\beta]_h = \ldots = [a(n-1)\beta]_h$ *for some* $a \in A$, $\beta \in \Omega$, $a < \beta$.[19] ⊣

[18] Recall that if a and b are two sets of ordinals, then the notation $a < b$ means that $\max(a) < \min(b)$.

[19] Here 'almost every' is to be interpreted by 'all except a nonstationary set'.

In fact, one can get a projection of this square-bracket operation with seemingly even more complex behaviour. Keeping the notation of Theorem 15.12, pick a function $\xi \longmapsto \xi^*$ from θ to ω such that $\{\xi \in \Gamma : \xi^* = n\}$ is stationary for all n. This gives us a way to consider the following projection of the trace function $\mathrm{Tr}^* : [\theta]^2 \longrightarrow \omega^{<\omega}$:

$$\mathrm{Tr}^*(\alpha, \beta) = \langle \min(C_\beta \setminus \alpha)^* \rangle^\frown \mathrm{Tr}^*(\alpha, \min(C_\beta \setminus \alpha)),$$

where we stipulate that $\mathrm{Tr}^*(\gamma, \gamma) = \langle \gamma^* \rangle$ for all $\gamma < \theta$. It is clear that the proof of Theorem 15.12 allows us to add the following conclusions:

15.12* Theorem. *Under the hypothesis of Theorem 15.12, its conclusion can be extended by adding the following two new statements:*

(4) $\mathrm{Tr}^*(\delta_1, \delta_0) = \ldots = \mathrm{Tr}^*(\delta_l, \delta_{l-1})$,

(5) *The maximal term of the sequence* $\mathrm{Tr}^*(\delta_1, \delta_0) = \ldots = \mathrm{Tr}^*(\delta_l, \delta_{l-1})$ *is bigger than the maximal term of any of the sequences* $\mathrm{Tr}^*(\delta_0, b(j))$ *or* $\mathrm{Tr}^*(a(i), \delta_l)$ *for* $i, j < n$. $\dashv$

15.15 Definition. For $\alpha < \beta < \theta$, let $[\alpha\beta]^* = \beta_t$ for t the minimal initial part of $\rho_0(\alpha, \beta)$ such that $\beta_t^* = \max(\mathrm{Tr}^*(\alpha, \beta))$.

Thus $[\alpha\beta]^*$ is the first place in the walk from β to α where the function $*$ reaches its maximum among all other places visited during the walk. Note that combining the conclusions (1)-(5) of Theorem $15.12^{(*)}$ we get:

15.12** Theorem. *Under the hypothesis of Theorem 15.12, its conclusion can be extended by adding the following:*

(6) $[a(i)b(j)]^* = [\delta_1\delta_0]^*$ *for all* $i, j < n$. $\dashv$

Having in mind the property of $[\cdot\cdot]_h$ stated in Lemma 15.14, the following variation is now quite natural.

15.16 Definition. $[\alpha\beta]_h^* = [\alpha[\alpha\beta]^*]_h$ for $\alpha < \beta < \theta$.

Using Theorem $15.12^{(**)}$(1)-(6) one easily gets the following conclusion.

15.17 Lemma. *Let A be a family of θ pairwise disjoint finite subsets of θ, all of some fixed size n. Then for all but nonstationarily many $\delta \in \Gamma$ one can find $a < b$ in A such that $[a(i)b(j)]_h^* = \delta$ for all $i, j < n$.* $\dashv$

15.18 Remark. Composing $[\cdot\cdot]_h^*$ with a mapping $\pi : \theta \longrightarrow \theta$ with the property that $\pi^{-1}(\{\xi\}) \cap \Gamma$ is stationary for all $\xi < \theta$, one gets a projection of $[\cdot\cdot]_h^*$ for which the conclusion of Lemma 15.17 is true for *all* $\delta < \kappa$. Assuming that θ is moreover a successor of a regular cardinal κ (of size at least continuum), in which case Γ can be taken to be $\{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\}$, and proceeding as in 15.11 above we get a projection $[\![\cdot\cdot]\!]_h^*$ with the following property:

15.19 Lemma. *For every family A of pairwise disjoint finite subsets of κ^+ all of some fixed size n and for every $q : n \times n \longrightarrow \kappa^+$ there exist $a < b$ in A such that $[\![a(i)b(j)]\!]^*_h = q(i,j)$ for all $i, j < n$.* $\dashv$

15.20 Remark. The first example of a cardinal with such a complex binary operation was given by the author [64] using the oscillation mapping described above in Section 14. It was the cardinal $\mathfrak{b}$, the minimal cardinality of an unbounded subset of ω^ω under the ordering of eventual dominance. The oscillation mapping restricted to some well-ordered unbounded subset W of ω^ω is perhaps still the most interesting example of this kind due to the fact that its properties are preserved in forcing extensions that do not change the unboundedness of W (although they can collapse cardinals and therefore destroy the properties of the square-bracket operations on them). This absoluteness of osc is the key feature behind its applications in various coding procedures (see e.g. [70]).

15.21 Theorem. *For every regular cardinal κ of size at least the continuum, the κ^+-chain condition is not productive, i.e. there exist two partially ordered sets $\mathcal{P}_0$ and $\mathcal{P}_1$ satisfying the κ^+-chain condition but their product $\mathcal{P}_0 \times \mathcal{P}_1$ fails to have this property.*

Proof. Fix two disjoint stationary subsets Γ_0 and Γ_1 of $\{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\}$. Let $\mathcal{P}_i$ be the collection of all finite subsets p of κ^+ with the property that $[\alpha\beta]^*_h \in \Gamma_i$ for all $\alpha < \beta$ in p. By Lemma 15.17, $\mathcal{P}_0$ and $\mathcal{P}_1$ are κ^+-cc posets. Their product $\mathcal{P}_0 \times \mathcal{P}_1$, however, contains a family $\langle\{\alpha\},\{\alpha\}\rangle$ $(\alpha < \kappa^+)$ of pairwise incomparable conditions. $\dashv$

15.22 Remark. Theorem 15.21 is due to Shelah [52] who proved it using similar methods. The first ZFC-examples of non-productiveness of the κ^+-chain condition were given by the author in [62] using what is today known under the name 'pcf theory'. After the full development of pcf theory it became apparent that the basic construction from [62] applies to every successor of a singular cardinal [53]. A quite different class of cardinals θ with θ-cc non-productive was given by the author in [61]. For example, $\theta = \mathrm{cf}(\mathfrak{c})$ is one of these cardinals. For an overview of recent advances in this area, the reader is referred to [41]. The following problem seems still open:

15.23 Question. Suppose that θ is a regular strong limit cardinal and the θ-chain condition is productive. Is θ necessarily a weakly compact cardinal?

16. Unbounded Functions on Successors of Regular Cardinals

In this section, κ is a regular cardinal and C_α $(\alpha < \kappa^+)$ is a fixed sequence with $\mathrm{tp}(C_\alpha) \leq \kappa$ for all $\alpha < \kappa^+$. Define $\rho^* : [\kappa^+]^2 \longrightarrow \kappa$ by

$$\rho^*(\alpha,\beta) = \sup\{\mathrm{tp}(C_\beta \cap \alpha), \rho^*(\alpha, \min(C_\beta \setminus \alpha)), \rho^*(\xi,\alpha) : \xi \in C_\beta \cap \alpha\}, \quad (\mathrm{I.15})$$

where we stipulate that $\rho^*(\gamma,\gamma) = 0$ for all $\gamma < \kappa^+$. Since $\rho^*(\alpha,\beta) \geq \rho_1(\alpha,\beta)$ for all $\alpha < \beta < \kappa^+$ by Lemma 7.1 we have the following:

16.1 Lemma. *For $\nu < \kappa$, $\alpha < \kappa^+$ the set $P_\nu(\alpha) = \{\xi \leq \alpha : \rho^*(\xi,\alpha) \leq \nu\}$ has size no more than $|\nu| + \aleph_0$.* ⊣

The proof of the following subadditivity properties of ρ^* is very similar to the proof of the corresponding fact for the function ρ from Section 11.

16.2 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $\rho^*(\alpha,\gamma) \leq \max\{\rho^*(\alpha,\beta), \rho^*(\beta,\gamma)\}$,

(b) $\rho^*(\alpha,\beta) \leq \max\{\rho^*(\alpha,\gamma), \rho^*(\beta,\gamma)\}$. ⊣

We mention a typical application of this function to the problem of existence of partial square sequences which, for example, have some applications in pcf theory (see [7]).

16.3 Theorem. *For every regular uncountable cardinal $\lambda < \kappa$ and stationary $\Gamma \subseteq \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \lambda\}$, there is a stationary set $\Sigma \subseteq \Gamma$ and a sequence C_α $(\alpha \in \Sigma)$ such that:*

(1) *C_α is a closed and unbounded subset of α,*

(2) *$C_\alpha \cap \xi = C_\beta \cap \xi$ for every $\xi \in C_\alpha \cap C_\beta$.*

Proof. For each $\delta \in \Gamma$, choose $\nu = \nu(\delta) < \kappa$ such that the set $P_{<\nu}(\delta) = \{\xi < \delta : \rho^*(\xi,\delta) < \nu\}$ is unbounded in δ and closed under taking suprema of sequences of size $< \lambda$. Then there are $\bar\nu, \bar\mu < \kappa$ and stationary $\Sigma \subseteq \Gamma$ such that $\nu(\delta) = \bar\nu$ and $\mathrm{tp}(P_{<\bar\nu}(\delta)) = \bar\mu$ for all $\delta \in \Sigma$. Let C be a fixed closed and unbounded subset of $\bar\mu$ of order-type λ. Finally, for $\delta \in \Gamma$ set

$$C_\delta = \{\alpha \in P_{<\bar\nu}(\delta) : \mathrm{tp}(P_{<\bar\nu}(\alpha)) \in C\}.$$

Using Lemma 16.2, one easily checks that C_α $(\alpha \in \Sigma)$ satisfies the conditions (1) and (2). ⊣

Another application concerns the fact described above in Section 13, that the inequalities 16.2(a),(b) are particularly useful when κ has cofinality ω. Also consider the well-known phenomenon first discovered by K.Prikry (see [29]), that in some cases, the cofinality of a regular cardinal κ can be changed to ω, while preserving all cardinals.

16.4 Theorem. *In any cardinal-preserving extension of the universe which has no new bounded subsets of κ, but in which κ has a cofinal ω-sequence diagonalizing the filter of closed and unbounded subsets of κ restricted to the ordinals of cofinality $> \omega$, there is a sequence $C_{\alpha n}(\alpha \in \lim(\kappa^+), n < \omega)$ such that for all $\alpha < \beta$ in $\lim(\kappa^+)$:*

(1) *$C_{\alpha n}$ is a closed subset of α for all n,*

(2) *$C_{\alpha n} \subseteq C_{\alpha m}$, whenever $n \leq m$,*

(3) *$\alpha = \bigcup_{n<\omega} C_{\alpha n}$,*

(4) *$\alpha \in \lim(C_{\beta n})$ implies $C_{\alpha n} = C_{\beta n} \cap \alpha$.*

Proof. For $\alpha < \kappa^+$, let D_α be the collection of all $\nu < \kappa$ for which $P_{<\nu}(\alpha)$ is σ-closed, i.e. closed under suprema of bounded countable subsets. Clearly, D_α contains a closed unbounded subset of κ, restricted to cofinality $> \omega$ ordinals. Note that $\nu \in D_\beta$ and $\rho^*(\alpha, \beta) < \nu$ imply that $\nu \in D_\alpha$. In the extended universe, pick a strictly increasing sequence ν_n $(n < \omega)$ which converges to κ and has the property that for each $\alpha < \kappa^+$ there is an $n < \omega$ such that $\nu_m \in D_\alpha$ for all $m \geq n$. Let $n(\alpha)$ be the minimal integer n with this property.

Given $\alpha < \kappa^+$ and $n < \omega$, we define $C_{\alpha n}$ according to the following cases. If there is a $\gamma \geq \alpha$ such that $n \geq n(\gamma)$ and $\sup P_{\nu_n}(\gamma) \cap \alpha = \alpha$, let $\gamma(\alpha, n)$ be the minimal such γ and let $C_{\alpha n} = \overline{P_{\nu_n}(\gamma(\alpha, n))} \cap \alpha$. If there is no such $\gamma \geq \alpha$, we let $C_{\alpha n} = \emptyset$ for $n < n(\alpha)$ and $C_{\alpha n} = \overline{P_{\nu_n}(\alpha)} \cap \alpha$ for $n \geq n(\alpha)$.

Then one can easily verify that $C_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$ satisfies the conditions (1),(2), (3) and (4). Detailed checking of this, however, can be found in [76]. $\dashv$

16.5 Remark. The combinatorial principle appearing in the statement of Theorem 16.4 is a member of a family of square principles that has been studied systematically by Schimmerling and others (see e.g. [49]). It is definitely a principle sufficient for all of the applications of $\square_\kappa$ appearing in Section 13 above.

16.6 Definition. A function $f : [\kappa^+]^2 \longrightarrow \kappa$ is *unbounded* if $f``[\Gamma]^2$ is unbounded in κ for every $\Gamma \subseteq \kappa^+$ of size κ. We shall say that such an f is *strongly unbounded* if for every family A of size κ^+, consisting of pairwise disjoint finite subsets of κ^+, and every $\nu < \kappa$ there exists an $A_0 \subseteq A$ of size κ such that $f(\alpha, \beta) > \nu$ for all $\alpha \in a$, $\beta \in b$ and $a \neq b$ in A_0.

16.7 Lemma. *If $f : [\kappa^+]^2 \longrightarrow \kappa$ is unbounded and subadditive (i.e. it satisfies the two inequalities 16.2(a),(b)), then f is strongly unbounded.*

Proof. For $\alpha < \beta < \kappa^+$, set $\alpha <_\nu \beta$ if and only if $f(\alpha, \beta) \leq \nu$. Then our assumption about f satisfying 16.2(a) and (b) reduces to the fact that each $<_\nu$ is a tree ordering on κ^+ compatible with the usual ordering on κ^+. Note that the unboundedness property of f is preserved by any forcing notion satisfying the κ-chain condition, so in particular no tree $(\kappa^+, <_\nu)$ can contain a subtree of height κ which is Souslin. In the proof of Lemma 12.4 above we have seen that this property of $(\kappa^+, <_\nu)$ alone is sufficient to conclude that every family A of κ many pairwise disjoint subsets of κ^+ contains a subfamily A_0 of size κ such that for every $a \neq b$ in A_0 every $\alpha \in a$ is $<_\nu$-incomparable to every $\beta \in b$, which is exactly the conclusion of f being strongly unbounded. $\dashv$

The following useful facts whose proof can be found in [76] relates the notions of unboundedness and subadditivity.

16.8 Lemma. *The following are equivalent:*

(1) *There is a structure* $(\kappa^+, \kappa, <, R_n)_{n<\omega}$ *with no substructure* B *of size* κ *such that* $B \cap \kappa$ *is bounded in* κ.

(2) *There is an unbounded function* $f : [\kappa^+]^2 \longrightarrow \kappa$,

(3) *There is a strongly unbounded, subadditive function* $f : [\kappa^+]^2 \longrightarrow \kappa$. $\dashv$

16.9 Remark. Recall that *Chang's conjecture* is the model-theoretic transfer principle asserting that every structure of the form $(\omega_2, \omega_1, <, \ldots)$ with a countable signature has an uncountable elementary submodel B with the property that $B \cap \omega_1$ is countable. This principle shows up in many considerations including the first two uncountable cardinals ω_1 and ω_2. For example, it is known that it is preserved by ccc forcing extensions, that it holds in the Silver collapse of an ω_1-Erdős cardinal, and that it in turn implies that ω_2 is an ω_1-Erdős cardinal in the core model of Dodd and Jensen (see e.g. [11], [29]).

16.10 Corollary. *The negation of Chang's conjecture is equivalent to the statement that there exists an* $e : [\omega_2]^2 \longrightarrow \omega_1$ *such that:*

(a) $e(\alpha, \gamma) \leq \max\{e(\alpha, \beta), e(\beta, \gamma)\}$ *whenever* $\alpha \leq \beta \leq \gamma$,

(b) $e(\alpha, \beta) \leq \max\{e(\alpha, \gamma), e(\beta, \gamma)\}$ *whenever* $\alpha \leq \beta \leq \gamma$,

(c) *For every uncountable family* A *of pairwise disjoint finite subsets of* ω_2 *and every* $\nu < \omega_1$ *there exists an uncountable* $A_0 \subseteq A$ *such that* $e(\alpha, \beta) > \nu$ *whenever* $\alpha \in a$ *and* $\beta \in b$ *for every* $a \neq b \in A_0$. $\dashv$

16.11 Remark. Note that if a mapping $e : [\omega_2]^2 \longrightarrow \omega_1$ has properties (a),(b) and (c) of Corollary 16.10, then $D_e : [\omega_2]^2 \longrightarrow [\omega_2]^{\aleph_0}$ defined by $D_e\{\alpha, \beta\} = \{\xi \leq \min\{\alpha, \beta\} : e(\xi, \alpha) \leq e\{\alpha, \beta\}\}$ satisfies the weak form of the Baumgartner-Shelah definition of a Δ-function considered above, where only the first condition is kept. It could be shown, however, that all three properties of a Δ-function cannot be achieved assuming only the negation of Chang's conjecture. This shows that the function ρ, based on a $\square_{\omega_1}$-sequence, is a considerably deeper object than an $e : [\omega_2]^2 \longrightarrow \omega_1$ satisfying 16.10(a),(b),(c).

Recall that the successor of the continuum is characterized as the minimal cardinal θ with the property that every $f : [\theta]^2 \longrightarrow \omega$ is constant on the square of some infinite set. We shall now see that in slightly weakening the partition property by replacing squares by rectangles one gets a characterization of a quite different sort. To see this, let us use the arrow notation

$$\binom{\theta}{\theta} \longrightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$$

to succinctly express the statement that for every map $f : \theta \times \theta \longrightarrow \omega$, there exist infinite sets $A, B \subseteq \theta$ such that f is constant on their product. Let θ_2 be the minimal θ which fails to satisfy this property. Note that $\omega_1 < \theta_2 \leq \mathfrak{c}^+$. The following result whose proof can be found in [76] shows that θ_2 can have the minimal possible value ω_2, as well as that θ_2 can be considerably smaller than the continuum.

16.12 Theorem. *Chang's conjecture is equivalent to the statement that*

$$\binom{\omega_2}{\omega_2} \longrightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$$

holds in every ccc forcing extension. ⊣

16.13 Remark. The relative size of θ_2 (or its higher-dimensional analogues $\theta_3, \theta_4, \ldots$) in comparison to the sequence of cardinals $\omega_2, \omega_3, \omega_4, \ldots$ is of considerable interest, both in set theory and model theory (see e.g. [51], [67], [72]). On the other hand, even the following most simple questions, left open by Theorem 16.12, are still unanswered.

16.14 Question. Can one prove any of the bounds like $\theta_2 \leq \omega_3$, $\theta_3 \leq \omega_4$, $\theta_4 \leq \omega_5$, etc. without appealing to additional axioms?

Note that by Corollary 16.10, Chang's conjecture is equivalent to the statement that within every decomposition of the usual ordering on ω_2 as an increasing chain of tree orderings, one of the trees has an uncountable chain. Is it possible to have decompositions of $\in\restriction(\omega_2 \times \omega_2)$ into an increasing ω_1-chain of tree orderings of countable heights? It turns out that the answer to this question is equivalent to a different well-known combinatorial statement about ω_2 rather than Chang's conjecture itself. Recall that $f : [\kappa^+]^2 \longrightarrow \kappa$ is *transitive* if $f(\alpha, \gamma) \leq \max\{f(\alpha, \beta), f(\beta, \gamma)\}$ whenever $\alpha \leq \beta \leq \gamma$. Given a transitive map $f : [\kappa^+]^2 \longrightarrow \kappa$, one defines $\rho_f : [\kappa^+]^2 \longrightarrow \kappa$ recursively on $\alpha \leq \beta < \kappa^+$ as follows

$$\begin{aligned}\rho_f(\alpha,\beta) = \sup\{f(\min(C_\beta \setminus \alpha),\beta), \mathrm{tp}(C_\beta \cap \alpha),\\ \rho_f(\alpha, \min(C_\beta \setminus \alpha)), \rho_f(\xi,\alpha) : \xi \in C_\beta \cap \alpha\},\end{aligned}$$

where we stipulate that $\rho_f(\alpha,\alpha) = 0$ for all $\alpha < \kappa^+$.

16.15 Lemma. *For every transitive map $f : [\kappa^+]^2 \longrightarrow \kappa$ the corresponding $\rho_f : [\kappa^+]^2 \longrightarrow \kappa$ has the following properties:*

(a) $\rho_f(\alpha,\gamma) \leq \max\{\rho_f(\alpha,\beta), \rho_f(\beta,\gamma)\}$ *whenever* $\alpha \leq \beta \leq \gamma$,

(b) $\rho_f(\alpha,\beta) \leq \max\{\rho_f(\alpha,\gamma), \rho_f(\beta,\gamma)\}$ *whenever* $\alpha \leq \beta \leq \gamma$,

(c) $|\{\xi \leq \alpha : \rho_f(\xi,\alpha) \leq \nu\}| \leq |\nu| + \aleph_0$ *for* $\nu < \kappa$ *and* $\alpha < \kappa^+$,

(d) $\rho_f(\alpha,\beta) \geq f(\alpha,\beta)$ *for all* $\alpha < \beta < \kappa^+$. ⊣

Transitive maps are frequently used combinatorial objects, especially when one works with quotient structures. Adding the extra subadditivity condition 16.15(b), one obtains a considerably more subtle object which is much less understood. For example, let $f_\alpha : \kappa \longrightarrow \kappa$ $(\alpha < \kappa^+)$ be a given sequence of functions such that $f_\alpha <^* f_\beta$ whenever $\alpha < \beta$.[20] Then the corresponding transitive map $f : [\kappa^+]^2 \longrightarrow \kappa$ is defined by $f(\alpha,\beta) = \min\{\mu < \kappa : f_\alpha(\nu) < f_\beta(\nu) \text{ for all } \nu \geq \mu\}$. Let ρ_f be the corresponding ρ-function that dominates this particular f and for $\nu < \kappa$ let $<^f_\nu$ be the corresponding tree ordering of κ^+, i.e., $\alpha <^f_\nu \beta$ if and only if $\rho_f(\alpha,\beta) \leq \nu$.

16.16 Lemma. *Suppose $f_\alpha \leq g$ for all $\alpha < \kappa^+$ where $\leq$ is the ordering of everywhere dominance. Then for every $\nu < \kappa$ the tree $(\kappa^+, <^f_\nu)$ has height $\leq g(\nu)$.*

Proof. Let P be a maximal chain of $(\kappa^+, <^f_\nu)$. $f(\alpha,\beta) \leq \rho_f(\alpha,\beta) \leq \nu$ for every $\alpha < \beta$ in P. It follows that $f_\alpha(\nu) < f_\beta(\nu) \leq g(\nu)$ for all $\alpha < \beta$ in P. So P has order-type $\leq g(\nu)$. ⊣

Note that if we have a function $g : \kappa \longrightarrow \kappa$ which bounds the sequence f_α $(\alpha < \kappa^+)$ in the ordering $<^*$ of eventual dominance, then the new sequence $\bar{f}_\alpha = \min\{f_\alpha, g\}$ $(\alpha < \kappa^+)$ is still strictly $<^*$-increasing but now bounded by g even in the ordering of everywhere dominance. So this proves the following result of Galvin (see [26],[46]).

16.17 Corollary. *The following two conditions are equivalent for every regular cardinal κ.*

[20] Here, $f_\alpha <^* f_\beta$ whenever $\{\nu < \kappa : f_\alpha(\nu) \geq f_\beta(\nu)\}$ is bounded in κ.

(1) *There is a sequence* $f_\alpha : \kappa \longrightarrow \kappa$ $(\alpha < \kappa^+)$ *which is strictly increasing and bounded in the ordering of eventual dominance.*

(2) *The usual order-relation of* κ^+ *can be decomposed into an increasing* κ*-sequence of tree orderings of heights* $< \kappa$. ⊣

16.18 Remark. The assertion that every strictly $<^*$-increasing κ^+-sequence of functions from κ to κ is $<^*$-unbounded is strictly weaker than Chang's conjecture and in the literature it is usually referred to as *weak Chang's conjecture*. This statement still has considerable large cardinal strength (see [12]). Also note the following consequence of Corollary 16.17 which can be deduced from Lemmas 16.1 and 16.2 above as well.

16.19 Corollary. *If* κ *is a regular limit cardinal (e.g.* $\kappa = \omega$*), then the usual order-relation of* κ^+ *can be decomposed into an increasing* κ*-sequence of tree orderings of heights* $< \kappa$. ⊣

17. Higher Dimensions

The reader must have noticed already that in this chapter so far, we have only considered functions of the form $f : [\theta]^2 \longrightarrow I$ or equivalently sequences $f_\alpha : \alpha \longrightarrow I$ $(\alpha < \theta)$ of one-place functions. To obtain analogous results about functions defined on higher-dimensional cubes $[\theta]^n$ one usually develops some form of *stepping-up procedure* that lifts a function of the form $f : [\theta]^n \longrightarrow I$ to a function of the form $g : [\theta^+]^{n+1} \longrightarrow I$. The basic idea seems quite simple. One starts with a coherent sequence $e_\alpha : \alpha \longrightarrow \theta$ $(\alpha < \theta^+)$ of one-to-one mappings and wishes to define $g : [\theta^+]^{n+1} \longrightarrow I$ as follows:

$$g(\alpha_0, \alpha_1, \ldots, \alpha_n) = f(e_{\alpha_n}(\alpha_0), \ldots, e_{\alpha_n}(\alpha_{n-1})). \tag{I.16}$$

In other words, we use e_{α_n} to send $\{\alpha_0, \ldots, \alpha_{n-1}\}$ to the domain of f and then apply f to the resulting n-tuple. The problem with such a simple-minded definition is that for a typical subset Γ of θ^+, the sequence of restrictions $e_\delta \restriction (\Gamma \cap \delta)$ $(\delta \in \Gamma)$ may not cohere, so we cannot produce a subset of θ that would correspond to Γ and on which we would like to apply some property of f. It turns out that the definition (I.16) is basically correct except that we need to replace e_{α_n} by $e_{\tau(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)}$, where $\tau : [\theta^+]^3 \longrightarrow \theta^+$ is defined as follows (see Definition 14.6):

$$\tau(\alpha, \beta, \gamma) = \gamma_t, \text{ where } t = \rho_0(\alpha, \gamma) \cap \rho_0(\beta, \gamma). \tag{I.17}$$

The function ρ_0 to which (I.17) refers is of course based on some C-sequence C_α $(\alpha < \theta^+)$ on θ^+. The following result shows that if the C-sequence is carefully chosen, the function τ will serve as a stepping-up tool. The following lemma whose proof can be found in [76] gives the basic idea behind this.

17.1 Lemma. *Suppose ρ_0 and τ are based on some $\Box_\theta$-sequence C_α ($\alpha < \theta^+$) and let κ be a regular uncountable cardinal $\leq \theta$. Then every set $\Gamma \subseteq \theta^+$ of order-type κ contains a cofinal subset Δ such that, if $\varepsilon = \sup(\Gamma) = \sup(\Delta)$, then $\rho_0(\xi, \varepsilon) = \rho_0(\xi, \tau(\alpha, \beta, \gamma))$ for all $\xi < \alpha < \beta < \gamma$ in Δ.*

Recall that for a given C-sequence C_α ($\alpha < \theta^+$) such that $\mathrm{tp}(C_\alpha) \leq \theta$ for all $\alpha < \theta^+$, the range of ρ_0 is the collection of all finite sequences of ordinals $< \theta$. There is a natural way to identify $\mathbb{Q}_\theta$ with θ itself via the well-ordering of $\mathbb{Q}_\theta$ of length θ: $s <_w t$ if and only if $\max(s) < \max(t)$, or $\max(s) = \max(t)$ and $t \subseteq s$, or $\max(s) = \max(t)$ and $s(i) \neq t(i)$ for some i in the common domain of s and t and $s(i) < t(i)$ for the minimal such i. This identification gives us a way to define a lift-up of an arbitrary map $f : [\theta]^n \longrightarrow I$ (really, $f : [\mathbb{Q}_\theta]^n \longrightarrow I$) to a map $f^+ : [\theta^+]^{n+1} \longrightarrow I$ by the following formula:

$$f^+(\alpha_0, \ldots, \alpha_{n-1}, \alpha_n) = f(\rho_0(\alpha_0, \varepsilon), \ldots, \rho_0(\alpha_{n-1}, \varepsilon)), \tag{I.18}$$

where $\varepsilon = \tau(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)$.

Let us examine how this stepping-up procedure works on a particular example, a combinatorial property of a function f which has been stepped up by Velleman [79] from $n = 3$ to $n = 4$ using his version of the gap-2 morass.

17.2 Theorem. *Suppose θ is an arbitrary cardinal for which $\Box_\theta$ holds. Suppose further that for some regular $\kappa > \omega$ and integer $n \geq 2$ there is a map $f : [\theta]^n \longrightarrow [[\theta]^{<\kappa}]^{<\kappa}$ such that:*

(1) *$A \subseteq \min(a)$ for all $a \in [\theta]^n$ and $A \in f(a)$.*

(2) *For all $\nu < \kappa$ and $\Gamma \subseteq \theta$ of size κ there exist $a \in [\Gamma]^n$ and $A \in f(a)$ such that* $\mathrm{tp}(A) \geq \nu$ *and $A \subseteq \Gamma$.*

Then θ^+ and κ satisfy the same combinatorial property, but with $n+1$ in place of n.

Proof. Identifying $\mathbb{Q}_\theta$ with θ using the wellordering $<_w$ defined above, we assume that actually $f : [\mathbb{Q}_\theta]^n \longrightarrow [[\mathbb{Q}_\theta]^{<\kappa}]^{<\kappa}$. Apply the idea of (I.18) and define $g : [\theta^+]^n \longrightarrow [[\theta^+]^{<\kappa}]^{<\kappa}$ by the formula

$$g(\alpha_0, \ldots, \alpha_{n-1}, \alpha_n) = (\rho_0)_\varepsilon^{-1}(f(\rho_0(\alpha_0, \varepsilon), \ldots, \rho_0(\alpha_{n-1}, \varepsilon))),$$

where $\varepsilon = \tau(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)$ and where τ is based on a fixed $\Box_\theta$-sequence.

Note that the transformation $(\rho_0)_\varepsilon^{-1}$ does not necessarily preserve (1), so we intersect each member of a given $g(a)$ with $\min(a)$ in order to satisfy this condition. To check (2), let $\Gamma \subseteq \theta^+$ be a given set of size κ. By Lemma 17.1, shrinking Γ we may assume that Γ has order-type κ and that if $\varepsilon = \sup(\Gamma)$, then

$$\rho_0(\xi, \varepsilon) = \rho_0(\xi, \tau(\alpha, \beta, \gamma)) \text{ for all } \alpha < \beta < \gamma \text{ in } \Gamma. \tag{I.19}$$

It follows that g restricted to $[\Gamma]^{n+1}$ satisfies the formula

$$g(\alpha_0, \ldots, \alpha_{n-1}, \alpha_n) = (\rho_0)_\varepsilon^{-1}(f(\rho_0(\alpha_0, \varepsilon), \ldots, \rho_0(\alpha_{n-1}, \varepsilon))). \tag{I.20}$$

Shrinking Γ further we assume that the mapping $(\rho_0)_\varepsilon : (\varepsilon, \in) \longrightarrow (\mathbb{Q}_\theta, <_w)$ is strictly increasing, when restricted to Γ. Given an ordinal $\nu < \kappa$, we apply (2) for f to the set $\Delta = \{\rho_0(\alpha, \varepsilon) : \alpha \in \Gamma\}$ and find $a \in [\Delta]^n$ and $A \in f(a)$ such that $\mathrm{tp}(A) \geq \nu$ and $A \subseteq \Delta$. Let $\{\alpha_0, \ldots, \alpha_{n-1}\}$ be the increasing enumeration of the preimage $(\rho_0)_\varepsilon^{-1}(a)$ and pick $\alpha_n \in \Gamma$ above α_{n-1}. Let B be the preimage $(\rho_0)_\varepsilon^{-1}(A)$. Then $B \in g(\alpha_0, \ldots, \alpha_{n-1}, \alpha_n)$, $\mathrm{tp}(B) \geq \nu$ and $B \subseteq \Gamma$. This completes the proof. $\dashv$

If we apply this stepping-up procedure to the projection $[\![\cdot\cdot]\!]$ of the square-bracket operation defined in 4.11, one obtains analogues of families $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ of Theorem 4.13 for ω_2 instead of ω_1. This will give us the following result whose proof can be found in [76].

17.3 Theorem. *Assuming $\square_{\omega_1}$, there is a reflexive Banach space E with a transitive basis of type ω_2 with the property that every bounded operator $T : E \longrightarrow E$ can be written as a sum of an operator with a separable range and a diagonal operator (relative to the basis) with only countably many changes of constants.* $\dashv$

17.4 Remark. In [32], Koszmider has shown that such a space cannot be constructed on the basis of the usual axioms of set theory. We refer the reader to that paper for more details about these kinds of examples of Banach spaces.

For the rest of this section we shall examine the stepping-up method with fewer restrictions on the given C-sequence C_α $(\alpha < \theta^+)$ on which it is based.

17.5 Theorem. *The following are equivalent for a regular cardinal θ such that* $\log \theta^+ = \theta$.[21]

(1) *There is a substructure of the form $(\theta^{++}, \theta^+, <, \ldots)$ with no substructure B of size θ^+ with $B \cap \theta^+$ of size θ.*

(2) *There is an $f : [\theta^{++}]^3 \longrightarrow \theta^+$ which takes all the possible values on the cube of any subset Γ of θ^{++} of size θ^+.*

[21] $\log \kappa = \min\{\lambda : 2^\lambda \geq \kappa\}$.

Proof. To prove the nontrivial direction from (1) to (2), we use Lemma 16.8 and choose a strongly unbounded and subadditive $e : [\theta^{++}]^2 \longrightarrow \theta^+$. We also choose a C-sequence C_α ($\alpha < \theta^+$) such that $\mathrm{tp}(C_\alpha) \leq \theta$ for all $\alpha < \theta^+$ and consider the corresponding function $\rho^* : [\theta^+]^2 \longrightarrow \theta$ defined above in (I.15). Finally, we choose a one-to-one sequence r_α ($\alpha < \theta^{++}$) of elements of $\{0,1\}^{\theta^+}$ and consider the corresponding function $\Delta : [\theta^{++}]^2 \longrightarrow \theta^+$:

$$\Delta(\alpha,\beta) = \Delta(r_\alpha, r_\beta) = \min\{\nu : r_\alpha(\nu) \neq r_\beta(\nu)\}. \tag{I.21}$$

The definition of $f : [\theta^{++}]^3 \longrightarrow \theta$ is given according to the following two rules applied to a given triple $x = \{\alpha, \beta, \gamma\} \in [\theta^{++}]^3$ ($\alpha < \beta < \gamma$):

Rule 1: If $\Delta(r_\alpha, r_\beta) < \Delta(r_\beta, r_\gamma)$ and $r_\alpha <_{\mathrm{lex}} r_\beta <_{\mathrm{lex}} r_\gamma$ or $r_\alpha >_{\mathrm{lex}} r_\beta >_{\mathrm{lex}} r_\gamma$, let

$$f(\alpha,\beta,\gamma) = \min(P_\nu(\Delta(\beta,\gamma)) \setminus \Delta(\alpha,\beta)),$$

where $\nu = \rho^*(\min\{\xi \leq \Delta(\alpha,\beta) : \rho^*(\xi, \Delta(\alpha,\beta)) \neq \rho^*(\xi, \Delta(\beta,\gamma))\}, \Delta(\beta,\gamma))$.

Rule 2: If $\alpha \in x$ is such that r_α is lexicographically between the other two r_ξ's for $\xi \in x$, if $\beta \in x \setminus \{\alpha\}$ is such that $\Delta(r_\alpha, r_\beta) > \Delta(r_\alpha, r_\gamma)$, where γ is the remaining element of x and if x does not fall under Rule 1, let

$$f(\alpha,\beta,\gamma) = \min(P_\nu(e(\beta,\gamma)) \setminus e(\alpha,\beta)),$$

where $\nu = \rho^*\{\Delta(\alpha,\beta), e(\beta,\gamma)\}$.

The proof of the theorem is complete once we show the following: for every stationary set Σ of cofinality θ ordinals $< \theta^+$ and every $\Gamma \subseteq \theta^{++}$ of size θ^+ there exist $\alpha < \beta < \gamma$ in Γ such that $f(\alpha,\beta,\gamma) \in \Sigma$. The details of this can again be found in [76]. ⊣

17.6 Theorem. *If θ is a regular strong limit cardinal carrying a nonreflecting stationary set, then there is an $f : [\theta^+]^3 \longrightarrow \theta$ which takes all the values from θ on the cube of any subset of θ^+ of size θ.*

Proof. This is really a corollary of the proof of Theorem 17.5, so let us only indicate the adjustments. By Corollary 16.19 and Lemma 16.7, we can choose a strongly unbounded subadditive map $e : [\theta^+]^2 \longrightarrow \theta$. By the assumption about θ we can choose a C-sequence C_α ($\alpha < \theta$) avoiding a stationary set $\Sigma \subseteq \theta$ and consider the corresponding notion of a walk, trace, ρ_0-function and the square-bracket operation $[\cdot\cdot]$ as defined in (I.13) in Section 15. As in the proof of Theorem 17.5, we choose a one-to-one sequence r_α ($\alpha < \theta^+$) of elements of $\{0,1\}^\theta$ and consider the corresponding function $\Delta : [\theta^+]^2 \longrightarrow \theta$:

$$\Delta(\alpha,\beta) = \Delta(r_\alpha, r_\beta) = \min\{\nu < \theta : r_\alpha(\nu) \neq r_\beta(\nu)\}.$$

The definition of $f : [\theta^+]^3 \longrightarrow \theta$ is given according to the following rules, applied to a given $x \in [\theta^+]^3$.

Rule 1: If $x = \{\alpha < \beta < \gamma\}$, $\Delta(r_\alpha, r_\beta) < \Delta(r_\beta, r_\gamma)$ and $r_\alpha <_{\text{lex}} r_\beta <_{\text{lex}} r_\gamma$, or $r_\alpha >_{\text{lex}} r_\beta >_{\text{lex}} r_\gamma$, let

$$f\{\alpha, \beta, \gamma\} = [\Delta(\alpha, \beta)\Delta(\beta, \gamma)].$$

Rule 2: If $\alpha \in x$ is such that r_α is lexicographically between the other two r_ξ's for $\xi \in x$, if $\beta \in x \setminus \{\alpha\}$ is such that $\Delta(r_\alpha, r_\beta) > \Delta(r_\alpha, r_\gamma)$, where γ is the remaining element of x, and they do not satisfy the conditions of Rule 1, set

$$f\{\alpha, \beta, \gamma\} = \min(\text{Tr}(\Delta(\alpha, \beta), e\{\beta, \gamma\}) \setminus e\{\alpha, \beta\}),$$

i.e. $f\{\alpha, \beta, \gamma\}$ is the minimal point on the trace of the walk from $e\{\beta, \gamma\}$ to $\Delta(\alpha, \beta)$ above the ordinal $e\{\alpha, \beta\}$; if such a point does not exist, set $f\{\alpha, \beta, \gamma\} = 0$.

Then it suffices to show that for every stationary $\Omega \subseteq \Sigma$ and every $\Gamma \subseteq \theta^+$ of size θ, there exists an $x \in [\Gamma]^3$ such that $f(x) \in \Omega$. The details of this are given in [76]. ⊣

Since $\log \omega_1 = \omega$, we get the following consequence of Theorem 17.5.

17.7 Theorem. *Chang's conjecture is equivalent to the statement that for every $f : [\omega_2]^3 \longrightarrow \omega_1$ there is an uncountable $\Gamma \subseteq \omega_2$ such that $f``[\Gamma]^3 \neq \omega_1$.* ⊣

17.8 Remark. Since this same statement is stronger for functions from higher dimensional cubes $[\omega_2]^n$ into ω_1 the Theorem 17.7 shows that they are all equivalent to Chang's conjecture. Note also that $n = 3$ is the minimal dimension for which this equivalence holds, since the case $n = 2$ follows from the Continuum Hypothesis, which has no relationship to Chang's conjecture.

For the rest of this section we examine the stepping-up procedure without the assumption that some form of Chang's conjecture is false. So let θ be a given regular uncountable cardinal and let C_α ($\alpha < \theta^+$) be a fixed C-sequence such that $\text{tp}(C_\alpha) \leq \kappa$ for all $\alpha < \theta^+$. Let $\rho^* : [\theta^+]^2 \longrightarrow \theta$ be the ρ^*-function defined above in (I.15). Recall that, in case C_α ($\alpha < \theta^+$) is a $\square_\theta$-sequence, the key to our stepping-up procedure was the function $\tau : [\theta^+]^3 \longrightarrow \theta^+$ defined by the formula (I.17). Without the assumption of C_α ($\alpha < \theta^+$) being a $\square_\theta$-sequence, the following related function turns out to be a good substitute: $\chi : [\theta^+]^3 \longrightarrow \omega$ defined by

$$\chi(\alpha, \beta, \gamma) = |\rho_0(\alpha, \gamma) \cap \rho_0(\beta, \gamma)|.$$

Thus $\chi(\alpha, \beta, \gamma)$ is equal to the length of the common part of the walks $\gamma \to \alpha$ and $\gamma \to \beta$.

17.9 Definition. A subset Γ of θ^+ is *stable* if χ is bounded on $[\Gamma]^3$.

The following result whose proof can be found in [76] relates this notion to the unboundedness property of ρ^*.

17.10 Lemma. *Suppose that Γ is a stable subset of θ^+ of size θ. Then $\{\rho^*(\alpha,\beta) : \{\alpha,\beta\} \in [\Omega]^2\}$ is unbounded in θ for every $\Omega \subseteq \Gamma$ of size θ.* ⊣

17.11 Definition. The 3-*dimensional version of the oscillation mapping*, osc : $[\theta^+]^3 \longrightarrow \omega$, is defined on the basis of the 2-dimensional version of Section 14 as follows

$$\operatorname{osc}(\alpha,\beta,\gamma) = \operatorname{osc}(C_{\beta_s} \setminus \alpha, C_{\gamma_t} \setminus \alpha),$$

where $s = \rho_0(\alpha,\beta)\restriction\chi(\alpha,\beta,\gamma)$ and $t = \rho_0(\alpha,\gamma)\restriction\chi(\alpha,\beta,\gamma)$.

In other words, we let n be the length of the common part of the two walks $\gamma \to \alpha$ and $\gamma \to \beta$, then we consider the walks $\gamma = \gamma_0 > \ldots > \gamma_k = \alpha$ and $\beta = \beta_0 > \ldots > \beta_l = \alpha$ from γ to α and β to α respectively; if both k and l are bigger than n, i.e. if γ_n and β_n are both defined, we let $\operatorname{osc}(\alpha,\beta,\gamma)$ be equal to the oscillation of the two sets $C_{\beta_n} \setminus \alpha$ and $C_{\gamma_n} \setminus \alpha$. If $\min\{k,l\} < n$, we let $\operatorname{osc}(\alpha,\beta,\gamma) = 0$. The proof of the following basic fact about the three-dimensional oscillation mapping can again be found in [76].

17.12 Lemma. *Suppose that Γ is a subset of θ^+ of size κ, a regular uncountable cardinal, and that every subset of Γ of size κ is unstable. Then for every integer $n \geq 1$, there exist $\alpha < \beta < \gamma$ in Γ such that $\operatorname{osc}(\alpha,\beta,\gamma) = n$.* ⊣

Applying the last two lemmas to the subsets of θ^+ of size θ, we get an interesting dichotomy:

17.13 Lemma. *Every $\Gamma \subseteq \theta^+$ of size θ can be refined to a subset Ω of size θ such that either:*

(1) *ρ^* is unbounded and therefore strongly unbounded on Ω, or*

(2) *the oscillation mapping takes all possible values on the cube of Ω.* ⊣

We finish the section with a typical application of this dichotomy.

17.14 Theorem. *Suppose θ is a regular cardinal such that $\log \theta^+ = \theta$. Then there is an $f : [\theta^{++}]^3 \longrightarrow \omega$ which takes all the values from ω on the cube of any subset of θ^{++} of size θ^+.*

Proof. We choose two C-sequences C_α $(\alpha < \theta^+)$ and C^+_α $(\alpha < \theta^{++})$ on θ^+ and θ^{++} respectively, such that $\operatorname{tp}(C_\alpha) \leq \theta$ for all $\alpha < \theta^+$ and $\operatorname{tp}(C^+_\alpha) \leq \theta^+$ for all $\alpha < \theta^{++}$. Let $\rho^* : [\theta^+]^2 \longrightarrow \theta$ and $\rho^{*+} : [\theta^{++}]^2 \longrightarrow \theta^+$ be the corresponding ρ^*-functions defined above in I.15. Also choose a one-to-one sequence r_α $(\alpha < \theta^{++})$ of elements of $\{0,1\}^{\theta^+}$ and consider the corresponding function $\Delta : [\theta^{++}]^2 \longrightarrow \theta^+$ defined in (I.21). We define $f : [\theta^{++}]^3 \longrightarrow \theta^+$ according to the following two cases for a given triple $\alpha < \beta < \gamma$ of elements of θ^{++}.

Case 1: $(C_{\beta_s} \cap C_{\gamma_t}) \setminus \alpha \neq \emptyset$, where $s = \rho_0(\alpha,\beta)\restriction\chi(\alpha,\beta,\gamma)$ and $t = \rho_0(\alpha,\gamma)\restriction\chi(\alpha,\beta,\gamma)$ of course assuming that $\rho_0(\alpha,\beta)$ has length at least $\chi(\alpha,\beta,\gamma)$.

Rule 1: If $\Delta(r_\alpha, r_\beta) < \Delta(r_\beta, r_\gamma)$ and $r_\alpha <_{\rm lex} r_\beta <_{\rm lex} r_\gamma$ or $r_\alpha >_{\rm lex} r_\beta >_{\rm lex} r_\gamma$, set

$$f(\alpha,\beta,\gamma) = \min(P_\nu(\Delta(\beta,\gamma)) \setminus \Delta(\alpha,\beta)),$$

where $\nu = \rho^*(\min\{\xi \leq \Delta(\alpha,\beta) : \rho^*(\xi,\Delta(\alpha,\beta)) \neq \rho^*(\xi,\Delta(\beta,\gamma))\}, \Delta(\beta,\gamma))$.

Rule 2: If $\bar\alpha \in \{\alpha,\beta,\gamma\}$ is such that $r_{\bar\alpha}$ is lexicographically between the other two r_ξ's for $\xi \in \{\alpha,\beta,\gamma\}$, if $\bar\beta \in \{\alpha,\beta,\gamma\} \setminus \{\bar\alpha\}$ is such that $\Delta(r_{\bar\alpha}, r_{\bar\beta}) > \Delta(r_{\bar\alpha}, r_{\bar\gamma})$, where $\bar\gamma$ is the remaining member of $\{\alpha,\beta,\gamma\}$, and if $\{\alpha,\beta,\gamma\}$ does not fall under Rule 1, let

$$f(\alpha,\beta,\gamma) = \min(P_\nu(\rho^{*+}\{\bar\beta,\bar\gamma\}) \setminus \rho^{*+}(\alpha,\beta)),$$

where $\nu = \rho^*\{\Delta(\alpha,\beta), \rho^{*+}(\beta,\gamma)\}$.

Case 2: $(C_{\beta_s} \cap C_{\gamma_t}) \setminus \alpha = \emptyset$, where $s = \rho_0(\alpha,\beta)\restriction\chi(\alpha,\beta,\gamma)$ and $t = \rho_0(\alpha,\gamma)\restriction\chi(\alpha,\beta,\gamma)$ of course assuming that $\rho_0(\alpha,\beta)$ has length at least $\chi(\alpha,\beta,\gamma)$. Let

$$f(\alpha,\beta,\gamma) = \mathrm{osc}(\alpha,\beta,\gamma).$$

If a given triple $\alpha < \beta < \gamma$ does not fall into one of these two cases, let $f(\alpha,\beta,\gamma) = 0$.

Then it suffices to show that for every $\Gamma \subseteq \theta^{++}$ of size θ^+, the image $f``[\Gamma]^3$ either contains all positive integers or almost all ordinals $< \theta^+$ of cofinality θ. The details of this are given in [76]. ⊣

17.15 Corollary. *There is an $f : [\omega_2]^3 \longrightarrow \omega$ which takes all the values on the cube of any uncountable subset of ω_2.* ⊣

17.16 Remark. Note that the dimension 3 in this Corollary cannot be lowered to 2 as long as one does not use some additional axioms to construct such f. Note also that the range ω cannot be replaced by a set of bigger size, as this would contradict Chang's conjecture. We have seen above that Chang's conjecture is equivalent to the statement that for every $f : [\omega_2]^3 \longrightarrow \omega_1$ there is an uncountable set $\Gamma \subseteq \omega_2$ such that $f``[\Gamma]^3 \neq \omega_1$. Is there a similar reformulation of the Continuum Hypothesis? More precisely, one can ask the following question.

17.17 Question. Is CH equivalent to the statement that for every $f : [\omega_2]^2 \longrightarrow \omega$ there exists an uncountable $\Gamma \subseteq \omega_2$ with $f``[\Gamma]^2 \neq \omega$?

Bibliography

[1] Pavel S. Alexandroff. Pages from an autobiography. *Russian Mathematical Surveys*, 34:267–302, 1979.

[2] Pavel S. Alexandroff and Pavel S. Urysohn. Mèmoire sur les espaces topologique compacts. *Verh. Akad. Wetensch. Amsterdam*, 14:VIII+96, 1929.

[3] Joan Bagaria. Fragments of Martin's Axiom and Δ^1_3 sets of reals. *Annals of Pure and Applied Logic*, 69:1–25, 1994.

[4] James E. Baumgartner and Saharon Shelah. Remarks on superatomic Boolean algebras. *Annals of Pure and Applied Logic*, 33:109–129, 1987.

[5] Mohamed Bekkali. *Topics in Set Theory*, volume 1476 of *Lecture Notes in Mathematics*. Springer-Verlag, 1992.

[6] Felix Bernstein. Zur Theorie der trigonometrischen Reihen. *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physikalische Klasse*, 60:325–338, 1908.

[7] Maxim Burke and Menachem Magidor. Shelah's pcf theory and its applications. *Annals of Pure and Applied Logic*, 50:207–254, 1990.

[8] W. Wistar Comfort and Stylianos Negrepontis. *Ultrafilters*. Springer-Verlag, 1974.

[9] Wayne J. Davis, Tadeusz Figiel, William B. Johnson, and Alexander Peł czyński. Factoring weakly compact operators. *Journal of Functional Analysis*, 17:311–327, 1974.

[10] Keith J. Devlin. *Constructibility*. Springer-Verlag, 1984.

[11] Hans-Dieter Donder, Ronald B. Jensen, and Bernd Koppelberg. Some applications of the core model. In *Set Theory and Model Theory*, volume 872 of *Lecture Notes in Mathematics*, pages 55–97. Springer-Verlag, 1979.

[12] Hans-Dieter Donder and Peter Koepke. On the consistency strength of 'accessible' Jonsson cardinals and of the weak Chang conjecture. *Annals of Pure and Applied Logic*, 25:233–261, 1983.

[13] Hans-Dieter Donder and Jean-Pierre Levinski. Some principles related to Chang's conjecture. *Annals of Pure and Applied Logic*, 45:39–101, 1989.

[14] Alan Dow. PFA and ω_1^*. *Topology and its Applications*, 28:127–140, 1988.

[15] Ben Dushnik and Edwin W. Miller. Partially ordered sets. *American Journal of Mathematics*, 63:600–610, 1941.

[16] Hans-Dieter Ebbinghaus and Jörg Flum. Eine Maximalitätseigenschaft der Prädikatenlogik erster Stufe. *Mathematisch-Physikalische Semesterberichte*, 21:182–202, 1974.

[17] Paul Erdős, Steve Jackson, and R. Daniel Mauldin. On partitions of lines and spaces. *Fundamenta Mathematicae*, 145:101–119, 1994.

[18] Paul Erdős, Steve Jackson, and R. Daniel Mauldin. On infinite partitions of lines and spaces. *Fundamenta Mathematicae*, 152:75–95, 1997.

[19] Paul Erdős and Alfred Tarski. On families of mutually exclusive sets. *Annals of Mathematics*, 44:315–329, 1943.

[20] Paul Erdős and Alfred Tarski. On some problems involving inaccessible cardinals. In Y. Bar-Hillel, E. Poznanski, M. Rabin, and A. Robinson, editors, *Essays on the Foundations of Mathematics*, pages 50–82. Magnes Press, 1961.

[21] Matthew Foreman and Menachem Magidor. A very weak square principle. *The Journal of Symbolic Logic*, 62:175–196, 1997.

[22] Gary F. Gruenhage and Yoshio Tanaka. Products of k-spaces. *Transactions of the American Mathematical Society*, 273:299–308, 1982.

[23] András Hajnal, Akihiro Kanamori, and Saharon Shelah. Regressive partition relations for infinite cardinals. *Transactions of the American Mathematical Society*, 299:145–154, 1987.

[24] William P. Hanf. On a problem of Erdős and Tarski. *Fundamenta Mathematicae*, 53:325–334, 1964.

[25] Leo Harrington and Saharon Shelah. Some exact equiconsistency results in set theory. *Notre Dame Journal of Formal Logic*, 26:178–187, 1985.

[26] Wilfrid Hodges and Saharon Shelah. Infinite games and reduced products. *Annals of Mathematical Logic*, 20:77–108, 1981.

[27] Ronald B. Jensen. The fine structure of the constructible hierarchy. *Annals of Mathematical Logic*, 4:229–308, 1972.

[28] Ronald B. Jensen and Karl Schlechta. Results on the generic Kurepa hypothesis. *Archive for Mathematical Logic*, 30:13–27, 1990.

[29] Akihiro Kanamori. *The Higher Infinite*. Springer Monographs in Mathematics. Springer-Verlag, 2005. Second edition, second printing.

[30] Piotr Koszmider. On coherent families of finite-to-one functions. *The Journal of Symbolic Logic*, 58:128–138, 1993.

[31] Piotr Koszmider. Forcing minimal extensions of Boolean algebras. *Transactions of the American Mathematical Society*, 351:3073–3117, 1999.

[32] Piotr Koszmider. On Banach spaces of large density but few operators. preprint, 2000.

[33] Kenneth Kunen. Saturated ideals. *The Journal of Symbolic Logic*, 43:65–76, 1978.

[34] Kenneth Kunen. *Set Theory. An introduction to independence proofs*. North-Holland, 1980.

[35] Djuro R. Kurepa. Ensembles ordonnés et ramifiés. *Publications Mathématiques de l"Université de Belgrade*, 4:1–138, 1935.

[36] Richard Laver. Better-quasi-orderings and a class of trees. In Gian Carlo Rota, editor, *Studies in Foundations and Combinatorics*, volume 1 of *Advances in Math. Suppl. Stud.*, pages 31–48. Academic Press, 1978.

[37] Richard Laver and Saharon Shelah. The $\aleph_2$-Souslin hypothesis. *Transactions of the American Mathematical Society*, 264:411–417, 1981.

[38] Jean-Pierre Levinski, Menachem Magidor, and Saharon Shelah. Chang's conjecture for $\aleph_\omega$. *Israel Journal of Mathematics*, 69:161–172, 1990.

[39] V.I. Malykhin. On Ramsey spaces. *Soviet Mathematics. Doklady*, 20:894–898, 1979.

[40] William J. Mitchell. Aronszajn trees and the independence of the transfer propery. *Annals of Mathematical Logic*, 5:21–46, 1972.

[41] J. Donald Monk. *Cardinal Invariants on Boolean Algebras.* Birkhäuser Verlag, Basel, 1996.

[42] Jaroslav Nešetril and Vojtěch Rődl. Ramsey topological spaces. In Jan Novak, editor, *General Toplogy and its Relation to Modern Algebra*, volume IV Part B, pages 333–337. Society of Czechoslovak Mathematicians and Physicists, Prague, 1977.

[43] T. Ohkuma. Comparability between ramified sets. *Proceedings of the Japan Academy*, 36:383–388, 1960.

[44] Martin Otto. EM constructions for a class of generalized quantifiers. *Archive for Mathematical Logic*, 31:355–371, 1992.

[45] Mariusz Rabus. An ω_2-minimal Boolean algebra. *Transactions of the American Mathematical Society*, 348:3235–3244, 1996.

[46] Judith Roitman. A thin atomic Boolean algebra. *Algebra Universalis*, 21:137–142, 1985.

[47] Mary Ellen Rudin. *Lectures in Set-Theoretic Topology.* Amer. Math. Soc. Providence R.I., 1975.

[48] Marion Scheepers. Concerning n-tactics in the countable-finite game. *The Journal of Symbolic Logic*, 56:786–794, 1991.

[49] Ernest Schimmerling. Combinatorial set theory and inner models. In Carlos A. DiPrisco et al., editors, *Set Theory: Techniques and Applications*, pages 207–212. Kluwer Academic Publishers, 1998.

[50] James H. Schmerl. A partition property characterizing cardinals hyper-inaccessible of finite type. *Transactions of the American Mathematical Society*, 188:281–291, 1974.

[51] James H. Schmerl. Transfer theorems and their applications to logics. In K. Jon Barwise et al., editors, *Model-Theoretic Logics*, pages 177–209. Springer-Verlag, 1985.

[52] Saharon Shelah. Strong negative partition above the continuum. *The Journal of Symbolic Logic*, 55:21–31, 1990.

[53] Saharon Shelah. *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides.* Oxford University Press, 1994.

[54] Saharon Shelah and Juris Steprans. A Banach space on which there are few operators. *Proceedings of the American Mathematical Society*, 104:101–105, 1986.

[55] Ivan Singer. *Bases in Banach Spaces, II.* Springer-Verlag, 1981.

[56] Ernst Specker. Sur un problème de Sikorski. *Colloquium Mathematicum*, 2:9–12, 1949.

[57] Daniel E. Talayco. Application of cohomology to set theory I. *Annals of Pure and Applied Logic*, 71:69–107, 1995.

[58] Daniel E. Talayco. Application of cohomology to set theory II. *Annals of Pure and Applied Logic*, 77:279–299, 1996.

[59] Stevo Todorcevic. Trees and linearly ordered sets. In Kenneth Kunen and Jerry E. Vaughan, editors, *Handbook of Set-Theoretic Topology*, pages 235–293. North-Holland, 1984.

[60] Stevo Todorcevic. Aronszajn trees and partitions. *Israel Journal of Mathematics*, 52:53–58, 1985.

[61] Stevo Todorcevic. Remarks on chain conditions in products. *Compositio Mathematica*, 55:295–302, 1985.

[62] Stevo Todorcevic. Remarks on cellularity in products. *Compositio Mathematica*, 57:357–372, 1986.

[63] Stevo Todorcevic. Partitioning pairs of countable ordinals. *Acta Mathematica*, 159:261–294, 1987.

[64] Stevo Todorcevic. Oscillations of real numbers. In *Logic Colloquium '86*, pages 325–331. North-Holland, Amsterdam, 1988.

[65] Stevo Todorcevic. *Partition Problems in Topology*, volume 84 of *Contemporary Mathematics*. Amer. Math. Soc. Providence R.I., 1989.

[66] Stevo Todorcevic. Special square sequences. *Proceedings of the American Mathematical Society*, 105:199–205, 1989.

[67] Stevo Todorcevic. Remarks on Martin's Axiom and the Continuum Hypothesis. *Canadian Journal of Mathematics*, 43:832–851, 1991.

[68] Stevo Todorcevic. Some partitions of three-dimensional combinatorial cubes. *Journal of Combinatorial Theory (A)*, 68:410–437, 1994.

[69] Stevo Todorcevic. Aronszajn orderings. *Institut Mathématique. Publications. Nouvelle Série (Beograd)*, 57(71):29–46, 1995.

[70] Stevo Todorcevic. Comparing the continuum with the first two uncountable cardinals. In M.L. Dalla Chiara et al., editors, *Logic and Scientific Methods*, pages 145–155. Kluwer Academic Publishers, 1997.

[71] Stevo Todorcevic. Basis problems in combinatorial set theory. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin)*, pages 43–52, 1998.

[72] Stevo Todorcevic. Countable chain condition in partition calculus. *Discrete Mathematics*, 188:205–223, 1998.

[73] Stevo Todorcevic. Oscillations of sets of integers. *Advances in Applied Mathematics*, 20:220–252, 1998.

[74] Stevo Todorcevic. A dichotomy for P-ideals of countable sets. *Fundamenta Mathematicae*, 166:251–267, 2000.

[75] Stevo Todorcevic. Lipschitz maps on trees. preprint, 2000.

[76] Stevo Todorcevic. *Walks on Ordinals and Their Characteristics.* Birkhauser, 2007+.

[77] Boban Veličković. PhD thesis, University of Wisconsin, 1986.

[78] Daniel J. Velleman. Simplified morasses. *The Journal of Symbolic Logic*, 49:257–271, 1984.

[79] Daniel J. Velleman. On a combinatorial principle of Hajnal and Komjáth. *The Journal of Symbolic Logic*, 51:1056–1060, 1986.

[80] H.M. Wark. A non-separable reflexive Banach space on which there are few operators. *Journal of the London Mathematical Society*, 64:675–689, 2001.

[81] N.M. Warren. Extending continuous functions in zero-dimensional spaces. *Proceedings of the American Mathematical Society*, 52:414–416, 1975.

[82] William Weiss. Partitioning topological spaces. In Jaroslav Nešetril and Vojtěch Rődl, editors, *Mathematics of Ramsey theory*, pages 155–171. Springer-Verlag, 1990.

[83] Kurt Wolfsdorf. Färbungen großer Würfel mit bunten Wegen. *Archiv der Mathematik*, 40:569–576, 1983.

[84] W. Hugh Woodin. *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal.* Walter de Gruyter & Co., Berlin, 1999.

Index