NOTES ON THE LOCAL THETA CORRESPONDENCE

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Introduction

These notes are intended to provide an introduction to the theory of the theta correspondence over a nonarchimdean local field. They form the basis and background for a series of ten lectures to be given as part of the European School in Group Theory at Schloß Hirschberg in September, 1996.

Here is a rough outline of what they contain:

Chapter I. The Heisenberg group, the metaplectic group, the Weil representation, the Rao cocycle, and Schrödinger models.

Chapter II. Reductive dual pairs, the theta correspondence, the Howe duality principle, restriction of the metaplectic cover, the Weil representation for the dual pair $(O(V), Sp(W))$, scaling.

Chapter III. Witt towers, parabolic subgroups, the Bernstein-Zelevinsky classification, persistence and stable range, first occurrence indices, the trivial representation, supercuspidals, compatibility with the Bernstein-Zelevinsky classification, Jacquet functors of the Weil representation.

Chapter IV. Seesaw dual pairs, degenerate principal series and their restrictions, proofs of some results of Chapter III.

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Chapter V. First occurrence indices for pairs of Witt towers, conservation and dichotomy conjectures, quotients of degenerate principal series, zeta integrals.

Chapter VI. Low dimensional examples, chains of supercuspidals, behavior of unramified representations in the correspondence, L-homomorphisms and functoriality

These notes overlap with the book of Moeglin, Vigneras and Waldspurger, [46], at a number of places, but the overall point of view is somewhat different. In particular, the role of Witt towers and of the degenerate principal series representations has been emphasized. We have included a discussion of conservation and dichotomy phenomena which have not yet been established in general. In addition, we have given a more detailed discussion of the cocycle(s) defining the metaplectic extension and their behavior upon restriction to the groups in a reductive dual pair. We have also provided a few low dimensional examples, cf. Chapter VI, and have described the unramified correspondence explicitly. The precise relationship between the local theta correspondence and Langlands functoriality is not yet fully understood. Some background concerning this problem is included in Chapter VI. On the other hand, we have limited our treatment to dual pairs of the form $(O(V), Sp(W))$, whereas the general irreducible type I and II dual pairs are considered in [46]. Our hope is that this restriction to one particular case will make these notes slightly more accessible. For example, each series of dual pairs $(O(V), Sp(W))$, $(U(V), U(W))$, etc. has its own set of small peculiarities which require separate comment. Such comments tend to gradually obscure the overall picture.

This theory is still not in its final form, nor, as will be evident, are these notes! All of the Chapters are incomplete, Chapter V is less than half finished. For this I apologize to the reader! But my time has run out.

Unfortunately, nothing is said about the rather well developed archimedean theta correspondence, nor about the global theory, Siegel-Weil formula, connections with L-functions and applications to geometry and arithmetic. It is the author's hope to cover some of these topics in a future version of these notes.

Finally, I would like to thank S. J. Patterson and J. Rohlfs, the organizers of the Schloß Hirschberg School, for the opportunity to give these lectures and for their patience, as a more realistic plan than that originally proposed gradually dawned on the author.

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CHAPTER I

In this section, we will review the basic facts about the Weil representation of the metaplectic group over a nonarchimedean local field. This material is very standard, and can be found in many places. Occasionally, we will follow the exposition of [46], Chapter 2 rather closely. The description of the metaplectic extension is roughly based on [64]. The reader can consult [46], [64], [33], and [83] for additional details.

I.1. The Weil representation. Let F be a nonarchimedean local field of characteristic not equal to 2, and let W, $< \, , \, >$ be a nondegenerate symplectic vector space of dimension $2n$ over over F. The Heisenberg group

$$H(W) := W \oplus F$$

has multiplication defined by

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2} < w_1, w_2 >),$$

and the symplectic group

$$Sp(W) = \{g \in GL(W) \mid < xg, yg > = < x, y >\},$$

acts as a group of automorphisms of $H(W)$ by the rule:

$$(w, t)^g = (wg, t).$$

This action is trivial on the center $Z = \{(0, t)\} \simeq F$ of $H(W)$.

Let ψ be a nontrivial additive character of F. The key fact in constructing the Weil representation is the following result.

Theorem 1.1. (Stone, von Neumann) *Up to isomorphism, there is a unique smooth irreducible representation* (ρ_ψ, S) *of* $H(W)$ *with central character* ψ, *i.e., such that*

$$\rho_\psi((0, t)) = \psi(t) \cdot id_S.$$

Remark. (i) This Theorem is usually stated in terms of unitary representations, [83]. The smooth version given here is taken from [46], Chapt 2, Théorème I.2, where the proof is given, pp. 28–31.
(ii) The Theorem implies that the interesting aspect of the representation theory

of the Heisenberg group is not the set of isomorphism classes of irreducible representations, but rather the various objects in one such class and the isomorphisms among them. We will refer to these objects as *models* of the representation (ρ_ψ, S).

Since the action of any $g \in Sp(W)$ is trivial on the center of $H(W)$, the representation (ρ_ψ^g, S) given by $\rho_\psi^g(h) = \rho_\psi(h^g)$ again has central character ψ, and so is isomorphic to (ρ_ψ, S). In particular, for each $g \in Sp(W)$, there is an automorphism $A(g) : S \longrightarrow S$ such that

$$(*_\psi) \qquad A(g)^{-1}\rho_\psi(h)A(g) = \rho_\psi(h^g) = \rho_\psi^g(h).$$

The automorphism $A(g)$ is only unique up to a scalar in $\mathbb{C}^\times$, and so the automorphisms $A(g_1)A(g_2)$ and $A(g_1g_2)$ need not coincide. But at least the map $g \mapsto A(g)$ defines a projective representation of $Sp(W)$, i.e., a homomorphism

$$Sp(W) \longrightarrow GL(S)/\mathbb{C}^\times.$$

For such a projective representation, consider the fiber product

$$\begin{array}{ccc} \widetilde{Sp(W)}_\psi & \longrightarrow & GL(S) \\ \downarrow & & \downarrow \\ Sp(W) & \longrightarrow & GL(S)/\mathbb{C}^\times, \end{array}$$

where

$$\widetilde{Sp(W)}_\psi := \{(g, A(g)) \in Sp(W) \times GL(S) \mid (*_\psi) \text{ holds } \}.$$

The group $\widetilde{Sp(W)}_\psi$ is then a central extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \widetilde{Sp(W)}_\psi \longrightarrow Sp(W) \longrightarrow 1$$

such that A may be lifted to a representation ω_ψ of $\widetilde{Sp(W)}_\psi$:

$$\omega_\psi((g, A(g)) = A(g).$$

To develope the picture further, we want to do the following:

(i) Construct (many) explicit realizations of ρ_ψ and hence of ω_ψ.

(ii) Show that the central extension $\widetilde{Sp(W)}_\psi$ does not depend on ψ.

(iii) Show that the metaplectic group $Mp(W) := \widetilde{Sp(W)}_\psi$ is isomorphic to the extension which is obtained from a nontrivial twofold topological central extension

$$1 \longrightarrow \mu_2 \longrightarrow Mp(W)^{(2)} \longrightarrow Sp(W) \longrightarrow 1$$

by the inclusion of $\mu_2 = \{\pm 1\}$ into $\mathbb{C}^\times$, i.e., that

$$Mp(W) \simeq Mp(W)^{(2)} \times_{\mu_2} \mathbb{C}^\times.$$

In fact, the existence of a unique nontrivial twofold topological central extension of $Sp(W)$ follows, [16], from the result of Moore [47], but we will give a direct construction instead. Once this construction has been made, we will refer to the representation (ω_ψ, S) of the metaplectic group $Mp(W)$ as *the Weil representation determined by the additive character* ψ .

We begin by observing some elementary facts about the dependence of ρ_ψ on ψ. Fix a nontrivial additive character $\psi = \psi_1$ of F, and recall that F is identified with its topological dual via the isomorphism $a \mapsto \psi_a$ where $\psi_a(x) = \psi(ax)$. For convenience, we write $(\rho_a, S) = (\rho_{\psi_a}, S)$ and $(\omega_a, S) = (\omega_{\psi_a}, S)$. For any $b \in F^\times$, there is an automorphism s_b of $H(W)$ given by $s_b(w,t) = (bw, b^2t)$. The central character of the irreducible representation $(\rho_a^{s_b}, S)$ defined by $\rho_a^{s_b}(h) = \rho_a(s_b(h))$ is ψ_{ab^2}, so that we can identify this representation with ρ_{ab^2}. Since the automorphism s_b commutes with the action of $Sp(W)$, we have

$$
\begin{aligned}
A(g)^{-1}\rho_{ab^2}(h)A(g) &= A(g)^{-1}\rho_a^{s_b}(h)A(g) = A(g)^{-1}\rho_a(s_b(h))A(g)\\
&= \rho_a(s_b(h)^g)\\
&= \rho_a(s_b(h^g)) = \rho_a^{s_b}(h^g) = \rho_{ab^2}(h^g).
\end{aligned}
$$

Thus the operators $A(g) \in GL(S)$ satisfying ($*_{\psi_a}$) also satisfy ($*_{\psi_{ab^2}}$). Hence (ω_a, S) and (ω_{ab^2}, S) are isomorphic (in fact, actually identical, once we identify $(\rho_a^{s_b}, S)$ with (ρ_{ab^2}, S)). We shall see later that the representations ω_a, as a runs over representatives for $F^\times/F^{\times,2}$, are inequivalent.

Since the contragradient $\rho_a^\vee$ of ρ_a has central character $\psi_a^{-1} = \psi_{-a}$, we see that

$$
\rho_a^\vee \simeq \rho_{-a} \qquad \text{and} \qquad \omega_a^\vee \simeq \omega_{-a}.
$$

If W_1 and W_2 are symplectic vector spaces and if $W = W_1 + W_1$ is their orthogonal direct sum, then there is an exact sequence

$$
1 \longrightarrow Z^{-\Delta} \longrightarrow H(W_1) \times H(W_2) \overset{j}{\longrightarrow} H(W) \longrightarrow 1,
$$

where $j((w_1,t_1),(w_2,t_2)) = (w_1+w_2, t_1+t_2)$, and $Z^{-\Delta} = \{\big((0,t),(0,-t)\big) \mid t \in F\}$. Then

$$
(**) \qquad (j^*\rho_\psi, S) \simeq (\rho_\psi^1 \otimes \rho_\psi^2, S_1 \otimes S_2).
$$

We also have an inclusion

$$
j : Sp(W_1) \times Sp(W_2) \longrightarrow Sp(W),
$$

compatible with the map j of Heisenberg groups:

$$j((h_1^{g_1}, h_2^{g_2})) = j(h_1, h_2)^{j(g_1,g_2)}.$$

If we identify the space S of ρ_ψ with the space $S_1 \otimes S_2$ of $\rho_\psi^1 \otimes \rho_\psi^2$, as in (**), we see that $A(g_1) \otimes A(g_2)$ and $A(j(g_1, g_2))$ coincide up to a scalar. Thus there is a homomorphism

$$\begin{aligned} &\tilde{j} : Mp(W_1) \times Mp(W_2) \longrightarrow Mp(W) \\ &((g_1, A(g_1)), (g_2, A(g_2)) \mapsto (j(g_1, g_2), A(g_1) \otimes A(g_2)), \end{aligned}$$

and

$$(\tilde{j}^*(\omega_\psi), S) \simeq (\omega_\psi^1 \otimes \omega_\psi^2).$$

The kernel of $\tilde{j}$ is $\{(1, z), (1, z^{-1}) \mid z \in \mathbb{C}^\times\}$.

I.2. Models. Models of the representation (ρ_ψ, S) can be constructed by induction. Here we follow the exposition of [46], Chapter 2, p.28-30 very closely.

The space W can be identified with its topological dual by means of the pairing

$$[w, w'] = \psi(< w, w' >)$$

valued in $\mathbb{C}^1$. For a closed subgroup Y in W, let

$$Y^\perp = \{w \in W \mid [w, y] = 1, \forall y \in Y\},$$

so that $Y^\perp$ is a closed subgroup of W and is isomorphic to the topological dual of W/Y. Let Y be a closed subgroup of W such that $Y^\perp = Y$. Let $Z_\psi = \{(0, t) \mid \psi(t) = 1\}$ be the subgroup of the center Z of $H(W)$ on which ψ is trivial. The Heisenberg group $H(Y) = Y \oplus F$ of Y is a closed subgroup of $H(W)$, and its image in $H(W)/Z_\psi$ is a maximal abelian subgroup. The character ψ of Z has a unique extension ψ_Y to $H(Y)$, given by $\psi_Y(y, t) = \psi(t)$. Let

$$S_Y = Ind_{H(Y)}^{H(W)} \psi_Y$$

be the representation of $H(W)$ obtained from the character ψ_Y of $H(Y)$ by smooth induced. Thus S_Y is the space of complex valued functions f on $H(W)$ such that

$$f(h_1 h) = \psi_Y(h_1) f(h)$$

for all $h_1 \in H(Y)$ and such that there exists an open subgroup $L \subset W$ such that

$$f(h(u, 0)) = f(h)$$

for all $u \in L$. The group $H(W)$ acts on S_Y by right translation. We denote this action by ρ and we note that for $(0, t) \in Z$,

$$\rho((0, t)) f(h) = f(h(0, t)) = f((0, t)h) = \psi(t) f(h),$$

so that Z acts by the character ψ.

Lemma 2.1. *(i) Any function* $f \in S_Y$ *has support in a set of the form* $H(Y)C$ *where* $C \subset H(W)$ *is compact (and depends on* f *).*
(ii) The representation S_Y *is smooth and irreducible with central character* ψ *. In particular,* (ρ, S_Y) *is a model of the representation* (ρ_ψ, S) *.*
(iii) The representation ρ_ψ *is admissible, i.e., the space of vectors* $f \in S_Y$ *fixed by a compact open subgroup of* $H(W)$ *is finite dimensional.*

Proof. cf. [46], Chapt.2.,p.29.

The most important special case of this construction is the following. Let $Y \subset W$ be a maximal isotropic subspace of W, which we view as a degenerate symplectic space over F. Then $H(Y) \simeq Y \oplus Z$ is a maximal abelian subgroup of $H(W)$. Let X be another maximal isotropic subspace of W such that $W = X+Y$ (necessarily a direct sum). Such a decomposition is refered to as a complete polarization of W. Then there is an isomorphism

$$S_Y \xrightarrow{\sim} S(X), \qquad f \mapsto \varphi$$

given by $\varphi(x) = f((x,0))$. Here $S(X)$ is the space of locally constant functions on X of compact support.

Lemma 2.2. *The resulting action of* $H(W)$ *in* $S(X)$ *is given by:*

$$\rho((x+y,t))\varphi(x_0) = \psi\big(t+ < x_0, y > +\frac{1}{2} < x, y > \big) \cdot \varphi(x_0 + x).$$

Proof.

$$\begin{aligned}\big(\rho((x+y,t))\varphi\big)(x_0) &= f\big((x_0,0)(x+y,t)\big) \\ &= f\big((x+y+x_0, t - \frac{1}{2} < y, x_0 >)\big) \\ &= f\big((y, t - \frac{1}{2} < y, x_0 > -\frac{1}{2} < y, x + x_0 >)(x + x_0, 0)\big) \\ &= \psi\big(t+ < x_0, y > +\frac{1}{2} < x, y > \big) \cdot \varphi(x + x_0).\end{aligned}$$

□

The representation of the group $Mp(W) = \widetilde{Sp(W)}_\psi$ (We have yet to show the independence of ψ.) in the space $S(X)$ is called the Schrödinger model of the Weil representation, associated to the complete polarization $W = X + Y$. We would like to have a more explicit description of the operators $A(g)$, for $g \in Sp(W)$, in this model.

Viewing elements of W as row vectors (x, y) with $x \in X$ and $y \in Y$, we can write $g \in Sp(W)$ as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a \in End(X)$, $b \in Hom(X, Y)$, $c \in Hom(Y, X)$ and $d \in End(Y)$. Recall that we are letting $Sp(W)$ act on W on the right. Let

$$M = M_{X,Y} = \{m(a) = \begin{pmatrix} a & \\ & a^\vee \end{pmatrix} \mid a \in GL(X)\},$$

and

$$N = N_Y = \{n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \mid b \in Hom(X, Y), \text{ symmetric}\}.$$

Here $a^\vee \in GL(Y)$ is determined by the condition $< xa, ya^\vee > = < x, y >$, for all $x \in X$ and $y \in Y$. Similarly, $b \in Hom(X, Y)$ is symmetric if $< x_1, x_2 b > = < x_2, x_1 b >$ for all x_1 and $x_2 \in X$. The stabilizer of Y in $Sp(W)$ is the (maximal parabolic) subgroup $P_Y = MN$, with unipotent radical N and Levi factor M. Note that the subgroup N depends only on Y, while M depends on the complete polarization.

Proposition 2.3. *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(W)$. *Then there is a unique choice of a Haar measure* $d\mu_g(y)$ *on* $ker(c)\backslash Y$ *such that the operator* $r(g)$ *defined for* $\varphi \in S(X)$ *by*

$$\big(r(g)\varphi\big)(x) := \int_{ker(c)\backslash Y} \psi\bigg(\frac{1}{2} < xa, xb > - < xb, yc > + \frac{1}{2} < yc, yd > \bigg) \cdot \varphi(xa + yc)\, d\mu_g(y),$$

preserves the L^2 *norm on* $S(X)$.

Proof. For any $g \in Sp(W)$, the isomorphism

$$A^0(g) : S_Y \longrightarrow S_{Yg^{-1}}$$

defined by

$$(**) \qquad (A^0(g)f)(h) = f(h^g)$$

satisfies

$$(***) \qquad \rho(h)A^0(g) = A^0(g)\rho(h^g),$$

for the action of $H(W)$, as is easily checked. If $g \in P_Y$, then $A^0(g)$ carries S_Y to itself, and the resulting operators on the space $S(X)$ are given by

$$A(m(a))\varphi(x) = |\det(a)|^{\frac{1}{2}}\varphi(xa),$$

and

$$A(n(b))\varphi(x) = \psi(\frac{1}{2} < x, xb >) \cdot \varphi(x).$$

The first of these has been adjusted by the factor $|\det(a)|^{\frac{1}{2}}$ so that the functions φ and $A(m(a))\varphi$ have the same L^2 norm on X.

To obtain operators on S_Y for any $g \in Sp(W)$, we must choose an isomorphism of the spaces S_Y and $S_{Yg^{-1}}$, compatible with the action of $H(W)$.

If Y_1 and Y_2 are maximal isotropic subspaces of W, we can define an explicit $H(W)$ intertwining map between the induced representations S_{Y_1} and S_{Y_2} as follows. Let $Y_{12} = Y_1 \cap Y_2$. For any function $f_1 \in S_{Y_1}$, we at least have left invariance under $Y_{12} \subset Y_2$. We then integrate to force invariance under all of Y_2. For $f_1 \in S_{Y_1}$, the integral

$$I_{Y_1,Y_2}(f)(h) = \int_{Y_{12}\backslash Y_2} f((y,0)h)\,dy$$

is absolutely convergent (because the image of the support of f_1 in $Y_1\backslash W$ is compact) and defines an element of S_{Y_2}. It is clear from the definition that

$$I_{Y_1,Y_2}(\rho(h)f) = \rho(h)(A_{Y_1,Y_2}(f)).$$

Since I_{Y_1,Y_2} is not identically zero, cf. [46], Chapt.2, p.32, and since (ρ, S_{Y_i}) is an irreducible representation of $H(W)$ for $i = 1$, 2, I_{Y_1,Y_2} defines an isomorphism

$$I_{Y_1,Y_2} : S_{Y_1} \xrightarrow{\sim} S_{Y_2}.$$

This isomorphism depends on a choice of the Haar measure on $Y_{12}\backslash Y_2$, and so is well defined up to a positive real number. We may then define

$$A(g) := I_{Yg^{-1},Y} \circ A^0(g),$$

and, in this way, we obtain operators on S_Y for all $g \in Sp(W)$.

Finally, we can transfer the operators $A(g)$ on S_Y to operators $r(g)$ on $S(X)$.

Lemma 2.4. *There is a unique choice of the Haar measure* $d\mu_g(y)$ *on* $(Yg^{-1} \cap Y)\backslash Y$ *, such that the operator* $r(g)$ *on* $S(X)$ *preserves the* L^2 *norm on this space.*

Proof. Fix a Haar measure dx on X. The space $S(X)$ of Schwartz-Bruhat functions on X is dense in the Hilbert space $L^2(X, dx)$. Since the action of $H(W)$ on $S(X)$ preserves the L^2 norm, this action extends, by continuity, to give an irreducible unitary representation of $H(W)$ on $L^2(X)$. Moreover, the space $S(X) \subset L^2(X)$ is precisely the space of smooth vectors in $L^2(X)$ for the action of $H(W)$, i.e., the space of vectors which are fixed by a compact open subgroup of $H(W)$. (Exercise: Check this.) The unitary version of the Stone, von-Neumann Theorem implies that, for each $g \in Sp(W)$, there is an isometry $U(g)$ of $L^2(X)$ such that

$$\rho_\psi(h)U(g) = U(g)\rho_\psi(h^g).$$

This isometry must preserve the space of smooth vectors $S(X)$, and so, its restriction to $S(X)$ differs from the transfer of $A(g)$ by a scalar. Adjusting $A(g)$ by a unique positive real number, i.e., by a unique choice of $d\mu_g(y)$, we obtain the required $r(g)$. □

These facts can be combined to yield an explicit formula for $r(g)$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, as above. Since, for $y \in Y$, $yg = yc + yd$ has X component yc, we have

$$Y \cap Yg = (ker(c)) \cdot g,$$

so that

$$Yg^{-1} \cap Y = ker(c).$$

Thus, for $\varphi \in S(X)$ corresponding to $f \in S_Y$, we have

$$\begin{aligned}(r(g)\varphi)(x) &= \int_{ker(c)\backslash Y} f((yc + yd, 0)(xa + xb, 0))\, d\mu_g(y) \\ &= \int_{ker(c)\backslash Y} \psi\Big(\frac{1}{2} < xa, xb > - < xb, yc > + \frac{1}{2} < yc, yd >\Big) \cdot \varphi(xa + yc)\, d\mu_g(y),\end{aligned}$$

for a suitable choice of the Haar measure $d\mu_g(y)$ on $ker(c)\backslash Y$, as in Lemma 2.4. This completes the proof of Proposition 2.3. □ □

I.3. The cocycle. In this section, we follow parts of Rao's paper [64] quite closely.

For a fixed complete polarization $W = X + Y$ and a fixed character ψ, we have now defined operators $r(g)$ on $S(X)$ for all $g \in Sp(W)$, and, by construction, these operators satisfy

$$\rho_\psi(h)r(g) = r(g)\rho_\psi(h^g).$$

Hence, we have chosen a section (not a homomorphism!)

$$\begin{aligned} Sp(W) &\longrightarrow Mp(W) \\ g &\mapsto (g, r(g)), \end{aligned}$$

and we would like to compute the resulting 2-cocycle $(g_1, g_2) \mapsto c(g_1, g_2)$ defined by

$$r(g_1)r(g_2) = c(g_1, g_2) \cdot r(g_1 g_2),$$

for g_1 and $g_2 \in Sp(W)$. To express the values of the cocycle, we need two additional quantities: the Leray invariant and the Weil index.

The Leray invariant is an isomorphism class of quadratic forms over F associated to a triple (Y_0, Y_1, Y_2) of maximal isotropic subspaces of W. First assume that the Y_j 's are pairwise transverse, i.e., that $Y_1 \cap Y_0 = Y_2 \cap Y_0 = Y_1 \cap Y_2 = 0$. Then $W = Y_1 + Y_0$ is a complete polarization of W, and there is a unique element $u \in N_{Y_0}$ such that $Y_2 = Y_1 u$. With the conventions above,

$$u = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$$

for a symmetric element $b \in Hom(Y_1, Y_0)$. Since Y_2 and Y_1 are transverse, b is an isomorphism. The function

$$q_b(x) = \frac{1}{2} < x, xb >$$

is a nondegenerate quadratic form on Y_1, and the isomorphism class of the quadratic space (Y_1, q_b) is the **Leray invariant**, $L(Y_0, Y_1, Y_2)$, of the triple. Note that, for any $g \in Sp(W)$, $L(Y_0 g, Y_1 g, Y_2 g) = L(Y_0, Y_1, Y_2)$.

One role played by the Leray invariant is the following. Let $\Omega(W)$ be the space of maximal isotropic subspaces of W, and let $\Omega(W)_0^2$ (resp. $\Omega(W)_0^3$) denote the subset of $\Omega(W) \times \Omega(W)$ (resp. $\Omega(W) \times \Omega(W) \times \Omega(W)$) consisting of transverse pairs (resp. pairwise transverse triples). The symplectic group $Sp(W)$ acts transitively on $\Omega(W)$ and on $\Omega(W)_0^2$. It has a finite number of orbits on $\Omega(W)_0^3$ and these are indexed by their Leray invariants, i.e.,there exists a $g \in Sp(W)$ such that $(Y_0 g, Y_1 g, Y_2 g) = (Y_0', Y_1', Y_2')$ if and only if $L(Y_0 g, Y_1 g, Y_2 g) = L(Y_0', Y_1', Y_2')$.

In general, let

$$R = Y_0 \cap Y_1 + Y_1 \cap Y_2 + Y_2 \cap Y_0,$$

and let

$$W_R = R^{\perp}/R \tag{A}$$

with the symplectic form induced by that of W. For $0 \le j \le 2$, let $Y_{j,R} = (Y_j \cap R^{\perp})/R$. Then, $(Y_{0,R}, Y_{1,R}, Y_{2,R})$ is triple of maximal isotropic subspaces of W_R which are pairwise transverse, and we let

$$L(Y_0, Y_1, Y_2) := L(Y_{0,R}, Y_{1,R}, Y_{2,R}).$$

For further properties of the Leray invariant, see [64].

The Weil index is defined as follows. Let X $(\ ,\ )$ be a vector space over F with a nondegenerate symmetric bilinear form. The associated quadratic form is $q(x) = \frac{1}{2}(x,x)$ satisfies

$$q(x+y) - q(x) - q(y) = (x,y).$$

Let $X^* = Hom(X,F)$ and denote the pairing of $x \in X$ and $x^* \in X^*$ by $[x,x^*]$. Let $\beta : X \to X^*$ be the isomorphism defined by

$$(x,y) = [x, \beta y].$$

Fix a nondegenerate additive character ψ of F, as before, and let dx and dx^* be Haar measures on X and X^* which are dual with respect to the Fourier transform defined using the pairing $\psi \circ [\ ,\ ]$, Thus, for any $f \in S(X)$, with Fourier transform

$$g(x^*) := \mathcal{F}(f)(x^*) = \int_X \psi([x,x^*])\, f(x)\, dx,$$

we have

$$f(-x) = \mathcal{F}(g)(x) = \int_{X^*} \psi([x,x^*])\, g(x^*)\, dx^*.$$

The Fourier transform of a tempered distribution ξ on $S(X)$ is defined by

$$\langle \mathcal{F}(\xi), g \rangle = \langle \xi, \mathcal{F}(g) \rangle.$$

The quadratic form q on X defines a character of second degree, [83],

$$\phi_q(x) = \psi(q(x)) = \psi(\frac{1}{2}(x,x)),$$

satisfying

$$\phi_q(x+y)\phi_q^{-1}(x)\phi_q^{-1}(y) = \psi((x,y)) = \psi([x,\beta y]).$$

This character of second degree gives a tempered distribution $\phi_q dx$ on $S(X)$;

$$\langle \phi_q dx, f \rangle = \int_X f(x)\, \phi_q(x)\, dx.$$

Similarly, the quadratic form q^* on X^* given by

$$q^*(x^*) = \frac{1}{2}[\beta^{-1}x^*, x^*]$$

defines a character of second degree ϕ_{q^*} and a tempered distribution $\phi_{q^*}^{-1}dx^*$ on $S(X^*)$. Note the additional inversion. These tempered distributions are related by the Fourier transform in the following way, [83]:

$$\mathcal{F}(\phi_q dx) = \gamma(\phi_q)\, |q|^{-\frac{1}{2}}\, \phi_{q^*}^{-1} dx^*,$$

where the modulus $|q| := |\beta|$ is defined by $d(\beta x) = |\beta|\, dx$, and where $\gamma(\phi_q) = \gamma(\psi\circ q) \in \mathbb{C}^1$ is the **Weil index** of the character of the second degree. In fact, $\gamma(\psi \circ q)$ is an eighth root of unity and depends only on the isomorphism class of q, cf. [64]. Note that, if the quadratic space (X,q) is an orthogonal direct sum (X_1+X_2, q_1+q_2), then

$$\gamma(\psi\circ q) = \gamma(\psi\circ q_1)\, \gamma(\psi\circ q_2).$$

This fact is immediate from the definition.

We now return to the cocycle c. The following result is due to Perrin,[50], and Rao, [64].

Theorem 3.1. *Let $W = X+Y$ be a complete polarization. The cocycle c_Y defined, for g_1 and $g_2 \in Sp(W)$, by the relation*

$$r(g_1)r(g_2) = c_Y(g_1,g_2)\cdot r(g_1g_2),$$

is given by

$$c_Y(g_1,g_2) = \gamma\big(\psi\circ L(Y, Yg_1^{-1}, Yg_2^{-1}g_1^{-1})\big),$$

where

$$L(Y, Yg_1^{-1}, Yg_2^{-1}g_1^{-1}) = L(Yg_1g_2, Yg_2, Y),$$

is the Leray invariant of the triple of maximal isotropic subspaces $(Y, Yg_1^{-1}, Yg_2^{-1}g_1^{-1})$.

Note that the cocycle c_Y depends only on the maximal isotropic subspace Y and not on the complete polarization.

Proof. Since the operators $r(g)$ preserve the L^2 norm, we have $c(g_1, g_2) \in \mathbb{C}^1$. This means that it suffices to compute this quantity modulo $\mathbb{R}_+^\times$. In particular, we can return to the space S_Y and use the relation:

$$A(g_1)A(g_2) \equiv c(g_1, c_2) \cdot A(g_1 g_2) \pmod{\mathbb{R}_+^\times}.$$

Thus, we must compare the operators

$$A(g_1)A(g_2) = I_{Y g_1^{-1}, Y} \circ A_Y^0(g_1) \circ I_{Y g_2^{-1}, Y} \circ A_Y^0(g_2)$$

and

$$\begin{aligned} A(g_1 g_2) &= I_{Y g_2^{-1} g_1^{-1}, Y} \circ A_Y^0(g_1 g_2) \\ &= I_{Y g_2^{-1} g_1^{-1}, Y} \circ A_{Y g_2^{-1}}^0(g_1) \circ A_Y^0(g_2) \end{aligned}$$

This gives

$$\text{(B)}\quad I_{Y g_1^{-1}, Y} \circ A_Y^0(g_1) \circ I_{Y g_2^{-1}, Y} \circ \left(A_{Y g_2^{-1}}^0(g_1) \right)^{-1} \equiv c(g_1, g_2) \cdot I_{Y g_2^{-1} g_1^{-1}, Y} \pmod{\mathbb{R}_+^\times}.$$

For any function $f \in S_{Y g_2^{-1} g_1^{-1}}$, the left had side of (B) is

$$\begin{aligned} &\int_{(Y g_1^{-1} \cap Y)\backslash Y} \left(\int_{(Y g_2^{-1} \cap Y)\backslash Y} f(\ ((y_2, 0)((y_1, 0)h)^{g_1})^{g_1^{-1}}\ )\, dy_2 \right) dy_1 \\ \equiv &\int_{(Y g_1^{-1} \cap Y)\backslash Y} \left(\int_{(Y g_2^{-1} g_1^{-1} \cap Y g_1^{-1})\backslash Y g_1^{-1}} f(\ (y_3, 0)(y_1, 0)h\ )\, dy_3 \right) dy_1, \end{aligned}$$

where we have written $y_3 = y_2 g_1^{-1}$, and $\equiv$ indicates that we are working modulo $\mathbb{R}_+^\times$, so that we need not specify the choice of Haar measures. The expression on the right side of (B) is simply

$$c(g_1, g_2) \cdot \int_{(Y g_2^{-1} g_1^{-1} \cap Y)\backslash Y} f((y, 0)h)\, dy.$$

The following key fact is due to Perrin [50] and Rao [64].

Proposition 3.2. *For a triple Y_0, Y_1 and Y_2 of maximal isotropic subspaces of W, and for any function $f \in S_{Y_2}$,*

$$\int_{Y_1 \cap Y_0 \backslash Y_0} \left(\int_{Y_2 \cap Y_1 \backslash Y_1} f((y_1, 0)(y_0, 0))\, dy_1 \right) dy_0 \equiv \gamma \cdot \int_{Y_2 \cap Y_0 \backslash Y_0} f((y_0, 0))\, dy_0 \pmod{\mathbb{R}_+^\times},$$

where

$$\gamma = \gamma(\psi \circ L(Y_0, Y_1, Y_2))$$

is the Weil index of the character of second degree $\psi \circ L$ *, where* $L = L(Y_0, Y_1, Y_2)$ *is the Leray invariant of the triple* (Y_0, Y_1, Y_2) *.*

Proof. First assume that Y_0 , Y_1 and Y_2 are pairwise transverse, and consider the integral

$$\int_{Y_0} \left(\int_{Y_1} f((y_1, 0)(y_0, 0))\, dy_1 \right) dy_0 = \int_{Y_0} \left(\int_{Y_1} \psi(\frac{1}{2} < y_1, y_0 >)\, f((y_1 + y_0, 0))\, dy_1 \right) dy_0.$$

Writing $W = Y_1 + Y_0$, there is a unique isomorphism $b \in Hom(Y_1, Y_0)$ such that $Y_2 = Y_1 n(b)$, where $n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in N_{Y_0}$. Recall that the Leray invariant of the triple (Y_0, Y_1, Y_2) is then the isometry class of the quadratic form q_b on Y_1 given by $q_b(x) = \frac{1}{2} < x, xb >$. *Note the factor of* $\frac{1}{2}$ *here!* Writing

$$(y_1 + y_0, 0) = (y_1 + y_1 b, -\frac{1}{2} < y_1 + y_1 b, y_0 - y_1 b >)(y_0 - y_1 b, 0),$$

in $H(W)$, the integral becomes

$$\int_{Y_0} \left(\int_{Y_1} \psi(\frac{1}{2} < y_1, y_1 b >)\, f((y_0 - y_1 b, 0))\, dy_1 \right) dy_0.$$

Substituting $y_1 + y_0 b$ for y_1 , this becomes

$$\int_{Y_0} \left(\int_{Y_1} \psi(\frac{1}{2} < y_1, y_1 b > + < y_1, y_0 > - \frac{1}{2} < y_0, y_0 b^{-1} >)\, f((-y_1 b, 0))\, dy_1 \right) dy_0.$$

We want to apply a slightly modified version of the Weil index identity, which we now describe. Let $W = X + Y$ be a complete polarization of W and let dx and dy be Haar measures on X and Y which are dual with respect to the pairing

$$[x, y] = \psi(< x, y >).$$

For any $n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in N_Y$, the element $b \in Hom(X, Y)$ is symmetric in the sense that

$$< x, x'b > = < x', xb > .$$

In particular, $x \mapsto q_b(x) := \frac{1}{2} < x, xb >$ is a quadratic form on X , associated to the bilinear form $< x, x'b >$, i.e.,

$$q_b(x + x') - q_b(x) - q_b(x') = < x, x'b > .$$

The function

$$\phi_b(x) = \psi(\frac{1}{2} < x, xb >) = \psi(q_b(x))$$

is a character of second degree on X, in Weil's terminology [83], associated to the bicharacter $[x, x'b]$, i.e.,

$$\phi_b(x + x')\phi_b(x)^{-1}\phi_b(x')^{-1} = [x, x'b].$$

If b is an isomorphism, we can define its modulus $|b|$ by the relation $b^*(dy) = d(xb) = |b|dx$, and we set $|q_b| := |b|$.

The function ϕ_b defines a tempered distribution $\phi_b \cdot dx$ on X. If b is an isomorphism, then b^{-1} defines a character of second degree on Y:

$$\phi_{b^{-1}}(y) := \psi(\frac{1}{2} < y, yb^{-1} >) = \phi_b(yb^{-1})^{-1}.$$

The tempered distributions $\phi_b \cdot dx$ on X and $\phi_{b^{-1}} \cdot dy$ on Y are related by the Weil index identity:

$$\mathcal{F}(\phi_b \cdot dx) = \gamma(\phi_b)|b|^{-\frac{1}{2}} \cdot \phi_{b^{-1}} \cdot dy.$$

It will be convenient to write this relation in a slightly different form. For a Schwartz function $f \in S(X)$, we have

$$\int_X \phi_b(x)\mathcal{F}(f)(x)\, dx = \gamma(\phi_b)|b|^{-\frac{1}{2}} \int_Y f(y)\, \phi_{b^{-1}}(y)\, dy.$$

Substituting the function $f \cdot \phi_{b^{-1}}^{-1}$ for f, this becomes

$$\int_X \left(\int_Y \psi(\frac{1}{2} < x, xb > + < x, y > -\frac{1}{2} < y, yb^{-1} >)\, f(y)\, dy\right) dx = \gamma(\phi_b)|b|^{-\frac{1}{2}} \int_Y f(y)\, dy.$$

Finally, setting $x = -y_1 b^{-1}$ and $y = -x_1 b$ on the left side, and using the fact that $< x, y > = < yb^{-1}, xb >$, we have (dropping the subscripts) the modified Weil index identity:

$$\int_Y \left(\int_X \psi(\frac{1}{2} < x, xb > + < x, y > -\frac{1}{2} < y, yb^{-1} >)\, f(xb)\, dx\right) dy = \gamma(\phi_b)|b|^{-\frac{1}{2}} \int_Y f(y)\, dy.$$

Applying this modified version of the Weil index identity, we find that the integral (X) above is equal to

$$\gamma(\psi \circ q_b)|b|^{-\frac{1}{2}} \cdot \int_{Y_0} f((-y_0, 0))\, dy_0 = \gamma(\psi \circ q_b)|b|^{-\frac{1}{2}} \cdot \int_{Y_0} f((y_0, 0))\, dy_0.$$

This proves the identity of Proposition 3.2 in the pairwise transverse case. We refer to reader to Rao's paper for the general case. □

Applying Proposition 3.2, with $Y_0 = Y$, $Y_1 = Yg_2^{-1}$ and $Y_2 = Yg_2^{-1}g_1^{-1}$, we obtain the result stated in Theorem 3.1 above. □

I.4. Reduction of the cocycle. In this section, we fix the maximal isotropic subspace Y, and we describe Rao's reduction of the $\mathbb{C}^1$-valued cocycle c_Y of the previous Theorem to a cocycle valued in μ_2. As a consequence of his formula, it will follow that the extension $\widetilde{Sp(W)}_\psi$ does not depend on the choice of ψ.

For g_1 and $g_2 \in Sp(W)$, let $q = L(Y, Yg_2^{-1}, Yg_1)$ be the Leray invariant, so that $c_Y(g_1, g_2) = \gamma(\psi \circ q)$. Since the Weil index $\gamma(\psi \circ q)$ depends only on the isomorphism class of q, this eighth root of unity can be expressed in terms of the quantities $\dim(q)$, $\det(q)$ and the Hasse invariant $\epsilon(q) \in \mu_2$ which determine this isomorphism class. First we need a little more notation.

For any nondegenerate additive character η of F, we let

$$\gamma(\eta) = \gamma(\eta \circ x^2)$$

be the Weil index of the character of second degree on F given by $\phi(x) = \eta(x^2)$. (Note that there is no factor of $\frac{1}{2}$ here.) For $a \in F^\times$, let

$$\gamma(a, \eta) = \gamma(\eta_a)/\gamma(\eta),$$

where $\eta_a(x) = \eta(ax)$, as before. The following facts may be found in [64], p.367.

Lemma 4.1.

$$\begin{aligned}
\gamma(ab^2, \eta) &= \gamma(a, \eta), \\
\gamma(ab, \eta) &= (a, b)_F\, \gamma(a, \eta)\, \gamma(b, \eta),
\end{aligned}$$

where $(\ ,\ )_F$ is the quadratic Hilbert symbol of F. In particular, a character of second degree on $F^\times/F^{\times,2}$.

$$\begin{aligned}
\gamma(a, \eta_b) &= (a, b)_F\, \gamma(a, \eta), \\
\gamma(a, \eta)^2 &= (-1, a)_F = (a, a)_F, \\
\gamma(a, \eta)^4 &= 1, \\
\gamma(\eta)^2 &= \gamma(-1, \eta)^{-1}, \\
\gamma(\eta)^8 &= 1.
\end{aligned}$$

Using some of these relations and the fact that the Weil index is multiplicative on direct sums, we obtain:

Lemma 4.2. *For a quadratic form* $q(x) = a_1x_1^2 + \cdots + a_nx_m^2$ *on* $X = F^m$ *,*

$$\gamma(\eta \circ q) = \gamma(\det(q), \eta)\, \gamma(\eta)^m\, \epsilon(q),$$

where

$$\epsilon(q) = \prod_{i<j} (a_i, a_j)_F$$

is the Hasse invariant of q *.*

Proof.

$$\begin{aligned}\gamma(\eta \circ q) &= \gamma(\eta_{a_1}) \ldots \gamma(\eta_{a_m})\\ &= \gamma(a_1, \eta) \ldots \gamma(a_m, \eta)\, \gamma(\eta)^m\\ &= \gamma(\det(q), \eta)\gamma(\eta)^m \prod_{i<j}(a_i, a_j)_F\\ &= \gamma(\det(q), \eta)\gamma(\eta)^m\, \epsilon(q).\end{aligned}$$

□

Returning to our cocycle and we have

$$c_Y(g_1, g_2) = \gamma(\psi \circ q) = \gamma(\det(q), \psi)\gamma(\psi)^\ell \epsilon(q),$$

where $\ell = \mathrm{rank}(q)$.

For the rest of this section the complete polarization $W = X + Y$ will be fixed, and we will write $P = P_Y$, with Levi decomposition $P = MN$, as above. Moreover, we will fix a basis

$$\mathbf{e} := \{e_1, \ldots, e_n, e'_1, \ldots, e'_n\},$$

for W such that X is the span of $e_1, \ldots, e_n$, Y is the span of $e'_1, \ldots, e'_n$, and $< e_i, e'_j >= \delta_{ij}$. We call such a basis **a standard symplectic basis** $\mathbf{e}$, adapted to the given complete polarization, and we obtain an identification

$$Sp(W) = \{g = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in GL_{2n}(F) \mid g \begin{pmatrix} & 1\\ -1 & \end{pmatrix} {}^tg = \begin{pmatrix} & 1\\ -1 & \end{pmatrix}\}.$$

For $0 \le r \le n$, let

$$w_j = \begin{pmatrix} 1_{n-j} & & & \\ & 0 & & 1_j \\ & & 1_{n-j} & \\ -1_j & & & 0 \end{pmatrix} \in Sp(W).$$

There is a double coset decomposition

$$Sp(W) = \coprod_{j=0}^{n} P w_j P,$$

where

$$\Omega_j := P w_j P = \{ g \in Sp(W) \mid \operatorname{rank}(c) = j \}.$$

If $g \in \Omega_j$, we write $j = j(g)$.

Rao defines a function

$$x : Sp(W) \longrightarrow F^\times / F^{\times,2},$$

by

$$x(p_1 w_j^{-1} p_2) := \det(p_1 p_2 | Y) \bmod F^{\times,2}.$$

In particular,

$$x(p_1 g p_2) = x(p_1) x(g) x(p_2),$$

and

$$x(n(b)m(a)) = \det(a) \bmod F^{\times,2}.$$

Also, if $g \in \Omega_n$, so that $\det(c) \neq 0$, then

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} {}^t c^{-1} & \\ & c \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ & 1 \end{pmatrix}.$$

This basic identity yields

$$x(g) = \det(c) \bmod F^{\times,2},$$

in this case. The following result is extracted for [64], cf. [33], Proposition 1.6.

Proposition 4.3. *For g_1 and $g_2 \in Sp(W)$, let $q = L(Y, Yg_2^{-1}, Yg_1)$ be the Leray invariant, and let $\ell = \dim(q)$ (so that $2\ell = \dim W_R$ in (A) above). Define an integer t by*

$$2t = j(g_1) + j(g_2) - j(g_1 g_2) - \ell.$$

Then

$$x(g_1 g_2) = x(g_1) x(g_2) (-1)^t \det(2q)$$

in $F^\times/F^{\times,2}$.

Remark 4.4. Note that, in Proposition 4.3, $\det(q)$ denotes the determinant of the matrix of the quadratic form $q(x) = \frac{1}{2}(x,x)$, so that $\det(2q)$ is the determinant of the matrix for the inner product (x,x) . Also note that, in Lemma 4.2, the quantity $\det(q)$ occurs. The cleanest way to absorb the extra factor of 2 is to write

$$\eta = \psi_{\frac{1}{2}},$$

and

$$\gamma(\psi \circ q) = \gamma(\eta \circ 2q).$$

Then

$$c_Y(g_1,g_2) = \gamma(\det(2q),\eta)\,\gamma(\eta)^\ell\,\epsilon(2q),$$

where q is the Leray invariant, as before. We then can use Proposition 4.3 to conveniently express the quantity $\det(2q)$, occuring here, in terms of the function $x(g)$.

We may now state Rao's result.

Theorem 4.5. *For* $g \in Sp(W)$ *, let*

$$\beta_{\mathbf{e},\psi}(g) := \gamma(x(g),\eta)^{-1}\gamma(\eta)^{-j(g)}.$$

Recall that $\eta = \psi_{\frac{1}{2}}$ *. Then*

$$c_Y(g_1,g_2) = \beta_{\mathbf{e},\psi}(g_1g_2)\beta_{\mathbf{e},\psi}(g_1)^{-1}\beta_{\mathbf{e},\psi}(g_2)^{-1}c_{Rao,\mathbf{e}}(g_1,g_2),$$

where

$$c_{Rao,\mathbf{e}}(g_1,g_2) = (x(g_1),x(g_2))_F(-x(g_1)x(g_2),x(g_1g_2))_F(-1,\det(2q))_F^t(-1,-1)_F^{\frac{t(t-1)}{2}}\,\epsilon(2q).$$

Proof. Since $\det(2q) = x(g_1g_2)x(g_1)x(g_2)(-1)^t$ and $\ell = j_1+j_2-j-2t$, we have

$$\begin{aligned}\gamma(\det(2q),\eta) &= \gamma(x(g_1g_2)x(g_1)x(g_2)(-1)^t,\eta)\\ &= (x(g_1g_2),-(-1)^t x(g_1)x(g_2))_F\,((-1)^t x(g_1),x(g_2))_F\,((-1)^t,x(g_1))_F\\ &\qquad\times\gamma(x(g_1g_2),\eta)^{-1}\,\gamma(x(g_1),\eta)\,\gamma(x(g_2),\eta)\,\gamma((-1)^t,\eta),\end{aligned}$$

and

$$\gamma(\eta)^\ell = \gamma(\eta)^{j_1}\gamma(\eta)^{j_2}\gamma(\eta)^{-j}\gamma(-1,\eta)^t.$$

Also,

$$\gamma((-1)^t,\eta)\gamma(-1,\eta)^t = (-1,-1)^{\frac{t(t+1)}{2}}.$$

Thus

$$\begin{aligned}
\gamma(\psi\circ q) = \gamma(\eta\circ 2q) &= ((-1)^t, x(g_1g_2)x(g_1)x(g_2))(-1,-1)_F^{\frac{t(t+1)}{2}} \\
&\quad \times (x(g_1g_2), -x(g_1)x(g_2))_F\,(x(g_1),x(g_2))_F\,\epsilon(2q) \\
&\quad \times \gamma(\eta)^{j_1}\gamma(\eta)^{j_2}\gamma(\eta)^{-j} \\
&\quad \times \gamma(x(g_1g_2),\eta)^{-1}\,\gamma(x(g_1),\eta)\,\gamma(x(g_2),\eta) \\
&= ((-1)^t, \det(2q))_F\,(-1,-1)_F^{t^2}\,(-1,-1)_F^{\frac{t(t+1)}{2}} \\
&\quad \times (x(g_1g_2), -x(g_1)x(g_2))_F\,(x(g_1),x(g_2))_F\,\epsilon(2q) \\
&\quad \times \gamma(\eta)^{j_1}\gamma(\eta)^{j_2}\gamma(\eta)^{-j} \\
&\quad \times \gamma(x(g_1g_2),\eta)^{-1}\,\gamma(x(g_1),\eta)\,\gamma(x(g_2),\eta) \\
&= ((-1)^t, \det(2q))_F\,(-1,-1)_F^{\frac{t(t-1)}{2}} \\
&\quad \times (x(g_1g_2), -x(g_1)x(g_2))_F\,(x(g_1),x(g_2))_F\,\epsilon(2q) \\
&\quad \times \gamma(\eta)^{j_1}\gamma(\eta)^{j_2}\gamma(\eta)^{-j} \\
&\quad \times \gamma(x(g_1g_2),\eta)^{-1}\,\gamma(x(g_1),\eta)\,\gamma(x(g_2),\eta).
\end{aligned}$$

The first two lines in the last expression here are $c_{Rao}(g_1,g_2)$, while the last two lines of that expression are clearly a coboundary. $\square$

Remark 4.6. The factor $(-1,-1)_F^{\frac{t(t+1)}{2}}$ in [64], Theorem 5.3, p.361, should be $(-1,-1)_F^{\frac{t(t-1)}{2}}$, as here.

Any choice of a standard basis for W determines a complete polarization and functions $j(g)$, $x(g)$, etc. on $Sp(W)$, as above. Hence, a choice of standard basis, which we will denote for a moment by $\mathbf{e}$, determines a Rao cocycle $c_{Rao,\mathbf{e}}$. Let

$$Mp^{(2)}(W)_{\mathbf{e}} = Sp(W)\times\mu_2,$$

with multiplication

$$(g_1,\epsilon_1)(g_2,\epsilon_2) = \big(g_1g_2, \epsilon_1\epsilon_2 c_{Rao,\mathbf{e}}(g_1,g_2)\big)$$

be the resulting twofold central extension of $Sp(W)$. Let

$$Mp(W)_{\mathbf{e}} = Mp^{(2)}(W)_{\mathbf{e}}\times_{\mu_2}\mathbb{C}^\times = Sp(W)\times\mathbb{C}^\times,$$

where the multiplication is again defined using the Rao cocycle $c_{Rao,\mathbf{e}}$. For any nondegenerate additive character ψ, the map

$$\begin{aligned} Mp(W)_{\mathbf{e}} &\longrightarrow \widetilde{Sp(W)}_{\psi} \\ (g,z) &\mapsto (g, \beta_{\mathbf{e},\psi}(g) r(g)) \end{aligned}$$

is an isomorphism of central extensions of $Sp(W)$, i.e, the diagram

$$\text{(C)} \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & Mp(W)_{\mathbf{e}} & \longrightarrow & Sp(W) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \widetilde{Sp(W)}_{\psi} & \longrightarrow & Sp(W) & \longrightarrow & 1 \end{array}$$

commutes.

Lemma 4.7. *Any automorphism of $Mp(W)_{\mathbf{e}}$ which restricts to the identity map on $\mathbb{C}^\times$ and which covers the identity map of $Sp(W)$ must be trivial.*

Proof. Indeed, such an automorphism would have the form $(g,z) \mapsto (g, z\lambda(g))$ for a character λ of $Sp(W)$, but, over a local field F, $Sp(W) = [Sp(W), Sp(W)]$, so that λ must be trivial.

Thus, the isomorphism (C) is unique. Since the extension on the top line of (C) does not depend on ψ, while the extension on the bottom line does not depend on $\mathbf{e}$, we obtain the following fact.

Corollary 4.8. *The extensions $Mp(W)_{\mathbf{e}}$, for various choice of standard basis $\mathbf{e}$, and the extensions $\widetilde{Sp(W)}_{\psi}$, for various choices of ψ, are all canonically isomorphic.*

Note that the group $Mp(W)_{\mathbf{e}}$ has a character

$$\begin{aligned} \lambda_{\mathbf{e}} : Mp(W)_{\mathbf{e}} &\longrightarrow \mathbb{C}^\times \\ (g,z) &\mapsto z^2, \end{aligned}$$

whose kernel is precisely the subgroup $Mp^{(2)}(W)_{\mathbf{e}} \subset Mp(W)_{\mathbf{e}}$. Since a character of $Mp(W)_{\mathbf{e}}$ is determined by its restriction to the central $\mathbb{C}^\times$, the $\lambda_{\mathbf{e}}$'s are carried to each other under the canonical isomorphisms.

Corollary 4.9. *The twofold central extensions $Mp^{(2)}(W)_{\mathbf{e}}$ defined by the various Rao cocycles $c_{Rao,\mathbf{e}}$ are canonically isomorphic.*

I.5. Summary. We may summarize this discussion as follows. For a finite dimensional nondegenerate symplectic vector space W over F, we have a unique metaplectic extension $Mp(W)$ of $Sp(W)$ and twofold cover $Mp^{(2)}(W)$,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & Mp(W) & \longrightarrow & Sp(W) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & Mp^{(2)}(W) & \longrightarrow & Sp(W) & \longrightarrow & 1. \end{array}$$

For any nondegenerate additive character ψ, the group $Mp(W)$ has an associated Weil representation (ω_ψ, S).

For a choice of standard basis $\mathbf{e}$, with corresponding complete polarization $W = X + Y$, the extensions $Mp(W)$ and $Mp^{(2)}(W)_{\mathbf{e}}$ can described explicitly in terms of the Rao cocycle $c_{Rao,\mathbf{e}}$, as above. For any nondegenerate additive character ψ, the associated Weil representation (ω_ψ, S) has a Schrödinger model $(\omega_\psi, S(X))$, where, for $(g, z) \in Mp(W) = Mp(W)_{\mathbf{e}}$,

$$\omega_\psi((g, z)) = z \cdot \beta_{\mathbf{e},\psi}(g) r(g),$$

with $r(g) = r_{Y,\psi}(g)$ given in Proposition 2.3 and $\beta_{\mathbf{e},\psi}(g)$ given in Theorem 4.5. (Here the subscripts indicate the dependence on our choices.)

I.6. Nontriviality of the extension $Mp(W)$. In this section, we will show that the metaplectic extension does not split. Recall that F is a nonarchimedean local field.

First, it will be useful later to note that the metaplectic extension $Mp(W)$ splits when restricted to certain subgroups of $Sp(W)$.

Lemma 6.1. *Let P_Y be the stabilizer in $Sp(W)$ of a maximal isotropic subspace Y of W, and let $\tilde{P}_Y$ be the inverse image of P_Y in $Mp(W)$. Then the extension*

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \tilde{P}_Y \longrightarrow P_Y \longrightarrow 1$$

is split, that is, there is a homomorphism $\tilde{i}$ such that

$$\begin{tikzcd} & Mp(W) \arrow[d] \\ P_Y \arrow[ur, "\tilde{i}"] \arrow[r, "i"] & Sp(W) \end{tikzcd}$$

commutes. The restriction of $\tilde{i}$ *to the subgroup* P_Y^1 *is unique, where* P_Y^1 *is the kernel of the character* $p \mapsto \det(p|Y)$ *of* P_Y *.*

Proof. If g_1 and g_2 are in P_Y , then $L(Y, Yg_2^{-1}, Yg_1) = 0$, and

$$c_Y(g_1, g_2) = \gamma(\psi \circ L(Y, Yg_2^{-1}, Yg_1)) = 1.$$

Thus the map $P_Y \to \widetilde{Sp(W)}_\psi$ given by $g \mapsto (g, r(g))$ is a homomorphism. □

Note that, if $\mathbf{e}$ is a standard basis compatible with Y , then the map $P_Y \to Mp(W)_{\mathbf{e}}$ given by

$$n(b)m(a) \mapsto \Big(n(b)m(a), \gamma(\det(a), \psi)\Big) \in Mp(W)_{\mathbf{e}}$$

is a homomorphism. Also note that, since $\gamma(x, \psi)$ is a 4th root of unity, the restriction of the twofold cover $Mp^{(2)}(W)$ to P does not split in general, while its restriction to the unipotent radical N is always uniquely split.

We now work out what all this says in the case when $\dim_F W = 2$, so that, for a fixed standard symplectic basis $\mathbf{e} = \{e, e'\}$, $Sp(W) \simeq SL_2(F)$. Let $X = Fe$ and $Y = Fe'$.

Suppose that $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ for $1 \le i \le 3$ and that

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}.$$

Lemma 6.2. *(i)*

$$L(Y, Yg_2^{-1}, Yg_1) = \frac{1}{2} c_1 c_2 c_3.$$

Note that this can be 0 *.*
(ii) The Leray cocycle is given by

$$c_Y(g_1, g_2) = \gamma(\frac{1}{2} c_1 c_2 c_3, \psi)\, \gamma(\psi),$$

where the notation is as in Lemma 4.1 above.
(iii) The Rao cocycle is given by

$$c_{Rao,\mathbf{e}}(g_1, g_2) = (x(g_1), x(g_2))_F\, (-x(g_1)x(g_2), x(g_3))_F,$$

where

$$x(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases}$$

Proof. First note that if any g_i is in P_Y, i.e., if any $c_i = 0$, then

$$L(Y, Yg_2^{-1}, Yg_1) = L(Yg_2, Y, Yg_1g_2) = 0,$$

since the isotropic subspace Y appears twice. Suppose, then, that $c_1c_2c_3 \neq 0$. Given isotropic lines Y_0, Y_1 and Y_2 in W with $Y_0 \cap Y_1 = 0$ and $Y_0 \cap Y_2 = 0$, choose basis vectors y_0, y_1 and y_2 such that $< y_1, y_0 >= 1 =< y_2, y_0 >$. If we write

$$y_2 = y_1 + y_0\rho,$$

then

$$\frac{1}{2}\rho = L(Y_0, Y_1, Y_2).$$

For the triple (Y, Yg_2^{-1}, Yg_1), we have $y_0 = (0,1)$, $y_1 = (1, -c_2^{-1}a_2)$ and $y_2 = (1, c_1^{-1}d_1)$. Thus

$$\begin{aligned} \rho &= c_1^{-1}d_1 + c_2^{-1}a_2 \\ &= c_1^{-1}c_2^{-1}(c_2d_1 + c_1a_2) \\ &\equiv c_1c_2c_3 \text{mod } F^{\times,2}. \end{aligned}$$

This proves (i) and (ii). The values of $x(g)$ for $g \in \Omega_0$ and Ω_1 are given by the formulas preceeding Proposition 4.3, and (iii) follows easily. $\square$

We now return to the general case, and, for a fixed standard basis $\mathbf{e}$ of W, let $W = X + Y$ be the associated complete polarization. For convenience, we let $Mp(W) \simeq Sp(W) \times \mathbb{C}^\times$ with multiplication given by the Leray cocycle c_Y. Note that $c_Y(p, g) = c_Y(g, p) = 1$ for all $p \in P_Y$ and $g \in Sp(W)$. In particular, the map $\mu : p \mapsto (p, 1)$ is a homomorphism from P_Y to lifting the natural inclusion.

Lemma 6.3. *The homomorphism $\mu : n \mapsto (n, 1)$ is the unique homomorphism lifting the natural inclusion*

$$\begin{array}{ccc} & & Mp(W) \\ & \overset{\mu}{\nearrow} & \downarrow \\ N_Y & \longrightarrow & Sp(W) \end{array}$$

and which is normalized by P_Y, i.e., for which

$$\tilde{p}\mu(n)\tilde{p}^{-1} = \mu(pnp^{-1}),$$

for all $p \in O_Y$. *Here* $\tilde{p} \in Mp(W)$ *is any lift of* $p \in P_Y$.

Proof. The homomorphism μ has the desired property. Any other homomorphism lifting the natural inclusion must have the form

$$n(b) \mapsto \Big(n(b), \psi(tr(\beta b))\Big),$$

for some $\beta \in Sym_n(F)$. But then

$$m(a)n(b)m(a)^{-1} = n(ab^t a) \mapsto (n(ab^t a), \psi(tr(\beta ab^t a))),$$

while

$$(m(a), 1)\,(n(b), \psi(tr(\beta b)))\,(m(a), 1)^{-1} = (n(ab^t a), \psi(tr(\beta b))).$$

Since these are required to be equal for all $a \in GL_n(F)$ and $b \in Sym_n(F)$, we conclude that $\beta = 0$. $\square$

Note that $P_X \cap P_Y = M_Y$, so that, by the Lemma, there is also a unique homomorphism $N_X \longrightarrow Mp(W)$ lifting the natural inclusion and normalized by M_Y .

Lemma 6.4. *For* $b \in Sym_n(F)$ *, let*

$$n'(b) = \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} \in N_X.$$

Then the homomorphism μ_X *is given by*

$$n'(b) \mapsto \big(n'(b), \gamma(\frac{1}{2}\psi \circ b_0)\big),$$

where

$$b = a \begin{pmatrix} 0 & \\ & b_0 \end{pmatrix} {}^t a$$

for $a \in GL_n(F)$ *and* $\det(b_0) \neq 0$ *.*

Proof. Note that, for any $g \in Sp(W)$, $c_Y(g, g^{-1}) = 1$, since

$$L(Y, Yg, Yg) = 0.$$

For $w = w_n$, we then have $(w, 1)^{-1} = (w^{-1}, 1) \in Mp(W)$, and so

$$n'(b) \mapsto (w, 1)\,(n(-b), 1)\,(w, 1)^{-1} = (n'(b), c_Y(wn(-b), w^{-1})),$$

is the desired homomorphism μ_X. But now,

$$L(Y, Yw, Ywn(-b)) = -\frac{1}{2}b_0,$$

where, on the right had side, we mean the isometry class of this symmetric matrix. Thus

$$c_Y(wn(-b), w^{-1})) = \gamma(-\frac{1}{2}\psi \circ b_0),$$

as required. $\square$

Remark 6.5. This Lemma can be viewed as giving another characterization of the Weil index. Suppose that $W = X + Y$ is a complete polarization of W and let $\underline{w} \in Mp(W)$ be any preimage of $w \in Sp(W)$. Let $\mu_Y : N_Y \to Mp(W)$ and $\mu_X : N_X \to Mp(W)$ be the unique splitting homomorphisms normalized by M_Y. Then, for $b \in Sym_n(F)$ with, say, $\det(b) \neq 0$, we have

$$\underline{w}\mu_Y(n(b))\underline{w}^{-1} = \mu_X(wn(b)w^{-1}) \cdot \gamma(-\frac{1}{2}\psi \circ b).$$

Recalling that F is a nonarchimdean local field of characteristic other than 2, we can now establish the following fundamental fact.

Proposition 6.6. *The extension $Mp(W)$ of $Sp(W)$ is nontrivial.*

Proof. Let $W_1 = Fe_n + Fe'_n$, where e_n and e'_n are in our standard basis $\mathbf{e}$. The restriction of the extension $Mp(W)$ to the subgroup $Sp(W_1)$ is the corresponding extension $Mp(W_1)$ defined by the Leray cocycle for $Y_1 = Fe'_n = Y \cap W_1$. Thus, it suffices to prove the Proposition when $\dim_F W = 2$.

Suppose that $s : Sp(W) \to Mp(W)$ is a homomorphism splitting the extension. The restriction of s to N_X and N_Y must coincide with μ_X and μ_Y respectively, by the uniqueness Lemma. For $b \neq 0$, note that

$$n(b)n'(-b^{-1})n(b) = \begin{pmatrix} & -b^{-1} \\ b & \end{pmatrix},$$

so that

$$h(b) = n(b)n'(-b^{-1})n(b)\Big(n(1)n'(-1)n(1)\Big)^{-1} = \begin{pmatrix} b & \\ & b^{-1} \end{pmatrix} = m(b).$$

Let

$$\tilde{h}(b) = \mu_Y(n(b))\mu_X(n'(-b^{-1}))\mu_Y(n(b))\big(\mu_Y(n(1))\mu_X(n'(-1))\mu_Y(n(1))\big)^{-1}.$$

Since $c_Y(n,g) = c_Y(g,n) = 1$, and since $\mu_Y(n(b)) = (n(b),1)$ and

$$\mu_X(n'(-b^{-1})) = (n'(-b^{-1}), \gamma(\frac{1}{2}\pi \circ b^{-1})),$$

we have

$$\begin{aligned}\tilde{h}(b) &= (m(b), \gamma(\frac{1}{2}\psi \circ b^{-1})\,\gamma(\frac{1}{2}\psi \circ 1)^{-1})\\ &= (m(b), \gamma(b,\eta)),\end{aligned}$$

with $\eta = \psi_{\frac{1}{2}}$, as before. But then

$$\begin{aligned}s(m(b)) &= s\Big(n(b)n'(-b^{-1})n(b)\big(n(1)n'(-1)n(1)\big)^{-1}\Big)\\ &= \mu_Y(n(b))\mu_X(n'(-b^{-1}))\mu_Y(n(b))\big(\mu_Y(n(1))\mu_X(n'(-1))\mu_Y(n(1))\big)^{-1}\\ &= (m(b), \gamma(b,\eta)) = \tilde{h}(b).\end{aligned}$$

However, this yields

$$\begin{aligned}s(m(b_1))\,s(m(b_2)) &= (m(b_1)m(b_2), \gamma(b_1,\eta)\gamma(b_2,\eta))\\ &= (m(b_1b_2), (b_1,b_2)_F\gamma(b_1b_2,\eta))\\ &= (b_1,b_2)_F\, s(m(b_1b_2)).\end{aligned}$$

Thus, since the Hilbert symbol $(\ ,\ )_F$ is nontrivial, the splitting homomorphism s cannot exist. $\square$

Remark 6.7. In fact, the calculation of the previous proof shows that the quadratic Hilbert symbol $(\ ,\ )_F$ is the Steinberg cocycle, [44], associated to the metaplectic extension $Mp(W)$ of $Sp(W)$. This proof is modeled on the calculations of Chapter II of [44], where a very readable account of the theory of Steinberg cocycles and central extensions for simply connected semisimple groups can be found.

CHAPTER II

In this section, we discuss reductive dual pairs, the theta correspondence, and Howe's duality conjecture. The basic setup is due to Howe [24].

II.1. Reductive dual pairs. Let W be a symplectic vector space over F, as in Chapter I. For a group A with subgroup B, we let

$$\mathrm{Cent}_A(B) = \{a \in A \mid ab = ba \text{ for all } b \in B\}$$

be the commutant of B in A. A pair of subgroups B and B' of A are said to be mutual commutants if $\mathrm{Cent}_A(B) = B'$ and $\mathrm{Cent}_A(B') = B$.

Definition 1.1. *A reductive dual pair (G, G') in $Sp(W)$ is a pair of subgroups G and G' of $Sp(W)$ such that G_1 and G_2 are reductive groups and*

$$\mathrm{Cent}_{Sp(W)}(G) = G' \qquad \textit{and} \qquad \mathrm{Cent}_{Sp(W)}(G') = G.$$

We will frequently drop the word reductive and simply refer to (G, G') as a dual pair in $Sp(W)$.

If $W = W_1 + W_2$ is an orthogonal decomposition, and if (G_1, G_1') and (G_2, G_2') are dual pairs in $Sp(W_1)$ and $Sp(W_2)$ respectively, then $(G, G') = (G_1 \times G_2, G_1' \times G_2')$ is a dual pair in $Sp(W)$. Such a pair is said to be reducible. A dual pair (G, G') which does not arise in this way is said to be irreducible. For example, if W is irreducible for the action of $G \times G'$, then the dual pair (G, G') is irreducible. Every dual pair can be written as a product of irreducible dual pairs.

The pair $(Sp(W), \{\pm 1_W\})$ is the most trivial example of an irreducible dual pair. At the other extreme, suppose that $\mathbf{e}$ is a standard basis for W, as in section 1, and let C be the subgroup of W which acts diagonally with respect to this basis. Then $\mathrm{Cent}_{Sp(W)}(C) = C$, so that (C, C) is a dual pair in $Sp(W)$. Note that the dual pair (C, C) is only irreducible if $\dim_F W = 2$.

Families of irreducible dual pairs can be constructed as tensor products. Let (D, ι) be one of the following F algebras with involution (F-linear isomorphism $\iota : D \to D$ with $\iota^2 = 1$ and such that $(ab)^\iota = b^\iota a^\iota$).

$$(i) \qquad (D, \iota) = (F, id),$$

$$(ii) \qquad (D, \iota) = (E, \sigma),$$

where E/F is a quadratic extension and σ is the nontrivial Galois automorphism

(Recall that the characteristic of F is not 2.)

$$(iii) \qquad (D,\iota) = (B,\sigma),$$

where B is a division quaternion algebra with center F, and ι is the main involution of this algebra.

Fix $\epsilon = \pm 1$. Let W, $<\ ,\ >$ be a finite dimensional left vector space over D with a nondegenerate ϵ-skew Hermitian form

$$<\ ,\ >: W \times W \longrightarrow D,$$

with

$$< ax, by > = \ a < x, y > b^{\iota} \qquad \text{and} \qquad < y, x > = -\epsilon \ < x, y >^{\iota}.$$

Let

$$G(W) = \{ g \in GL_D(W) \mid < xg, yg > = < x, y > \quad \text{for all } x, y \in W \ \}$$

be the isometry group of W.

Similarly, let V, $(\ ,\ )$ be a finite dimensional right vector space over D with a nondegenerate ϵ-Hermitian form

$$(\ ,\ ) : V \times V \longrightarrow D,$$

with

$$(xa, yb) = a^{\iota}(x,y)b \qquad \text{and} \qquad (y,x) = \epsilon(x,y)^{\iota}.$$

Let

$$G(V) = \{ g \in GL_D(V) \mid (gx, gy) = (x,y) \text{ for all } x,\ y \in V \ \}$$

be the isometry group of V.

The F vector space

$$\mathbb{W} = V \otimes_D W$$

has a nondegenerate F-bilinear alternating form

$$<< x_1 \otimes y_1, x_2 \otimes y_2 >>= tr_{D/F}\big(\big((x_1,y_1) < x_2, y_2 >^{\iota}\ \big),$$

and there is a natural map

$$\begin{aligned} G(V) \times G(W) &\longrightarrow Sp(\mathbb{W}) \\ (h,g) &\mapsto h \otimes g. \end{aligned}$$

Thus we obtain an irreducible dual pair $(G(V), G(W))$ in $Sp(\mathbb{W})$. The dual pairs constructed on this way are said to be of type I.

Another type of dual pair can be constructed as follows. Let D be a division algebra with center F, and let A and B be a finite dimensional right vector spaces over D. Let $A^* = \mathrm{Hom}_D(A, D)$ where D is viewed as a right vector space over itself. Note that A^* is a left vector space over D, with the action given by postmultiplication on the left: $[d\cdot a^*, a] = d\cdot[a^*, a]$. Define the left D vector space $B^* = \mathrm{Hom}_D(B, D)$ similarly. There are natural pairings

$$\begin{aligned} A^* \times A &\longrightarrow D \qquad (a^*, a) \mapsto [a^*, a], \\ B^* \times B^* &\longrightarrow D \qquad (b^*, b) \mapsto [b^*, b]. \end{aligned}$$

Let

$$X = A \otimes_D B^* \qquad \text{and} \qquad Y = B \otimes A^*,$$

and note that there is a nondegenerate pairing

$$X \times Y \longrightarrow F$$

given by

$$\big[a \otimes b^*, b \otimes a^*\big] = tr_{D/F}\big(\,[a^*, a][b^*, b]\,\big),$$

where $tr_{D/F}$ is the reduced trace from D to F. For $g \in GL_D(A)$, define $g^* \in GL_D(A^*)$ by $[a^*g^*, a] = [a^*, ga]$, and similarly for $g \in GL_D(B)$. Let $\mathbb{W} = X+Y$, with the symplectic form

$$<< x_1 + y_1, x_2 + y_2 >>= \big[x_1, y_2\big] - \big[y_1, x_2\big].$$

There is then a natural homomorphism

$$\begin{aligned} GL_D(A) \times GL_D(B) &\longrightarrow Sp(\mathbb{W}) \\ &(g, h) \mapsto g \otimes h^* + h^{-1} \otimes (g^*)^{-1}. \end{aligned}$$

This construction gives rise to an irreducible dual pair $(GL_D(A), GL_D(B))$ in $Sp(\mathbb{W})$; such dual pairs are said to be of type II.

Up to restriction of scalars, every irreducible dual pair is of one of these types, [46],p.15. For a much more thorough discussion of the classification of reductive dual pairs, see Chapter 1 of [46].

II.2. The theta correspondence. Again, the foundations of this theory are due to Howe, [24].

Let (G, G') be a dual pair in $Sp(W)$. Let $\tilde{G}$ (resp. $\tilde{G}'$) be the inverse image of G (resp. G') in the metaplectic group $Mp(W)$, so that these groups are extensions

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

and

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \tilde{G}' \longrightarrow G \longrightarrow 1.$$

As explained in [46], Chapter II, $\tilde{G}$ and $\tilde{G}'$ are mutual commutants in $Mp(W)$, and there is a natural homomorphism

$$j : \tilde{G} \times \tilde{G}' \longrightarrow Mp(W).$$

For an additive character ψ of F, we may then consider the pullback $(j^*(\omega_\psi), S)$ of the Weil representation (ω_ψ, S) to $\tilde{G} \times \tilde{G}$. Note that the central $\mathbb{C}^\times$ in $\tilde{G}$ and $\tilde{G}'$ both act in S by the identity character, i.e.,

$$\omega_\psi(j(z_1, z_2)) = z_1 z_2 \cdot id_S.$$

The basic idea is that, because the Weil representation of $Mp(W)$ is very 'small', its restriction to the subgroup $\tilde{G} \times \tilde{G}$, consisting of mutual commutants, should decompose into irreducibles in a reasonable way. Since there are many nontrivial extensions in the category of smooth or even admissible representations of p-adic groups, it is best to study quotients of $(j^*(\omega_\psi), S)$ rather than summands or submodules. Of course, only irreducible representations π of $\tilde{G}$ for which $\pi(z) = z \cdot id_\pi$ can play a role, and similarly for $\tilde{G}'$.

For more details in the following discussion, the reader can consult [46], Chapter 2, pp.44-49. Let $(\pi, \mathcal{V}_\pi)$ be an irreducible admissible representation of $\tilde{G}$ for which $\pi(z) = z \cdot id_\pi$. For convenience, we will frequently write π instead of $\mathcal{V}_\pi$. Let $S(\pi)$ be the maximal quotient of S on which $\tilde{G}$ acts as a multiple of π. A little more concretely, let

$$\mathcal{N}(\pi) = \bigcap_{\lambda \in Hom_{\tilde{G}}(S,\pi)} \ker(\lambda)$$

be the intersection of the kernels of all $\tilde{G}$ equivariant maps from S to π. Then

$$S(\pi) = S/\mathcal{N}(\pi).$$

Note that, since the action of $\tilde{G}'$ on S permutes the elements $\lambda \in Hom_{\tilde{G}}(S, \pi)$, the subspace $\mathcal{N}(\pi)$ is stable under the action of $\tilde{G}'$, and hence $\tilde{G}'$ acts on the space $S(\pi)$. Since each $\ker(\lambda)$ is stable under the action of $\tilde{G}$, $\mathcal{N}(\pi)$ is as well, so that $S(\pi)$ is a representation of $\tilde{G} \times \tilde{G}'$.

By Lemma III.4 of Chapter 2 of [46], there is a smooth representation $\Theta_\psi(\pi)$ of $\tilde{G}'$ such that

$$S(\pi) \simeq \pi \otimes \Theta_\psi(\pi),$$

and $\Theta_\psi(\pi)$ is unique up to isomorphism. If π does not occur as a quotient of (ω_ψ, S) , we set both $S(\pi)$ and $\Theta_\psi(\pi)$ equal to zero. The most fundamental (and deep) structural fact about the restriction of S to $\tilde{G} \times \tilde{G}'$ is due to Howe.

Howe Duality Principle. (Howe, [46],[26], Waldspurger, [80]) *Assume that the residue characteristic of F is not 2 . Then, for any irreducible admissible representation π of $\tilde{G}$*
(i) Either $\Theta_\psi(\pi) = 0$ or $\Theta_\psi(\pi)$ is an admissible representation of $\tilde{G}'$ of finite length.
(ii) If $\Theta_\psi(\pi) \neq 0$, There is a unique $\tilde{G}'$ invariant submodule $\Theta^0_\psi(\pi)$ of $\Theta_\psi(\pi)$ such that the quotient

$$\theta_\psi(\pi) := \Theta_\psi(\pi)/\Theta^0_\psi(\pi)$$

is irreducible. The irreducible admissible representation $\theta_\psi(\pi)$ of $\tilde{G}'$ is uniquely determined by π . If $\Theta_\psi(\pi) = 0$, set $\theta_\psi(\pi) = 0$.
(iii) If $\theta_\psi(\pi_1)$ and $\theta_\psi(\pi_2)$ are nonzero and isomorphic, then $\pi_1 \simeq \pi_2$.

It is expected (The Howe duality conjecture) that this result also holds when the residue characteristic of F is 2 . Certain facts in this direction are known, some of which will be discussed below, cf. also [46], Chapter 2, pp. 48–9.

Let

$$\mathrm{Howe}_\psi(\tilde{G};\tilde{G}') = \{\pi \in \mathrm{Irr}(\tilde{G}) \mid \theta_\psi(\pi) \neq 0 \ \},$$

be the set of all (isomorphism classes of) irreducible admissible representations of $\tilde{G}$ such that $\mathrm{Hom}_{\tilde{G}}(S, \pi) \neq 0$. Note that this set can depend both on the dual pair (G, G') and on the choice of ψ . The Howe duality principle asserts that the map $\pi \mapsto \theta_\psi(\pi)$ defines a *bijection*

$$\mathrm{Howe}_\psi(\tilde{G};\tilde{G}') \xrightarrow{\sim} \mathrm{Howe}_\psi(\tilde{G}';\tilde{G}).$$

We will refer to this bijection as the **local theta correspondence**.

In the rest of the local part of these lectures, we will discuss some of what is known to be or conjectured to be true about this correspondence.

Warning. *To simplify the exposition, we will, from now on, specialize to the case of an irreducible dual pair of type I,* $(G,G')=(G(V),G(W))$ *with* $(D,\iota)=(F,id)$ *, and with* $\epsilon=+1$ *. Thus, we have* $G(V)=O(V)$ *and* $G(W)=Sp(W)$ *.*

II.3. Restriction of the metaplectic cover. In this section, we describe the extensions $\tilde{G}$ and $\tilde{G}'$. Recall that we are specializing to the case of the dual pair $G(W)=Sp(W)$ and $G(V)=O(V)$ in $Sp(\mathbb{W})$ where $\mathbb{W}=V\otimes_F W$, so that we have the homomorphism $j: G(V)\times G(W)\rightarrow Sp(\mathbb{W})$. We will write j_V for the restriction of this homomorphism to $1\times Sp(W)$ and j_W for its restriction to $O(V)\times 1$.

Let $\dim_F V=m$ and $\dim_F W=2n$.

Let $\Omega(W)$ (resp. $\Omega(\mathbb{W})$) be the space of maximal isotropic subspaces of W (resp. $\mathbb{W}$). There is a natural map

$$\Omega(W)\longrightarrow\Omega(\mathbb{W})\qquad Y\mapsto\mathbb{Y}=V\otimes Y.$$

This map is equivariant for the action of $Sp(W)$, and its image is precisely the set of maximal isotropic subspaces of $\mathbb{W}$ which are fixed by $j(O(V))$, [46], Chapter 1, . Recall that, if $\dim_F W=2n$, the Leray invariant gives a map

$$\begin{aligned}L:\Omega(W)^3&\longrightarrow Sym_{\le n}(F)/\sim\\(Y_0,Y_1,Y_2)&\mapsto L(Y_0,Y_1,Y_2),\end{aligned}$$

where $Sym_{\le n}(F)/\sim$ is the set of isomorphism classes of nondegenerate symmetric bilinear forms over F of dimension at most n . Note that, for the moment, we are viewing $L(Y_0,Y_1,Y_2)$ as a symmetric bilinear form, rather than as a quadratic form. There is an analogous map

$$L:\Omega(\mathbb{W})^3\longrightarrow Sym_{\le mn}(F)/\sim .$$

The relation between these two maps on the image of $\Omega(W)^2$ is given by the following result, which is easily checked.

Lemma 3.1.

$$L(V\otimes Y_0,V\otimes Y_1,V\otimes Y_2)=V\otimes L(Y_0,Y_1,Y_2),$$

i.e., the following diagram commutes:

$$\begin{array}{ccc}\Omega(W)^3 & \xrightarrow{V\otimes} & \Omega(\mathbb{W})^3\\ L\downarrow & & L\downarrow\\ Sym_{\le n}(F)/\sim & \xrightarrow{V\otimes} & Sym_{\le mn}(F)/\sim .\end{array}$$

We are now in a position to describe the pullback to $Sp(W)$, via j_V, of the metaplectic cover $Mp(\mathbb{W})$. It will be easiest to use the Leray cocycle.

Proposition 3.2. *Fix $Y \in \Omega(W)$ and let $\mathbb{Y} = V \otimes_F Y \in \Omega(\mathbb{W})$. For $g \in Sp(W)$, let*

$$\mu_V(g) = \epsilon(V)^{j(g)}\,(\det(V), x(g))_F\,\gamma(\det(V), \eta)^{-j(g)},$$

where $j(g)$ and $x(g)$ are defined as above. Here $\det(V)$ and $\epsilon(V)$ are computed with respect to the matrix for the bilinear form on V. Then, for g_1 and $g_2 \in Sp(W)$,

$$c_{\mathbb{Y}}(j_V(g_1), j_V(g_2)) = \mu_V(g_1g_2)\mu_V(g_1)^{-1}\mu_V(g_2)^{-1}c_Y(g_1,g_2)^m.$$

Proof. We compute:

$$\begin{aligned} c_{\mathbb{Y}}(j_V(g_1), j_V(g_2)) &= \gamma\big(\psi \circ L(\mathbb{Y}, \mathbb{Y}g_2^{-1}, \mathbb{Y}g_1)\big) \\ &= \gamma\big(\psi \circ V \otimes_F L(Y, Yg_2^{-1}, Yg_1)\big). \end{aligned}$$

Here By Lemma I.4.2, this last expression is

$$\gamma(\det(V \otimes L), \eta)\,\gamma(\eta)^{m\ell}\,\epsilon(V \otimes L),$$

where $L = L(Y, Yg_2^{-1}, Yg_1)$ and ℓ is the dimension of L. In this expression the determinants and Hasse invariants are computed with respect to the matrix for the bilinear form! Now it is easy to check that, for a tensor product of bilinear forms:

$$\begin{aligned} \epsilon(V \otimes L) &= \epsilon(V)^{\ell}\epsilon(L)^m(\det(V), \det(L))_F^{m\ell-1} \\ &\qquad \times (-1, \det(V))_F^{\frac{\ell(\ell-1)}{2}}(-1, \det(L))_F^{\frac{m(m-1)}{2}}, \\ \det(V \otimes L) &= \det(V)^{\ell}\det(L)^m, \end{aligned}$$

and hence

$$\begin{aligned} \gamma(\det(V \otimes L), \eta) = (\det(V), \det(L))_F^{\ell m}\,&(-1, \det(V))_F^{\frac{\ell(\ell-1)}{2}}\,(-1, \det(L))_F^{\frac{m(m-1)}{2}} \\ &\times \gamma(\det(V), \eta)^{\ell}\,\gamma(\det(L), \eta)^m. \end{aligned}$$

Thus

$$\begin{aligned}
c_{\mathbb{Y}}(j(g_1), j(g_2)) = (\det(V), \det(L))_F^{\ell m}\, &(-1, \det(V))_F^{\frac{\ell(\ell-1)}{2}}\, (-1, \det(L))_F^{\frac{m(m-1)}{2}} \\
&\times \gamma(\det(V), \eta)^{\ell}\, \gamma(\det(L), \eta)^{m} \\
&\times \gamma(\eta)^{\ell m} \\
&\times \epsilon(V)^{\ell} \epsilon(L)^{m}\, (\det(V), \det(L))_F^{m\ell - 1} \\
&\qquad \times (-1, \det(V))_F^{\frac{\ell(\ell-1)}{2}}\, (-1, \det(L))_F^{\frac{m(m-1)}{2}} \\
= \Big(\epsilon(L)\gamma(\det(L), \eta)\gamma(\eta)^{\ell}\Big)^{m} & \\
&\times (\det(V), \det(L))_F\, \epsilon(V)^{\ell}\, \gamma(\det(V), \eta)^{\ell} \\
= c_Y(g_1, g_2)^m & \\
\times (\det(V), x(g_1 g_2)x(g_1)x(g_2))_F & \big(\epsilon(V)\gamma(\det(V), \eta)\big)^{j_1 + j_2 - j} \\
\times (\det(V), (-1)^t)_F \gamma(\det(V), \eta)^{-2t}. &
\end{aligned}$$

Since $(\det(V), (-1)^t)_F \gamma(\det(V), \eta)^{-2t} = 1$, this gives the required relation. $\square$

Corollary 3.3. *The homomorphism $j_V : Sp(W) \to Sp(\mathbb{W})$ lifts uniquely to a homomorphism $\tilde{j_V} : Mp(W) \to Mp(\mathbb{W})$ whose restriction to $\mathbb{C}^\times$ takes z to z^m. In particular, the restriction of this homomorphism to the twofold cover $Mp^{(2)}(W)$ factors through $Sp(W)$ if and only if $m = \dim_F(V)$ is even.*

II.4. The Weil representation for $(O(V), Sp(W))$. We can now consider the pullback $j_V^*(\omega_\psi)$ to $Mp(W)$ of the Weil representation of $Mp(\mathbb{W})$. The restriction of this pullback to $\mathbb{C}^\times$ is given by $j_V^*(\omega_\psi)(z) = z^m \cdot id_S$. It is inconvenient to have this central character depend on $m = \dim_F V$, so we will make a slight modification.

Remark 4.1. As explained in Chapter I, the group $Mp(W)$ has a unique character λ whose restriction to $\mathbb{C}^\times$ is given by $\lambda(z) = z^2$ and whose kernel is the twofold metaplectic cover $Mp^{(2)}(W)$. If $\mathcal{A}_m(Mp(W))$ is the category of smooth representations of $Mp(W)$ whose restriction to $\mathbb{C}^\times$ is $z \mapsto z^m$, then, for $m - m'$ even, the categories $\mathcal{A}_m(Mp(W))$ and $\mathcal{A}_{m'}(Mp(W))$ are naturally identified via the tensor product with $\lambda^{\pm\frac{m-m'}{2}}$. Thus, we need only consider representations in the two categories: $\mathcal{A}_0(Mp(W)) \simeq \mathcal{A}(Sp(W))$, the category of representations of $Mp(W)$ which factor through $Sp(W)$, and $\mathcal{A}(Mp(W))^{gen} := \mathcal{A}_1(Mp(W))$, the category of *genuine* representations of $Mp(W)$. Tensor products and contragredients are defined in $\mathcal{A}(Mp(W))^{gen}$ by twisting the usual operations by a suitable power of λ. For example, if π_1 and $\pi_2 \in \mathcal{A}(Mp(W))^{gen}$, then

$$\pi_1 \otimes^{gen} \pi_2 = \pi_1 \otimes \pi_2 \otimes \lambda^{-2} \in \mathcal{A}(Sp(W)).$$

Similarly, for $\pi \in \mathcal{A}(Mp(W))^{gen}$, $\pi^{\vee,gen} := (\pi^\vee) \otimes \lambda^2$, where $\pi^\vee \in \mathcal{A}_{-1}(Mp(W))$ is the usual contragredient. These conventions allow us to avoid having to constantly keep track of tensor products with λ in the notation. These twists must be kept in mind when doing computations, however.

Definition 4.2. *The* **Weil representation** $\omega_{\psi,V}$ *associated to* ψ *and* V *is the image of* $j_V^*(\omega_\psi)$ *either in* $\mathcal{A}(Sp(W))$ *if* $m = \dim_F V$ *is even, or in* $\mathcal{A}(Mp(W))^{gen}$ *if* m *is odd. Thus,*

$$\omega_{V,\psi} = \begin{cases} \lambda^{-\frac{m}{2}} \otimes j_V^*(\omega_\psi) & \textit{if } m \textit{ is even,} \\ \lambda^{-\frac{m-1}{2}} \otimes j_V^*(\omega_\psi) & \textit{if } m \textit{ is odd.} \end{cases}$$

For given ψ, we obtain an explicit description of the Weil representation $\omega_{\psi,V}$ of $Mp(W)$ determined by V as follows. Let $\mathbf{e}$ be a standard basis for W, with associated complete polarization $W = X + Y$. Note that $\mathbb{W} = V \otimes X + V \otimes Y$ is a complete polarization of $\mathbb{W}$, and that

$$V \otimes X \simeq V^n = \{x = (x_1, \dots x_n) \mid x_j \in V\}.$$

For x and $y \in V^n$, we write

$$(x, y) = ((x_i, y_j)) \in Sym_m(F),$$

and we let $a \in GL_n(F)$ act on V^n by right multiplication: $x \mapsto xa$. Also, we make the identification $Mp(W) = Mp(W)_{\mathbf{e}} = Sp(W) \times \mathbb{C}^\times$.

Proposition 4.3. *Let* χ_V *be the quadratic character of* $F^\times$ *defined by*

$$\chi_V(x) = (x, (-1)^{\frac{m(m-1)}{2}} \det(V))_F,$$

where $\det(V)$ *is the determinant of the matrix of the bilinear form on* V. *For* $z \in \mathbb{C}^\times$, *let*

$$\chi_V^\psi(x, z) = \chi_V(x) \cdot \begin{cases} z \cdot \gamma(x, \eta)^{-1} & \textit{if } m \textit{ is odd, and} \\ 1 & \textit{if } m \textit{ is even.} \end{cases}$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(W)$ *and for* $\varphi \in S(V^n)$,

$$\big(\omega_{\psi,V}(g,z)\varphi\big)(x) = \chi_V^\psi(x(g), z)\, \gamma(\eta \circ V)^{-j(g)} \cdot \big(r(j_V(g))\varphi\big)(x),$$

where

$$\big(r(j_V(g))\varphi\big)(x) = \int_{ker(c)\backslash V^n} \psi\Big(\frac{1}{2}tr((xa,xb)) - tr((xb,yc)) + \frac{1}{2}tr((yc,yd))\Big)\cdot\varphi(xa+yc)\,d\mu_g(y),$$

for a suitable choice of the Haar measure $d\mu_g(y)$ *on* $ker(c)\backslash V^n$.

Proof. We have, literally,

$$\big(j_V^*\omega_\psi(g,z)\varphi\big)(x) = z^m\cdot\mu_V(g)\beta_{\mathbf{e},\psi}(g)^m\,\big(r(j_V(g))\varphi\big)(x),$$

where $\big(r(j_V(g))\varphi\big)(x)$ is given by the integral above. The twist by a suitable power of the character λ reduces the factor z^m to either z, if m is odd, or 1 if m is even. For the other factors, we have

$$\beta_{\mathbf{e},\psi}(g) = \gamma(x(g),\eta)^{-1}\gamma(\eta)^{-j(g)},$$

and

$$\mu_V(g) = \epsilon(V)^{j(g)}\,(x(g),\det(V))_F\,\gamma(\det(V),\eta)^{-j(g)},$$

so that

$$\begin{aligned}\mu_V(g)\beta_{\mathbf{e},\psi}(g)^m &= (x(g),\det(V))_F\,\gamma(x(g),\eta)^{-m}\cdot\Big(\epsilon(V)\,\gamma(\eta)^m\,\gamma(\det(V),\eta)\Big)^{-j(g)}\\ &= \chi_V^\psi(x(g),1)\,\gamma(\psi\circ V)^{-j(g)},\end{aligned}$$

as claimed. In this last line, the factor $\gamma(\psi\circ V)$ is the Weil index of the quadratic space associated to the bilinear form on V :

$$\gamma(\psi\circ V) = \epsilon(V)\,\gamma(\eta)^m\,\gamma(\det(V),\eta).$$

$\square$

In particular, for $a\in GL_n(F)$,

$$\big(\omega_{\psi,V}(m(a),z)\varphi\big)(x) = \chi_V^\psi(\det(a),z)\,|\det(a)|^{\frac{m}{2}}\,\varphi(xa),$$

and, for $b\in Sym_n(F)$,

$$\big(\omega_{\psi,V}(n(b),1)\varphi\big)(x) = \psi(\frac{1}{2}tr((x,x)b))\cdot\varphi(x).$$

Note that, for a_1 and $a_2\in GL_n(F)$,

$$c_{Rao,\mathbf{e}}(m(a_1),m(a_2)) = (\det(a_1),\det(a_2))_F,$$

so that, since

$$(a,b)_F = \gamma(ab,\psi)\gamma(a,\psi)^{-1}\gamma(b,\psi)^{-1},$$

$$(m(a),z) \mapsto \chi_V^{\psi}(\det(m(a)),z)$$

is a character of the inverse image $\widetilde{GL_n(F)}$ of $M_Y \simeq GL_n(F)$ in $Mp(W)$.

Finally, for $w = w_n = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$,

$$\big(\omega_{\psi,V}(w,1)\varphi\big)(x) = \gamma(\psi\circ V)^{-n}\int_{V^n}\psi(-tr((x,y))\,\varphi(y)\,dy,$$

where dy is the measure on V^n which is self dual for this Fourier transform.

Next consider the action of $h \in O(V)$. Since, with respect to the complete polarization $\mathbb{W} = V\otimes X + V\otimes Y$,

$$j_W(h) = \begin{pmatrix} h\otimes 1 & \\ & h\otimes 1\end{pmatrix} \in M_{\mathbb{Y}},$$

we have $c_{\mathbb{Y}}(h_1,h_2) = 1$. Thus, the map $h \mapsto (j_W(h), r(j_W(h)))$ defines a homomorphism from $\tilde{j}_W : O(V) \to \widetilde{Sp(\mathbb{W})}$ lifting j_W. Setting $\omega_{\psi,W} = \tilde{j}_W^*(\omega_\psi)$, we have, for $\varphi \in S(V^n)$,

$$\omega_{\psi,W}(h)\varphi(x) = \varphi(h^{-1}x).$$

Note that the homomorphism $\tilde{j}_W$ is not unique, since it can be twisted by any character of $O(V)$. We make the traditional choice, since it yields the natural linear action of $O(V)$ in the Schrödinger model $S(V^n)$.

We have taken some care in our derivation of these formulas for the action of the dual pair $(O(V),Sp(W))$ on $S(V^n)$, since they are usually quoted in the literature without proof.

For a general irreducible dual pair (G,G') in $Sp(W)$ of type I, the extensions $\tilde{G}$ and $\tilde{G}'$ are always split, with the exception of the case we have just discussed, when $m = \dim_F(V)$ is odd. Explicit splittings can be constructed by a suitable elaboration of the method used above for $(O(V),Sp(W))$. This construction can be found in [33], and [22].

II.5. Summary: For any ψ, we obtain a Weil representation of $O(V)\times Sp(W)$, when m is even and of $O(V)\times Mp(W)$, when m is odd. A choice of standard basis $\mathbf{e}$ for W yields a Schrödinger model of this representation on $S(V^n)$. When m is even, we obtain a theta correspondence

$$\begin{array}{rcl} \mathrm{Howe}_\psi(O(V);Sp(W)) & \overset{\sim}{\longleftrightarrow} & Howe_\psi(Sp(W);O(V)) \\ \pi & \mapsto & \theta_\psi(\pi,W) \\ \theta_\psi(\pi',V) & \leftarrow\!\shortmid & \pi'. \end{array}$$

When m is odd, we obtain a theta correspondence

$$\begin{array}{rcl} \mathrm{Howe}_\psi(O(V);Mp(W)) & \overset{\sim}{\longleftrightarrow} & Howe_\psi(Mp(W);O(V)) \\ \pi & \mapsto & \theta_\psi(\pi,W) \\ \theta_\psi(\pi',V) & \leftarrow\!\shortmid & \pi'. \end{array}$$

Here $Howe_\psi(Mp(W);O(V)) \subset Irr(Mp(W))^{gen}$, where $Irr(Mp(W))^{gen}$ is the set of (isomorphism classes of) irreducible representations in the category $\mathcal{A}(Mp(W))^{gen}$.

II.6. Scaling. We conclude this section with an observation, which will be important later, about the behavior of the Weil representation under scaling. For $a \in F^\times$, let V_a be the quadratic space V with inner product $a(\ ,\ )$. Recall that $\psi_a(x) = \psi(ax)$. By inspecting the explicit formulas for the Weil representation of Proposition 4.3 above, we obtain

Corollary 6.1. *As representations of $Mp(W)$,*

$$\omega_{\psi,V_a} \simeq \omega_{\psi_a,V}.$$

In fact, for any choice of standard basis $\mathbf{e}$ for W,

$$\omega_{\psi,V_a}(g) = \omega_{\psi_a,V}(g)$$

as operators in the Schrödinger model $S(V^n)$.

Proof. The only point to check is that

$$(x(g),\det(V_a))_F\,\gamma(x(g),\eta)^{-m}\,\gamma(\psi\circ V_a)^{-m} = (x(g),\det(V))_F\,\gamma(x(g),\eta_a)^{-m}\,\gamma(\psi_a\circ V)^{-m},$$

and this is immediate from the relation $\gamma(x,\psi_a) = (x,a)_F\,\gamma(x,\psi)$. $\square$

Corollary 6.2. *The theta correspondence for the dual pair $(O(V_a),Sp(W))$ with respect to ψ is the same as the theta correspondence form the dual pair $(O(V),Sp(W))$ with respect to the character ψ_a.*

CHAPTER III

In this chapter, we will discuss the behavior of the local theta correspondence as the groups $(O(V), Sp(W))$ in a dual pair vary in Witt towers.

III.1. Witt towers. For any $r > 0$, let $V_{r,r}$ be the quadratic space over F of dimension $2r$ with quadratic form

$$\begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix}.$$

We will refer to such a space as a split space. Every quadratic space V over F has a Witt decomposition

$$V = V_0 + V_{r,r}$$

where V_0 is anisotropic. The anisotropic space V_0 is unique up to isomorphism and r is called the Witt index of V. The Witt tower associated to a given anisotropic space V_0 is the collection of spaces $\{V_0 + V_{r,r} \mid r \geq 0\}$. Note that

$$\det(V_0 + V_{r,r}) = (-1)^r \det(V_0) \in F^\times/F^{\times,2},$$

so that the quadratic character

$$\chi_V(x) = (x, (-1)^{\frac{m(m-1)}{2}} \det(V))_F$$

is the same for all spaces V in a Witt tower.

In this section, we fix an anisotropic quadratic space V_0, and we let

$$V_r = V_0 + V_{r,r} = \mathrm{span}(u_1, \ldots, u_r) + V_0 + \mathrm{span}(u'_1, \ldots, u'_r),$$

where $(u_i, u'_j) = \delta_{ij}$. Thus $\{V_r \mid r \geq 0\}$ is a Witt tower. Similarly, we let W_n be the symplectic space over F of dimension $2n$, with a fixed standard basis $\mathbf{e}$, and we refer to the collection $\{W_n \mid n \geq 0\}$ as a (the) Witt tower of symplectic spaces.

III.2. Parabolic subgroups. Before discussing the theta correspondence, we recall a few facts about parabolic subgroups.

For any s with $1 \leq s \leq r$, may write

$$V_r = U_s + V_{r-s} + U'_s,$$

where

$$U_s = \mathrm{span}(u_1, \ldots, u_s) \qquad \text{and} \qquad U'_s = \mathrm{span}(u'_1, \ldots, u'_s)$$

are isotropic subspaces of V_r. The subgroup Q_s of $O(V_r)$ which stabilizes U_s is a maximal parabolic subgroup of $O(V_r)$. It has a Levi decomposition $Q_s = M_s N_s$, where

$$M_s \simeq GL_s(F) \times O(V_{r-s}),$$

via restriction to $U_s + V_{r-s}$. More generally, if $\mathbf{s} = (s_1, s_2, \dots, s_j)$ is a sequence of positive integers with $|\mathbf{s}| = s_1 + \cdots + s_j \le r$, then

$$U'_{s_1} \subset U'_{s_1+s_2} \subset \cdots \subset U_{|\mathbf{s}|}$$

is an isotropic flag in V_r. The stabilizer of this flag in $O(V_r)$ is a parabolic subgroup $Q_{\mathbf{s}}$. There is a Levi decomposition $Q_{\mathbf{s}} = M_{\mathbf{s}} N_{\mathbf{s}}$ with

$$M_{\mathbf{s}} \simeq GL_{s_1}(F) \times \dots GL_{s_j}(F) \times O(V_{r-|\mathbf{s}|}).$$

For example, if $\mathbf{s} = (1, \dots, 1)$ with $|\mathbf{s}| = r$, then

$$U'_1 \subset U'_2 \subset \cdots \subset U'_r$$

is a maximal isotropic flag in V_r, and the stabilizer $Q_{\mathbf{s}} = Q_{min}$ of this flag is a minimal parabolic subgroup of $O(V_r)$. Note that for any $\mathbf{s}$, $Q_{\mathbf{s}} \supset Q_{min}$. Every parabolic subgroup of $O(V_r)$ is conjugate to a unique $Q_{\mathbf{s}}$. In particular, the maximal parabolic subgroups $Q_1, \dots, Q_r$ are a set of representatives for the conjugacy classes of maximal parabolic subgroups of $O(V_r)$. **Remark 2.1.** Since the group $O(V_r)$ is not connected as an algebraic group, the notion of parabolic subgroup is not standard. The definition we have given is the most natural in our situation. For additional discussion see [32], Remark 1.2.1, [46], Chapter 1.

Similarly, for any t with $1 \le t \le n$, write

$$W_n = X_t + W_{n-t} + Y_t,$$

where

$$X_t = \mathrm{span}(e_1, \dots, e_t) \qquad \text{and} \qquad Y_t = \mathrm{span}(e'_1, \dots, e'_t).$$

For any collection $\mathbf{t} = (t_1, \dots, t_j)$ with $|\mathbf{t}| \le n$, we have the parabolic subgroup $P_{\mathbf{t}}$ of $Sp(W_n)$ which stabilizes the isotropic flag

$$Y_{t_1} \subset Y_{t_2} \subset \dots Y_{t_j}.$$

There is a Levi decomposition $P_{\mathbf{t}} = M_{\mathbf{t}} N_{\mathbf{t}}$ with

$$M_{\mathbf{t}} \simeq GL_{t_1}(F) \times GL_{t_j}(F) \times Sp(W_{n-|\mathbf{t}|}).$$

Since $Sp(W_n)$ is a split semisimple algebraic group, $P_{(1,\ldots,1)} = P_{min}$ is a Borel subgroup. Again the $P_{\mathbf{t}}$'s give a standard collection of parabolic subgroups of $Sp(W_n)$ containing P_{min}, and give representatives for the conjugacy classes of (proper) parabolic subgroups of $Sp(W_n)$, [12].

Finally, consider $Mp(W_n)$. For any $\mathbf{s}$, let $\tilde{P}_{\mathbf{s}}$ be the full inverse image in $Mp(W_n)$ of the parabolic subgroup $P_{\mathbf{s}}$ of $Sp(W_n)$. The following result is proved in [46], p.43.

Lemma 2.2. *There is a unique homomorphism* $\tilde{j} : N_{\mathbf{s}} \longrightarrow \tilde{P}_{\mathbf{s}}$ *which lifts the natural inclusion* $j : N_{\mathbf{s}} \longrightarrow P_{\mathbf{s}}$ *and which is normalized by* $P_{\mathbf{s}}$.

There is a Levi decomposition $\tilde{P}_{\mathbf{s}} = \tilde{M}_{\mathbf{s}} N_{\mathbf{s}}$

III.3. Supercuspidals and the Bernstein-Zelevinsky classification. Also as a preliminary to our study of the local theta correspondence, we will review some facts about the classification of irreducible admissible representations, due to Bernstein and Zelevinsky,[11]. The results assert that all the irreducible admissible representations of (the p-adic points of) a connected reductive group over F can be obtained from the supercuspidal representations of Levi components of parabolic subgroups by the process of induction and taking constituents.

Suppose that G is (the p-adic points of) a connected reductive group over F, and that P is a parabolic subgroup with Levi decomposition $P = MN$. Let $\mathcal{A}(G)$ (resp. $\mathcal{A}(M)$) be the category of smooth representations of G (resp. of M). For $(\tau, V_\tau) \in \mathcal{A}(M)$, we extend τ to a representation of P by letting N act trivially, and we let $I_P^G(\tau) \in \mathcal{A}(G)$ be the smooth induced representation. The space of $I_P^G(\tau)$ is

$$\{f : G \longrightarrow V_\tau \mid f(pg) = \delta_P^{\frac{1}{2}}(p)\tau(p)f(g) \text{ for all } p \in P\},$$

where f is a smooth function on G; and G acts by right translation. Here $\delta_P : P \to \mathbb{R}_+^\times$ is the modular function for P, defined by

$$d(pnp^{-1}) = \delta_P(p)\, dn$$

for any Haar measure on N. This induction is normalized so that unitarizable representations go to unitarizable representations. The Jacquet functor R is the adjoint of the I :

$$\mathrm{Hom}_G(\pi, I_P^G(\tau)) = \mathrm{Hom}_M(R_P^G(\pi), \tau).$$

More concretely, let $\pi[N]$ be the subspace of π spanned by the vectors $\pi(n)v-v$, and let $\pi_N = \pi/\pi[N]$. Note that π_N is a representation of M. We then have

$$R_P^G(\pi) = \pi_N \cdot \delta_P^{-\frac{1}{2}},$$

so that the space of $R_P^G(\pi)$ is π_N where the action of M is twisted by the character $\delta_P^{-\frac{1}{2}}$. This twist is required to compensate for the normalization factor in the definition of induction. Recall that a representation $\pi \in \mathcal{A}(G)$ is said to be supercuspidal if $R_P^G(\pi) = 0$ for all proper parabolic subgroups P of G.

Let $\mathrm{Irr}(G)$ be the set of (isomorphism classes of) irreducible admissible representations of G and let $\mathrm{Irr}_{sc}(G)$ denote the subset of supercuspidal representations. For a Levi subgroup M in G and an irreducible admissible representation $\tau \in \mathrm{Irr}(M)$, the representation $I_P^G(\tau)$ has finite length and its set of Jordan-Holder constituents $JH(I_P^G(\tau)) =: JH(M,\tau)$ depends only on M and not on the choice of P with M as Levi factor.

In the case of $Sp(W_n)$, the result of [11] yields:

Proposition 3.1. *For any $\pi \in \mathrm{Irr}(Sp(W_n))$, there exists an $\mathbf{s} = (s_1, \dots, s_j)$ and supercuspidal representations $\sigma_i \in \mathrm{Irr}_{sc}(GL_{s_i}(F))$ and $\tau \in \mathrm{Irr}_{sc}(Sp(W_{n-|\mathbf{s}|}))$ such that*

$$\pi \in JH(I_{P_{\mathbf{s}}}^{Sp(W_n)}\big(\sigma_1 \otimes \dots \sigma_j \otimes \tau\big).$$

The data $\big(\sigma_1, \dots, \sigma_j, \tau\big)$ is uniquely determined by π up to a permutation of the σ_j's.

In this situation, we will write

$$[\pi] = \big[\sigma_1, \dots, \sigma_j, \tau\big]$$

and refer ot this collection as the Bernstein-Zelevinsky data associated to π. Note that the index $\mathbf{s}$ is implicit in this data.

Similarly, by a slight extension of [11], cf. [32] and [46], Chapter 3, we can associate Bernstein-Zelevinsky data to elements of $\mathrm{Irr}(O(V_r))$ and of $\mathrm{Irr}(Mp(W_n))$.

III.4. Persistence and stable range. We now return to the theta correspondence. In this section, we will fix the additive character ψ and omit it from the notation.

For the moment, fix $W = W_n$ and a Witt tower $\{V_r \mid r \geq 0\}$ of orthogonal spaces. Note that $m = \dim_F V_r = m_0 + 2r$, so that the parity of this dimension is the same for all spaces in the tower. Let

$$G = G_n = \begin{cases} Sp(W_n) & \text{if } m_0 \text{ is even,} \\ Mp(W_n) & \text{if } m_0 \text{ is odd.} \end{cases}$$

In our discussion of the theta correspondence, we will use the conventions explained in Remark II.4.1 concerning representations of $Mp(W)$. Let

$$\mathrm{Irr}(G) = \begin{cases} \mathrm{Irr}(Sp(W)) & \text{if } m \text{ is even,} \\ \mathrm{Irr}(Mp(W))^{gen} & \text{if } m \text{ is odd.} \end{cases}$$

For a given $\pi \in \mathrm{Irr}(G)$, we can consider the images $\theta(\pi, V_r)$ for $r \geq 0$. It turns out that, once π occurs in the correspondence at some level, then it continues to occur further up the tower. This **persistence principle** can be made can be formulated more precisely.

Proposition 4.1. *(i) Suppose that* $\Theta(\pi, V_{r_0}) \neq 0$. *Then* $\Theta(\pi, V_r) \neq 0$ *for all* $r \geq r_0$.
(ii) Let $t = r - r_0$. *If* $\Theta(\pi, V_{r_0}) \neq 0$, *then there is a nonzero* $G \times O(V_r)$ *equivariant map*

$$S(V_r^n) \longrightarrow \pi \otimes I_{Q_t}^{O(V_r)}\Big(\,|\det|^{n-\frac{m-t-1}{2}} \otimes \Theta(\pi, V_{r_0})\Big).$$

(iii) Assume the Howe duality conjecture (e.g., suppose that the residue characteristic of F *is not* 2*). Then,* $\theta(\pi, V_r)$ *is a subquotient of*

$$I_{Q_t}^{O(V_r)}\big(\,|\det|^{n-\frac{m-t-1}{2}} \otimes \theta(\pi, V_{r_0})\big).$$

Proof. Let $W = X + Y$ be the compete polarization determined by the standard basis $\mathbf{e}$ of $W = W_n$, so that $\mathbb{W} = V \otimes X + V \otimes Y = \mathbb{X} + \mathbb{Y}$ is a compete polarization of $\mathbb{W} = V \otimes W$. From the Witt decompostion

$$V_r = U' + V_{r_0} + U,$$

of V_r, we obtain another complete polarization $\mathbb{W} = \mathbb{X}' + \mathbb{Y}'$

$$\mathbb{Y}' = V_{r_0} \otimes Y + U' \otimes W = U' \otimes X + V_{r_0} \otimes Y + U' \otimes Y$$

and

$$\mathbb{X}' = V_{r_0} \otimes X + U \otimes W = V_{r_0} \otimes X + U \otimes X + U \otimes Y.$$

Note that

$$\mathbb{X} = U' \otimes X + V_{r_0} \otimes X + U \otimes X,$$

and that the spaces $U \otimes Y$ and $U' \otimes X$ paired via the symplectic form $<< \, , \, >>$ on $\mathbb{W}$. Let $t = r - r_0 = \dim_F U = \dim_F U'$. The intertwining map

$$I : S(V^n) = S(\mathbb{X}) \longrightarrow S(\mathbb{X}') \simeq S(V_{r_0}^n) \otimes S(W^t)$$

is given by a partial Fourier transform: for $v \in V_{r_0}^n$, $x \in U \otimes X \simeq M_{t,n}(F)$, $y \in U \otimes Y \simeq M_{t,n}(F)$, and $z \in U' \otimes X \simeq M_{t,n}(F)$,

$$(I\varphi)(v, x, y) = \int_{M_{t,n}(F)} \psi(tr({}^t zy))\, \varphi\left(\begin{pmatrix} z \\ v \\ x \end{pmatrix}\right) dz.$$

Let Q_t be the maximal parabolic subgroup of $O(V_r)$ which stabilizes U', and let N_t be its unipotent radical. Then

$$N_t = \{ n(c,d) = \begin{pmatrix} 1 & -{}^t c & d - \frac{1}{2}(c,c) \\ & 1 & c \\ & & 1 \end{pmatrix} \mid c \in Hom(U, V_{r_0}),\ d \in Hom(U, U') \ \}.$$

Here we will identify

$$c = (c_1, \ldots, c_t) \in V_{r_0}^t$$

with an element of $Hom(U, V_{r_0})$ by sending the column vector $u = {}^t(u_1, \ldots, u_t)$ to $c \cdot u = c_1 u_1 + \cdots + c_t u_t$. Similarly, we identify the column vector ${}^t c$ with the element of $Hom(V_{r_0}, U')$ given by

$$v \mapsto \begin{pmatrix} (c_1, v) \\ \vdots \\ (c_t, v) \end{pmatrix} \in F^t \simeq U'.$$

Also $d = -{}^t d \in M_t(F)$. The Levi factor of Q_t is

$$M_t = \{ m(a,h) = \begin{pmatrix} a & & \\ & h & \\ & & {}^t a^{-1} \end{pmatrix} \mid a \in GL_t(F),\ h \in O(V_{r_0}) \ \}.$$

Note that the modular function of Q_t is $\delta_{Q_t}(m(a, h_0)) = |a|^{m_0 + t - 1}$, where $m_0 = \dim V_{r_0}$, so that $m_0 + t - 1 = m - t - 1$, where $m = \dim V_r$.

Lemma 4.2. *The intertwining operator I transfers the linear action of Q_t in $S(V_r^n)$ to the action in $S(V_{r_0}^n)\otimes S(W^t)$ given by:*

$$\omega(n(c,d))\varphi'(v,x,y)=\psi\Big(-tr((x,c)y)-tr({}^txdy)+\frac{1}{2}tr({}^tx(c,c)y)\Big)\cdot\varphi'(v-cx,x,y),$$

and

$\omega(m(a,h_0))\varphi'(v,x,y)=|\det a|^n\,\varphi'(h_0^{-1}v,{}^tax,{}^tay).$

Proof. For $h=n(c,d)$ and noting that $n(c,d)^{-1}=n(-c,-d)$, we calculate

$$\begin{aligned}\big(I\omega(h)\varphi\big)(v,x,y)&=\int_{M_{t,n}(F)}\psi(tr({}^tzy))\,\varphi\left(\begin{pmatrix}z+(c,x)-dx-\frac{1}{2}(c,c)x\\ v-cx\\ x\end{pmatrix}\right)dz\\&=\psi\Big(-tr((x,c)y)-tr({}^txdy)+\frac{1}{2}tr({}^tx(c,c)y)\Big)\\&\qquad\times\int_{M_{t,n}(F)}\psi(tr({}^tzy))\,\varphi\left(\begin{pmatrix}z\\ v-cx\\ x\end{pmatrix}\right)dz\\&=\psi\Big(-tr((x,c)y)-tr({}^txdy)+\frac{1}{2}tr({}^tx(c,c)y)\Big)\cdot\big(I\varphi\big)(v-cx,x,y).\end{aligned}$$

If $h=m(a,h_0)$, then

$$\begin{aligned}\big(I\omega(h)\varphi\big)(v,x,y)&=\int_{M_{t,n}(F)}\psi(tr({}^tzy))\,\varphi\left(\begin{pmatrix}a^{-1}z\\ h_0^{-1}v\\ {}^tax\end{pmatrix}\right)dz\\&=|\det a|^n\big(I\varphi\big)(h_0^{-1}v,{}^tax,{}^tay).\end{aligned}$$

□

Now let $\lambda_0:S(V_{r_0}^n)\to\pi$ be a nonzero G-equivariant map, and let δ_0 be the delta function at $0\in W^t$. Consider the composition:

$$\lambda:=(\lambda_0\otimes\delta_0)\circ I:S(V_r^n)\longrightarrow\pi.$$

This map is clearly G equivariant and nonzero, so (i) of the Proposition is proved.

Next, let $\mu_0:S(V_{r_0}^n)\to\Theta(\pi,V_{r_0})\otimes\pi$ be the natural quotient map. For $\varphi\in S(V_r^n)$, and $h\in O(V_r)$, let

$$\mu(\varphi)(h)=(\mu_0\otimes\delta_0)(I(\omega(h)\varphi).$$

Thus, $\mu(\varphi)$ is a smooth function on $O(V_r)$, valued in $\Theta(\pi, V_{r_0})\otimes\pi$ and such that,

$$\mu(\varphi)\big(n(c,d)m(a,h_0)h\big) = |a|^n\,\Theta(h_0)\mu(\varphi),$$

where $\Theta(h_0)$ denotes the action of h_0 in $\Theta(\pi, V_{r_0})$. Shifting to normalized induction, we obtain statement (ii).

Finally, for (iii), we can replace μ_0 by the analogous map to $\theta(\pi, V_r)\otimes\pi$ in the previous construction. $\square$

The next result shows that, every representation $\pi \in \mathrm{Irr}(G)$ occurs, once you go far enough up the tower.

Proposition 4.3. (stable range) *If $r \geq 2n$, then for every $\pi \in \mathrm{Irr}(G)$, $\Theta(\pi, V_r) \neq 0$.*

Here recall that if $\dim_F V_r$ is even, we are considering $\pi \in \mathrm{Irr}(Sp(W))$, while, if $\dim_F V_r$ is odd, we are considering *genuine* representations of $Mp(W)$, i.e., $\pi \in \mathrm{Irr}(Mp(W))^{gen}$, so that $\pi(z) = z\cdot id_\pi$.

Proof. The fixed standard basis $\mathbf{e}$ of W can be viewed as an element of W^{2n}, and the map $g \mapsto \mathbf{e}\cdot g$ defines an inclusion $Sp(W)\hookrightarrow W^{2n}$ which is equivariant for the right action of $Sp(W)$ and whose image is closed. Take $r = 2n$, so that the intertwining map I above followed by restriction yields a surjective equivariant map

$$S(V_r^n) \longrightarrow S(V_0^n)\otimes S(W^{2n}) \longrightarrow S(V_0^n)\otimes S(Sp(W)).$$

Lemma 4.4. *For any $\pi \in \mathrm{Irr}(G)$,*

$$\mathrm{Hom}_G\Big(S(V_0^n)\otimes S(Sp(W)), \pi\Big) \neq 0.$$

Proof. Note that, in the case $G = Mp(W)$, we are assuming that the central character of π and of ω_0 is $z\mapsto z$. For fixed $\xi_0 \in \pi$ and $x \in V_0^n$, consider the map, $f \in S(Sp(W))$,

$$\begin{aligned} S(V_0^n)\otimes S(Sp(W)) &\longrightarrow \pi \\ \varphi_0\otimes f &\mapsto \int_G \omega_0(g)\varphi_0(x)\, f(g)\, \pi(g^{-1})\xi_0\, dg. \end{aligned}$$

Since the function $\omega_0(g)\varphi_0(x)\, f(g)$ is a smooth function on $Sp(W)$ of compact support, this map is well defined and is equivariant for the action of G. Since the function $\omega_0(g)\varphi_0(x)\,\pi(g^{-1})\xi_0$ is nonzero and smooth on $Sp(W)$, we can choose f so that the integral is nonzero. $\square$

Thus we have produced a nonzero G equivariant map $S(V_r^n) \to \pi$, as required. $\square$

The analogous results hold for the theta correspondence from a fixed $O(V_r)$ as we vary W_n; the proofs are the same, except that the intertwining operator I is not needed. Let

$$G_n = \begin{cases} Sp(W_n) & \text{if } \dim_F V_r \text{ is even,} \\ Mp(W_n) & \text{if } \dim_F V_r \text{ is odd.} \end{cases}$$

Proposition 4.5. *(i) Given* $\pi' \in \mathrm{Irr}(O(V_r))$, *if* $\Theta(\pi', W_{n_0}) \neq 0$, *then* $\Theta(\pi', W_n) \neq 0$ *for all* $n \geq n_0$.
(ii) Let $s = n - n_0$. *If* $\Theta(\pi', W_{n_0}) \neq 0$, *then there is a nonzero* $G_n \times O(V_r)$ *equivariant map*

$$S(V_r^n) \longrightarrow I_{P_s}^{G_n}\Big(\chi_V^\psi\,|\det|^{\frac{m}{2}-n_0-\frac{s+1}{2}} \otimes \Theta(\pi', W_{n_0})\Big) \otimes \pi.$$

(iii) By the Howe duality principle, $\theta(\pi', W_n)$ *is a subquotient of* $I_{P_s}^{G_n}\big(\chi_V^\psi\,|\det|^{\frac{m}{2}-n_0-\frac{s+1}{2}} \otimes \theta(\pi', W_{n_0})\big)$.
(iv) If $n \geq \dim_F V_r$, *then* $\Theta(\pi', W_n) \neq 0$ *for all* $\pi' \in \mathrm{Irr}(O(V_r))$.

Proof. We only check (ii). Consider

$$\lambda : S(V_r^n) \simeq S(V_r^{n_0}) \otimes S(V_r^s) \overset{\lambda_0\otimes\delta_0}{\longrightarrow} \pi',$$

where $\lambda_0 : S(V_r^s) \to \pi'$ is a nonzero $O(V_r)$ intertwining map, and δ_0 is the delta function at $0 \in V_r^s$. For $a \in GL_s(F)$ and $g_0 \in Sp(W_{n_0})$ let

$$m(a, g_0) = \begin{pmatrix} a & & \\ & g_0 & \\ & & {}^t a^{-1} \end{pmatrix} \in M_s$$

Note that $\delta_{P_s}(m(a, g_0)) = |\det(a)|^{2n_0+s+1}$, and that,

$$\lambda(\omega_{\psi,V}((m(a,g_0),z)) = z^m \cdot \chi_V^\psi(\det(a))\,|\det(a)|^{\frac{m}{2}}\,\lambda_0 \otimes \delta_0\big((\omega_{\psi,V}(g_0)\varphi\big).$$

The preceeding results allow us to make the following:

Definition 4.6. *(i) Given*

$$\pi \in \mathrm{Irr}(G) = \begin{cases} \mathrm{Irr}(Sp(W_n)) & \textit{if } \dim_F V_r \textit{ is even,} \\ \mathrm{Irr}(Mp(W_n))^{gen} & \textit{if } \dim_F V_r \textit{ is odd,} \end{cases}$$

the **first occurrence index** $r(\pi)$ *of* π *for the Witt tower* $\{V_r\}$ *is the smallest* $r \geq 0$ *such that* $\Theta(\pi, V_r) \neq 0$ *. In particular,* $r(\pi) \leq 2n = \dim_F W_n$ *.*
(ii) Similarly, if $\pi' \in \mathrm{Irr}(O(V_r))$ *, the first occurrence index* $n(\pi')$ *is the smallest value of* $n \geq 0$ *such that* $\Theta(\pi', W_n) \neq 0$ *. Then* $n(\pi') \leq \dim_F V_r$ *.*

On of the most basic problems in the local theory is to undertand the first occurrence index. We will describe some curious results about such indices in Chapter V, below.

III.5. Where does the trivial representation go?

We begin with the simplest situation and consider the trivial representation $1\!\!1_V \in \mathrm{Irr}(O(V))$ of $O(V)$ for quadratic space V . It turns out that, formally, we should view $1\!\!1_V$ as being the Weil representation of $O(V)$ for the dual pair $(O(V), Sp(W_0))$! Thus, the first occurrence index of the trivial representation is $n(1\!\!1_V) = 0$. The problem, then, is to identify $\Theta(1\!\!1_V, W_n)$ for any n . Since the distribution δ_0 on $S(V^n)$ is invariant under $O(V)$, we obtain a nonzero $O(V) \times G_n$ equivariant map

$$\lambda_V : S(V^n) \longrightarrow 1\!\!1_V \otimes I_{P_n}^{G_n}(\chi_V^{\psi} |\det|^{\frac{m}{2} - \frac{n+1}{2}}),$$

where $m = \dim_F V$ and

$$\lambda_V(\varphi)(g) = \big(\omega_{\psi,V}(g)\varphi\big)(0).$$

Similarly, consider the trivial representation $1\!\!1_W \in \mathrm{Irr}(Sp(W))$ for some symplectic space W . For the moment, we will only consider the case of the Witt tower of split orthogonal spaces $\{V_{r,r} \mid r \geq 0\}$, and, again, we formally view $1\!\!1_W$ as being the Weil representation of $Sp(W)$ for the dual pair $(Sp(W), O(V_{0,0}))$ at the bottom of the tower. The Weil representation of the dual pair $(Sp(W), O(V_{r,r}))$ can be realized in the space $S(W^r)$ with $Sp(W)$ acting linearly by right multiplication. Again using the delta distribution at $0 \in W^r$, we obtain a nonzero $Sp(W) \times O(V_{r,r})$ equivariant map

$$\lambda_W : S(W^r) \longrightarrow 1\!\!1_W \otimes I_{Q_r}^{O(V_{r,r})}(|\det|^{n - \frac{r-1}{2}}).$$

The following fundamental result is due to Rallis, [60].

Theorem 5.1. *(i) Let* $\Theta(\mathbb{1}_V, W_n) = S(V^n)_{O(V)}$ *be the maximal quotient of* $S(V^n)$ *on which* $O(V)$ *acts trivially. Then* λ_V *induces an injection:*

$$\begin{array}{ccc} S(V^n) & & \\ \downarrow & \overset{\lambda_V}{\searrow} & \\ \Theta(\mathbb{1}_V, W_n) & \hookrightarrow & I_{P_n}^{G_n}(\chi_V^{\psi}|\det|^{\frac{m}{2}-\frac{n+1}{2}}). \end{array}$$

(ii) Let $\Theta(\mathbb{1}_W, V_{r,r}) = S(W^r)_{Sp(W)}$ *be the maximal quotient of* $S(W^r)$ *on which* $Sp(W)$ *acts trivially. Then* λ_W *induces an injection*

$$\begin{array}{ccc} S(W^r) & & \\ \downarrow & \overset{\lambda_W}{\searrow} & \\ \Theta(\mathbb{1}_W, V_{r,r}) & \hookrightarrow & I_{Q_r}^{O(V_{r,r})}(|\det|^{n-\frac{r-1}{2}}). \end{array}$$

We will not give the proof here. The analogous result for any irreducible dual pair of type I is proved in [46], Chapter 3. The occurrence of $\mathbb{1}_W \in \mathrm{Irr}(Sp(W))$ for an arbitrary Witt tower $\{V_r \mid r \geq 0\}$ of even dimensional orthogonal groups will be discussed later.

Definition 5.2. *Let*

$$R_n(V) := \lambda_V(S(V^n)) \simeq \Theta(\mathbb{1}_V, W_n),$$

and let

$$R_r(W) := \lambda_W(S(W^r)) \simeq \Theta(\mathbb{1}_W, V_r).$$

These representations will play an important role in Chapter IV and elsewhere. For example, they account for all of the reducibility of the degenerate principal series representations in which they occur, see below.

Corollary 5.3. *(i) The Bernstein-Zelevinsky data of any constituent of* $R_n(V)$ *is*

$$\left[\chi_V|\ |^{\frac{m}{2}-n}, \chi_V|\ |^{\frac{m}{2}-n+1}, \dots, \chi_V|\ |^{\frac{m}{2}-1}\right].$$

(ii) The Bernstein-Zelevinsky data of any constituent of $R_r(W)$ *is*

$$[|\ |^{n-r+1}, |\ |^{n-r+2}, \dots, |\ |^{n}].$$

Proof. Consider $R_n(V)$, and, for simplicity, assume that $m = \dim_F V$ is even. The point is that the degenerate principal series representation $I_{P_n}^{G_n}(\chi_V|\det|^{\frac{m}{2}-\frac{n+1}{2}})$

containing $R_n(V)$ is a subspace of a certain representation induced from a character of the Borel subgroup $B \subset P_n$. More precisely, if $f \in I_{P_n}^{G_n}(\chi_V|\det|^{\frac{m}{2}-\frac{n+1}{2}})$, then for $a = \mathrm{diag}(a_1,\ldots,a_n)$, we have

$$f(m(a)g) = \chi_V(a_1)|a_1|^{\frac{m}{2}} \ldots \chi_V(a_n)|a_n|^{\frac{m}{2}} f(g),$$

since the induction is normalized. Since f is also left invariant under the unipotent radical of B, and since the modulus character of B is $(|\ |^n, |\ |^{n-1},\ldots,|\ |)$, we see that the function f lies in the induced representation

$$I_B^{G_n}\big(\chi_V|\ |^{\frac{m}{2}-n} \otimes \chi_V|\ |^{\frac{m}{2}-n+1} \otimes \ldots \otimes \chi_V|\ |^{\frac{m}{2}-1}\big).$$

Of course, these characters are to be thought of as supercuspidal representations of $GL_1(F)$. The collection is only unique up to the action of the Weyl group, i.e., up to permutation and componentwise inversions. $\square$

Example 5.4. Let $V = F$ with quadratic form $q(x) = x^2$. Then, writing $W_n = W = X + Y$, we have $S(V^n) = S(X)$ and $\omega_{\psi,V} = \omega_\psi$ is just the Weil representation of $Mp(W)$ determined by ψ. The space $S(V^n)_{O(V)}$ is just the space of even functions in $S(V^n)$. This representation of $Mp(W)$ is irreducible, and Theorem 5.1 identifies it as a subrepresentation of the degenerate principal series:

$$\omega_\psi^+ \hookrightarrow I_{P_n}^{Mp(W)}(\chi_V^\psi|\det|^{-\frac{n}{2}}),$$

where

$$\chi_V^\psi((m(a),z)) = z \cdot \gamma(\det(a),\psi)^{-1}.$$

Later on, we will explain where the odd part ω_ψ^- of the basic Weil representation occurs as a constituent of an induced representation of $Mp(W)$.

Example 5.5. Let $W = F^2$ be the two dimensional symplectic space over F. For $r \geq 3$, the representation $R_r(W) = \Theta(\mathbb{1}_W, V_r)$ of $O(V_{r,r})$ is irreducible and is known as the Kepler representation [???]. It is an interesting 'small' representation of $O(V_{r,r})$, which is, in some sense, the analogue of the even Weil representation ω_ψ^+.

III.6. Supercuspidals We now consider the case of a supercuspidal representation $\pi \in \mathrm{Irr}_{sc}(G)$ or $\tau \in \mathrm{Irr}_{sc}(O(V_r))$. These results can be found in [32] and [46], Chapter 3.

Theorem 6.1. *Fix a Witt tower $\{V_r \mid r \geq 0\}$ of quadratic spaces. Let $\pi \in \mathrm{Irr}_{sc}(G)$ be a supercuspidal representation, and let $r_0 = r(\pi)$ be the first occurrence index of π for this tower.*

(i) For all $r \geq r_0$ *,* $\Theta(\pi, V_r)$ *is an irreducible admissible representation of* $O(V_r)$ *. In particular,* $\Theta(\pi, V_r) = \theta(\pi, V_r)$ *.*

(ii) The representation $\tau = \theta(\pi, V_{r_0})$ *is a supercuspidal representation of* $O(V_{r_0})$ *.*

(iii) For $t = r - r_0 > 0$ *, the representation* $\theta(\pi, V_r)$ *is a subrepresentation of*

$$I_{Q_t}^{O(V_r)}(\,|\det|^{n-\frac{m-t-1}{2}} \otimes \tau\,).$$

(iv) In particular, the Bernstein-Zelevinsky data of $\theta(\pi, V_r)$ *is*

$$\begin{aligned}\left[\ \theta(\pi, V_r)\ \right] &= \left[\ |\ |^{n-\frac{m_0}{2}}, |\ |^{n-\frac{m_0}{2}-1}, \ldots, |\ |^{n-\frac{m}{2}+1}; \tau\ \right] \\ &= \left[\ |\ |^{\frac{m}{2}-n-1}, |\ |^{\frac{m}{2}-n-2}, \ldots, |\ |^{\frac{m_0}{2}-n}; \tau\ \right],\end{aligned}$$

and this representation is never supercuspidal for $r > r_0$ *. Here* $m = \dim_F V_r$ *and* $m_0 = \dim_F V_{r_0}$ *.*

Theorem 6.2. *Let* $\tau \in \mathrm{Irr}_{sc}(O(V))$ *be a supercuspidal representation, and let* $n_0 = n(\tau)$ *be the first occurrence index of* τ *for the symplectic Witt tower* $\{W_n \mid n \geq 0\}$ *. Let*

$$G_n = \begin{cases} Sp(W_n) & \text{if } m = \dim_F V \text{ is even,} \\ Mp(W_n) & \text{if } m \text{ is odd.}\end{cases}$$

(i) For all $n \geq n_0$ *,* $\Theta(\tau, W_n)$ *is an irreducible admissible representation of* G_n *. In particular,* $\Theta(\tau, W_n) = \theta(\tau, W_n)$ *.*

(ii) The representation $\pi = \theta(\tau, W_{n_0})$ *is a supercuspidal representation of* G_{r_0} *.*

(iii) For $s = n - n_0 > 0$ *, the representation* $\theta(\tau, W_n)$ *is a subrepresentation of*

$$I_{P_s}^{G_n}(\,\chi_V^{\psi}\,|\det|^{\frac{m}{2}-n_0-\frac{s+1}{2}} \otimes \pi\,).$$

(iv) In particular, the Bernstein-Zelevinsky data of $\theta(\tau, W_n)$ *is*

$$\begin{aligned}\left[\ \theta(\pi', W_n)\right] &= \left[\ \chi_V|\ |^{\frac{m}{2}-n_0-1}, \chi_V|\ |^{\frac{m}{2}-n_0-2}, \ldots, \chi_V|\ |^{\frac{m}{2}-n}; \pi\ \right] \\ &= \left[\ \chi_V|\ |^{\frac{m}{2}-n}, \chi_V|\ |^{\frac{m}{2}-n+1}, \ldots, \chi_V|\ |^{\frac{m}{2}-n_0-1}; \pi\ \right],\end{aligned}$$

and this representation is never supercuspidal for $n > n_0$ *.*

We will sketch the proof of these Theorems in Chapter IV.

Remark 6.3. The Weil representation of a dual pair $(GL_m(F), GL_n(F))$ of type II, can be taken to be the action on $S(M_{m,n}(F))$ given by

$$\omega_\psi(a, b)\varphi(x) = |\det(a)|^{-\frac{n}{2}}|\det(b)|^{\frac{m}{2}}\varphi(a^{-1}xb),$$

where $a \in GL_m(F)$ and $b \in GL_n(F)$. It will be slightly more convenient to remove the determinant characters and to consider the theta correspondence defined using the natural linear action:

$$\omega_\psi^0(a,b)\varphi(x) = \varphi(a^{-1}xb).$$

We fix m and vary n.

Proposition 6.4. *If* $\pi \in \mathrm{Irr}_{sc}(GL_m(F))$, *then*
(i) $\Theta^0(\pi, GL_n(F)) = 0$ *for* $n < m$.
(ii) For $n = m$, $\Theta^0(\pi, GL_n(F)) = \pi$.
(iii) For $n > m$, $\Theta^0(\pi, GL_n(F))$ *has no supercuspidal component.*

Here the superscript ^{0} indicates that we are using the natural linear action in our definition of the correspondence.

For more information about the theta correspondence for type II dual pairs, see [46], Chapter 3.

III.7. Compatibility with Bernstein-Zelevinsky data. Next we turn to the case of an arbitrary irreducible admissible representation $\pi \in \mathrm{Irr}(G_n)$, or $\tau \in \mathrm{Irr}(O(V_r))$.

First, fix a Witt tower of quadratic spaces $\{V_r \mid r \geq 0\}$, and let G_n be either $Sp(W_n)$ or $Mp(W_n)$, depending on the parity of $\dim_F(V_r)$, determined by the tower.

For $\pi \in \mathrm{Irr}(G_n)$, let

$$[\pi] = [\,\sigma_1, \ldots, \sigma_j; \pi_0\,]$$

be its Bernstein-Zelevinsky data. Recall that $\mathbf{s} = (s_1, \ldots, s_j)$ and that $\sigma_i \in \mathrm{Irr}_{sc}(GL_{s_i}(F))$ and $\pi_0 \in \mathrm{Irr}_{sc}(G_{n-|\mathbf{s}|})$. Let $r(\pi_0)$ be the first occurrence index of π_0 for the Witt tower $\{V_r\}$, and let $\tau_0 = \theta(\pi_0, V_{r_0})$ be the first nonvanishing theta lift for this tower. Note that $\tau_0 \in \mathrm{Irr}_{sc}(O(V_{r_0}))$ is supercuspidal. Also let $r_0 = r(\pi_0) + |\mathbf{s}|$. The following result describes the Bernstein-Zelevinsky data of any irreducible quotient τ of $\Theta(\pi, V_r)$. If $r \geq r_0 = r(\pi_0) + |\mathbf{s}|$, then this data is obtained by replacing π_0 by τ_0, transfering the σ_i 's the GL blocks of the same size and filling any remaining space with a certain fixed sequence of characters of GL_1 's. We will often refer to this sequence of characters as the 'tail'. If $r < r_0 = r(\pi_0) + |\mathbf{s}|$, it may still be possible for $\Theta(\pi, V_r)$ to be nonzero, but this can only occur if a certain sequence of characters occurs within the σ_i 's, and the Bernstein-Zelvinsky data of an irreducible quotient τ of $\Theta(\pi, V_r)$ is then obtained by cancelling this 'tail' and replacing π_0 by τ_0. Of course, this cancellation just reverses a 'filling' process.

Theorem 7.1. *(i) Suppose that $r \geq r_0 = r_0(\pi_0) + |\mathbf{s}|$ and that τ is a nonzero irreducible quotient of $\Theta(\pi, V_r)$. Then the Bernstein-Zelevinsky data of τ is given by*

$$[\,\tau\,] = [\ |\ |^{\frac{m}{2}-n-1}, |\ |^{\frac{m}{2}-n-2} \dots, |\ |^{\frac{m_0}{2}-n}, \chi_V^{-1}\sigma_1, \dots, \chi_V^{-1}\sigma_j, \tau_0\ ],$$

where $m_0 = \dim_F V_{r_0}$.
(ii) Suppose that $r < r_0$, and that τ is a nonzero irreducible quotient of $\Theta(\pi, V_r)$. Then, there exists a sequence $i_1, \dots, i_t$, with $t = r_0 - r$, such that

$$\sigma_{i_1} = \chi_V|\ |^{\frac{m}{2}-n}, \sigma_{i_2} = \chi_V|\ |^{\frac{m}{2}-n+1}, \dots, \sigma_{i_t} = \chi_V|\ |^{\frac{m_0}{2}-n-1},$$

and the Bernstein-Zelevinsky data of τ is given by

$$[\,\tau\,] = [\ \chi_V^{-1}\sigma_1, \dots, \widehat{\chi_V^{-1}\sigma_{i_1}}, \dots, \widehat{\chi_V^{-1}\sigma_{i_t}}, \dots, \chi_V^{-1}\sigma_j, \tau_0\ ].$$

Similarly, if $\tau \in \mathrm{Irr}(O(V_r))$ is an irreducible admissible representation with Bernstein-Zelevinsky data

$$[\tau] = [\sigma_1', \dots, \sigma_j'; \tau_0],$$

recall that $\mathbf{t} = (t_1, \dots, t_j)$ and that $\sigma_i' \in \mathrm{Irr}_{sc}(GL_{t_i}(F))$ and $\tau_0 \in \mathrm{Irr}_{sc}(O(V_{r-|\mathbf{t}|}))$. Let $n(\tau_0)$ be the first occurrence index of τ_0 for the symplectic Witt tower, and let $\pi_0 = \theta(\tau_0, W_{n(\pi_0')})$. Note that $\pi_0 \in \mathrm{Irr}_{sc}(G_{n(\tau_0)})$.

Theorem 7.2. *Let $n_0 = n(\tau_0) + |\mathbf{s}|$.*
(i) Suppose that $n \geq n_0$ and that π is a nonzero irreducible quotient of $\Theta(\tau, W_n)$. Then the Bernstein-Zelevinsky data of π is given by

$$[\,\pi\,] = [\ \chi_V|\ |^{\frac{m}{2}-n}, \chi_V|\ |^{\frac{m}{2}-n+1}, \dots, \chi_V|\ |^{\frac{m}{2}-n_0-1}, \chi_V\sigma_1, \dots, \chi_V\sigma_j, \pi_0\ ].$$

(ii) Suppose that $n < n_0$, and that π is a nonzero irreducible quotient of $\Theta(\tau, W_n)$. Then, there exists a sequence $i_1, \dots, i_t$, with $t = n_0 - n$, such that

$$\sigma_{i_1}' = |\ |^{\frac{m}{2}-n-1}, \sigma_{i_2}' = |\ |^{\frac{m}{2}-n-2}, \dots, \sigma_{i_t}' = |\ |^{\frac{m}{2}-n_0},$$

and the Bernstein-Zelevinsky data of π is given by

$$[\,\pi\,] = [\ \chi_V\sigma_1', \dots, \widehat{\chi_V\sigma_{i_1}'}, \dots, \widehat{\chi_V\sigma_{i_t}'}, \dots, \chi_V\sigma_j', \pi_0\ ].$$

Note that the 'tails' to be added in one direction are just those which are cancelled in the other. Theorem 7.1 and Theorem 7.2 here are Corollary 2.6 and Theorem 2.5 of [32]. The case of an arbitrary irreducible dual pair of type I is considered in [46], Chapter 3.

These Theorems contain quite a lot of useful information about the correspondence. We will discuss examples in Chapter VI, below.

III.8. Jacquet functors of the Weil representation. In this section, we will describe the Jacquet functors with respect to maximal parabolics applied to the Weil representation for the dual pair $(Sp(W_n), O(V_r))$.

We will use the following notation. As before,

$$G_n = \begin{cases} Sp(W_n) & \text{if } m = \dim_F V_r \text{ is even,} \\ Mp(W_n) & \text{if } m \text{ is odd.} \end{cases}$$

Let $(\omega_{n,r}, S)$ be the Weil representation of $G_n \times O(V_r)$. For the moment, we do not specify the model. Let P_i, with $1 \le i \le n$, be the maximal parabolic subgroups of G_n, and let Q_j, with $1 \le j \le r$, be the maximal parabolic subgroups of $O(V_r)$, as defined in section III.2 above. Let

$$R_{P_i} : \mathcal{A}(G_n) \longrightarrow \mathcal{A}(GL_i(F)) \times \mathcal{A}(G_{n-i}),$$

and

$$R_{Q_j} : \mathcal{A}(O(V_r)) \longrightarrow \mathcal{A}(GL_j(F)) \times \mathcal{A}(O(V_{r-j}))$$

be the normalized Jacquet functors –cf, III.3, where these functors are denoted by $R_{P_i}^{G_n}$ and $R_{Q_j}^{O(V_r)}$, respectively. We want to compute

$$R_{P_i}(\omega_{n,r}) \in \mathcal{A}(GL_i(F)) \times \mathcal{A}(G_{n-i}) \times \mathcal{A}(O(V_r)),$$

and

$$R_{Q_j}(\omega_{n,r}) \in \mathcal{A}(G_n) \times \mathcal{A}(GL_j(F)) \times \mathcal{A}(O(V_{r-j})).$$

The answer is quite nice in the sense that it reveals something about the inductive properties of the Weil representation. For example, the representation $R_{P_i}(\omega_{n,r})$ has a natural filtration, whose successive quotients, roughly speaking, are induced from Weil representations of (reducible) dual pairs of the form $(GL_k \times G_{n-i}, GL_k \times O(V_{r-k}))$. This inductive structure is the basis for the proof of Theorems 7.1 and 7.2. The following Theorem gives the precise result.

Theorem 8.1. *Let $J_i = R_{P_i}(\omega_{n,r})$ be the Jacquet module with respect to the maximal parabolic P_i of G_n of the Weil representation $\omega_{n,r}$ of $G_n \times O(V_r)$. There is a $GL_i(F)) \times G_{n-i} \times O(V_r)$ invariant filtration:*

$$J_i = J_i^{(0)} \supset \cdots \supset J_i^{(s)} \supset J_i^{(s+1)} = 0,$$

where $s = \min(i,r)$. The successive quotients

$$J_{ik} := J_i^{(k)}/J_i^{(k+1)},$$

for $0 \leq k \leq s$, can be described as follows. Let Q_k be the maximal parabolic subgroup of $O(V_r)$ with Levi factor isomorphic to $GL_k(F) \times O(V_{r-k})$. Let P'_{ik} be the maximal parabolic subgroup of M_i whose Levi factor has image in $Sp(W_n)$ isomorphic to $GL_k(F) \times GL_{i-k}(F) \times Sp(W_{n-i})$. Let $\omega_{n-i,r-k}$ denote the Weil representation (associated to the fixed additive character ψ) of $G_{n-i} \times O(V_{r-k})$. Let σ_k be the representation of $GL_k(F) \times GL_k(F)$ on $S(GL_k(F))$ given by

$$\sigma_k(h_1,h_2)f(x) = f({}^t h_1 x h_2).$$

This may be viewed as a slight twist of the Weil representation for the type II dual pair $(GL_k(F), GL_k(F))$, cf. Remark 6.3. Let α_{ik} be the character of $P'_{ik} \times Q_k$ whose restriction are:

$$\alpha_{ik} = \begin{cases} \chi_V^{\psi}|\ |^{\frac{m}{2}-\frac{k+1}{2}} & \text{on } \widetilde{GL}_k(F) \subset P'_{ik} \\ \chi_V^{\psi}|\ |^{\frac{m}{2}-n+\frac{i-k-1}{2}} & \text{on } \widetilde{GL}_{i-k}(F) \subset P'_{ik} \\ |\ |^{\frac{m}{2}-\frac{k+1}{2}} & \text{on } GL_k(F) \subset Q_k. \end{cases}$$

Extend the representation $\alpha_{ik} \cdot \sigma_k \otimes \omega_{n-i,r-k}$ of the Levi factor of $P'_{ik} \times Q_k$ to a representation of $P'_{ik} \times Q_k$, trivial on the unipotent radical. Then

$$J_{ik} \simeq I_{P'_{ik} \times Q_k}^{M_i \times O(V_r)}\Big(\alpha_{ik} \cdot \sigma_k \otimes \omega_{n-i,r-k}\Big).$$

Note that, except for the character χ_V^{ψ}, the two $GL_k(F)$'s have the same twist.

The next Theorem gives the analogous description of $R_{Q_i}(\omega_{n,r})$.

Theorem 8.2. *Let $I_j = R_{Q_j}(\omega_{n,r})$ be the Jacquet module with respect to the maximal parabolic Q_j of $O(V_r)$ of the Weil representation $\omega_{n,r}$ of $G_n \times O(V_r)$. There is a $G_n \times GL_j(F)) \times O(V_{r-j})$ invariant filtration:*

$$I_j = I_j^{(0)} \supset \cdots \supset I_j^{(t)} \supset I_j^{(t+1)} = 0,$$

where $t = \min(j,n)$. *The successive quotients*

$$I_{jk} := I_j^{(k)}/I_j^{(k+1)},$$

for $0 \le k \le t$ *, can be described as follows. Let* P_k *be the maximal parabolic subgroup of* G_n *with Levi factor having image in* $Sp(W_n)$ *isomorphic to* $GL_k(F)\times G_{n-k}$ *. Let* Q'_{jk} *be the maximal parabolic subgroup of* $M_j \subset Q_j \subset O(V_r)$ *whose Levi factor is isomorphic to* $GL_k(F)\times GL_{j-k}(F)\times O(V_{r-j})$ *. Let* $\omega_{n-k,r-j}$ *denote the Weil representation (associated to the fixed additive character* ψ *) of* $G_{n-k}\times O(V_{r-j})$ *. Let* σ_k *be as in Theorem 8.1. Let* β_{jk} *be the character of* $P_k \times Q'_{jk}$ *whose restriction are:*

$$\beta_{jk} = \begin{cases} |\ |^{\frac{m}{2}-j+\frac{k-1}{2}} & \text{on } GL_k(F) \subset Q'_{jk} \\ |\ |^{\frac{m}{2}-n-\frac{j-k+1}{2}} & \text{on } GL_{j-k}(F) \subset Q'_{jk} \\ \chi_V^{\psi}|\ |^{\frac{m}{2}-j+\frac{k-1}{2}} & \text{on } \widetilde{GL}_k(F) \subset P_k. \end{cases}$$

Extend the representation $\beta_{jk}\cdot\sigma_k\otimes\omega_{n-k,r-j}$ *of the Levi factor of* $P_k\times Q'_{jk}$ *to a representation of* $P_k\times Q'_{jk}$ *, trivial on the unipotent radical. Then*

$$I_{jk} \simeq I_{P_k\times Q'_{jk}}^{G_n\times M_j}\Big(\beta_{jk}\cdot\sigma_k\otimes\omega_{n-k,r-j}\Big).$$

Theorem 8.1 and Theorem 8.2 are Theorem 2.9 and Theorem 2.8 in [32]. The analogue for any irreducible dual pair of type I is proved in [46], Chapter 3, Théorème 5, p .70. We will not give the proof here.

Note that the top quotient is particularly simple in both cases. For example, J_{i0} is the representation of $GL_i(F)\times G_{n-i}\times O(V_r)$

$$J_{i0} = \chi_V^{\psi}|\ |^{\frac{m}{2}-n+\frac{i-1}{2}}\otimes\omega_{n-i,r}.$$

Similarly, I_{j0} is the representation of $G_n\times O(V_{r-j})\times GL_j(F)$

$$I_{j0} = \omega_{n,r-j}\otimes|\ |^{\frac{m}{2}-n-\frac{j+1}{2}}.$$

These facts will be useful later.

CHAPTER IV

In this Chapter we will sketch the proofs of some of the main results of Chapter III. In order to do this, we must first introduce seesaw dual pairs and degenerate principal series. These objects will also play an important role in the global theory.

IV.1. Seesaw dual pairs. Before giving some ideas about the proofs of the Theorems of sections III.6, III.7, we need to introduce a structure, the seesaw dual pair, which will play a extensive role in both the local and global theta correspondence. The basic idea is the following. Suppose that (G, G') is a dual pair in a symplectic group $Sp(\mathbb{W})$, and let $H \subset G$ be a reductive subgroup. Let $H' = \mathrm{Cent}_{Sp(\mathbb{W})}(H)$ be the centralizer of H in $Sp(\mathbb{W})$, and note that $H' \supset G$. If the group H' is again reductive, and if (H, H') also is a dual pair in $Sp(\mathbb{W})$, we will call the pair of dual pairs $(G \supset H, H' \supset G')$ a **seesaw dual pair**. The terminology is motivated by the following picture:

$$\begin{array}{ccc} G & & H' \\ \uparrow & \times & \uparrow \\ H & & G', \end{array}$$

in which the sloping arrows, connecting groups in a dual pair, resemble the two positions of a common playground apparatus.

The examples which we need at the moment are constructed by direct sums.

Example 1.1. First suppose that we are give a symplectic space W of dimension $2n$ and an orthogonal space V over F, so that $(G, G') = (O(V), Sp(W))$ form a dual pair in $Sp(\mathbb{W})$. Given an orthogonal direct sum decomposition of the orthogonal vector space: $V = V_1 + V_2$, let $H = O(V_1) \times O(V_2) \subset O(V)$. Since

$$\mathbb{W} = V \otimes W = V_1 \otimes W + V_2 \otimes W = \mathbb{W}_1 + \mathbb{W}_2,$$

it is clear that the centralizer of H in $Sp(\mathbb{W})$ is two copies of $Sp(W)$, one acting on the second factor of $V_1 \otimes W$ and the other acting on the second factor of $V_2 \otimes W$. Thus, we obtain a seesaw dual pair

$$\begin{array}{ccc} O(V) & & Sp(W) \times Sp(W) \\ \uparrow & \times & \uparrow \\ O(V_1) \times O(V_2) & & Sp(W). \end{array}$$

Example 1.2. Similarly, if $W = W_1 + W_2$ is a direct sum decomposition of the symplectic vector space, where W_1 and W_2 nondegenerate and perpendicular to each other for the symplectic form, we obtain a seesaw dual pair

$$\begin{array}{ccc} O(V)\times O(V) & & Sp(W) \\ \uparrow & \times & \uparrow \\ O(V) & & Sp(W_1)\times Sp(W_2). \end{array}$$

Here the relevant decomposition of $\mathbb{W}$ is

$$\mathbb{W} = \mathbb{W}_1 + \mathbb{W}_2 = V\otimes W_1 + V\otimes W_2.$$

There are many other interesting examples of seesaw dual pairs. For further discussion of some of these, see [31].

We now give an example of the way in which the seesaw pair construction can be put into play.

As explained in section I.1, there is a homomorphism

$$\tilde{j}: Mp(\mathbb{W}_1)\times Mp(\mathbb{W}_2) \longrightarrow Mp(\mathbb{W}),$$

and a natural isomorphism

$$(\tilde{j}^*(\omega_\psi), S) = (\omega_\psi^1\otimes\omega_\psi^2, S_1\otimes S_2)$$

for the Weil representations.

Consider Example 1.2, and, for convenience, write G_1 for G_{n_1}, G_2 for G_{n_2}, and G for G_n, where $n = n_1+n_2$. Let $\tau\in\mathrm{Irr}(O(V))$ be a fixed irreducible representation, and suppose that $\pi_1\in\mathrm{Irr}(G_1)$ is a quotient of $\Theta(\tau, W_1)$ and $\pi_2\in\mathrm{Irr}(G_2)$ is a quotient of $\Theta(\tau,W_2)$. Thus we have surjective equivariant maps:

$$S_1\longrightarrow \tau\otimes\pi_1 \qquad \text{and} \qquad S_2\longrightarrow\tau\otimes\pi_2.$$

These can be combined to give a surjective map

$$S = S_1\otimes S_2\longrightarrow \tau\otimes\pi_1\otimes\tau\otimes\pi_2,$$

which is equivariant for $O(V)\times G_1\times O(V)\times G_2$. This group is, up to a rearrangement of factors, $H\times H'$ for the (H,H') dual pair in our seesaw. If we restrict the action of $O(V)\times O(V)$ on S to the diagonal $O(V)$, i.e., to the group G' in the seesaw, then the larger group $G\supset G_1\times G_2$ acts on S, commuting with the action of the diagonal $O(V)$. Recall that every $\tau\in\mathrm{Irr}(O(V))$ is isomorphic to its contragradient, $\tau \simeq \tau^\vee$, [46], p.91. Thus, after rearranging, we obtain a surjective map

$$S \longrightarrow \pi_1 \otimes \pi_2 \otimes \tau \otimes \tau^\vee \longrightarrow \pi_1 \otimes \pi_2 \otimes \mathbb{1}_V,$$

equivariant for $G_1 \times G_2 \times O(V)$. This map factors through the maximal quotient $R_n(V)$ of S on which $O(V)$ acts trivially. Recall that $R_n(V)$ is discussed in section III.5. Thus we obtain a quotient

$$R_n(V) \longrightarrow \pi_1 \otimes \pi_2,$$

equivariant for the subgroup $G_1 \times G_2$ of G. Note that, in fact,

Lemma 1.3.

$$\mathrm{Hom}_{G_1 \times G_2 \times O(V)}(S, \pi_1 \otimes \pi_2 \otimes \mathbb{1}_V) \simeq \mathrm{Hom}_{G_1 \times G_2}(R_n(V), \pi_1 \otimes \pi_2).$$

The role of the seesaw can be pictured in the following way.

$$\begin{array}{ccccc} \tau \otimes \tau & O(V) \times O(V) & & Sp(W) & R_n(V) \\ \downarrow & \uparrow & \times & \uparrow & \downarrow \\ \mathbb{1}_V & O(V) & & Sp(W_1) \times Sp(W_2) & \pi_1 \otimes \pi_2. \end{array}$$

We have proved the first part of:

Lemma 1.4. *(i) If* $\tau \in \mathrm{Irr}(O(V))$ *and if* π_i, *for* $i = 1, 2$, *is a nonzero quotient of* $\Theta(\tau, W_i)$, *then*

$$\mathrm{Hom}_{G_1 \times G_2}(R_n(V), \pi_1 \otimes \pi_2) \neq 0.$$

(ii) If $\pi_1 \in \mathrm{Irr}(G_1)$ *and* $\pi_2 \in \mathrm{Irr}(G_2)$, *and if*

$$\mathrm{Hom}_{G_1 \times G_2}(R_n(V), \pi_1 \otimes \pi_2) \neq 0,$$

then there is a nonzero bilinear pairing

$$\Theta(\pi_1, V) \times \Theta(\pi_2, V) \longrightarrow \mathbb{C},$$

invariant for the diagonal action of $O(V)$.
(iii) Suppose that $\pi \in \mathrm{Irr}(G)$ *and that the representation* $\Theta(\pi, V)$ *of* $O(V)$ *has a nonzero irreducible quotient. Then*

$$\mathrm{Hom}_{G \times G}(R_{2n}(V), \pi \otimes \pi) \neq 0.$$

(iv) Suppose that $\pi \in \mathrm{Irr}(G)$ *and that*

$$\mathrm{Hom}_{G\times G}(R_{2n}(V), \pi \otimes \pi) \neq 0.$$

Then $\Theta(\pi, V) \neq 0$.

Here we have avoided the use of the Howe duality principle, so that these results are valid if the residue characteristic is 2 . In fact, imilar techniques prove that $\Theta(\pi, V)$ and $\Theta(\tau, W)$ have finite length, [46], so that these representations always will have an irreducible quotient whenever they are nonzero. This will yield a criterion for the occurrence of π in the theta correspondence with V .

Proposition 1.5.

$$\Theta(\pi, V) \neq 0 \iff \mathrm{Hom}_{G\times G}(R_{2n}(V), \pi \otimes \pi) \neq 0.$$

Proof of Lemma 1.4. Given a $G_1 \times G_2$ equivariant surjection $R_n(V) \longrightarrow \pi_1 \otimes \pi_2$, we obtain a $G_1 \times G_2 \times O(V)$ equivariant surjection $S_1 \otimes S_2 = S \longrightarrow \pi_1 \otimes \pi_2 \otimes 1\!\!1_V$. Forgetting the $O(V)$, action for a moment, and writing $\Theta_1 = \Theta(\pi_1, V)$ and $\Theta_2 = \Theta(\pi_2, V)$, we obtain:

$$\begin{array}{ccc} S_1 \otimes S_2 & \longrightarrow & \pi_1 \otimes \Theta_1 \otimes \pi_2 \otimes \Theta_2 \\ || & & \downarrow \mu \\ S & \longrightarrow & \pi_1 \otimes \pi_2 \otimes 1\!\!1_V, \end{array}$$

where the vertical map μ on the right side is $G_1 \times G_2 \times O(V)$ equivariant. Now for any $f_1 \in \Theta_1$ and $f_2 \in \Theta_2$, there is a $G_1 \times G_2$ equivariant map

$$\pi_1 \otimes \pi_2 \longrightarrow \pi_1 \otimes \Theta_1 \otimes \pi_2 \otimes \ \Theta_2 \xrightarrow{\mu} \ \pi_1 \otimes \pi_2$$

taking $\xi_1 \otimes \xi_2$ to $\mu(f_1 \otimes \xi_1 \otimes f_2 \otimes \xi_2)$. Since π_1 and π_2 are irreducible, and since the composite map is $G_1 \times G_2$ equivariant, Schur's lemma implies that

$$\mu(f_1 \otimes \xi_1 \otimes f_2 \otimes \xi_2) = B(f_1, f_2) \cdot \xi_1 \otimes \xi_2,$$

and the map

$$\Theta(\pi_1, V) \otimes \Theta(\pi_2, V) \longrightarrow \mathbb{C}, \qquad f_1 \otimes f_2 \mapsto B(f_1, f_2),$$

is the required nonzero $O(V)$ -invariant bilinear form. This proves (ii).

Now, for $\pi \in \mathrm{Irr}(G_n)$, suppose that there is a nonzero irreducible quotient $\lambda : \Theta(\pi, V) \to \tau$. Thus we have a $O(V) \times G_n$ equivariant quotient $S \to \pi \otimes \tau$. Taking two copies and contracting τ with $\tau^\vee \simeq \tau$, yields

$$S \otimes S \longrightarrow \pi \otimes \tau \otimes \pi \otimes \tau \longrightarrow \pi \otimes \pi \otimes \mathbb{1}_V,$$

equivariant for $G_n \times G_n \times O(V)$. By Lemma 3.1, this yields a nonzero element of $\mathrm{Hom}_{G_n \times G_n}(R_{2n}(V), \pi \otimes \pi)$, and (iii) is proved.

Finally, a nonzero element of $\mathrm{Hom}_{G_n \times G_n}(R_{2n}(V), \pi \otimes \pi)$ yields a nonzero quotient $\lambda : S \otimes S \to \pi \otimes \pi \otimes \mathbb{1}_V$. This map must be nonzero on some element of $S \otimes S$ of the form $f_1 \otimes f_2$. Choose an element $\xi^\vee \in \pi^\vee$ such that $(1 \otimes \xi^\vee)\big(\lambda(f_1 \otimes f_2)\big) \neq 0$. Then the map $S \to \pi$ which sends $f \in S$ to $(1 \otimes \xi^\vee)\big(\lambda(f \otimes f_2)\big) \in \pi$ is nonzero and G_n equivariant. Thus $\Theta(\pi, V) \neq 0$, as required. $\square$

Of course, the situation is fairly symmetric, and we can proceed in the much same way using the first seesaw,

$$\begin{array}{ccc} O(V) & & Sp(W) \times Sp(W) \\ \uparrow & \times & \uparrow \\ O(V_1) \times O(V_2) & & Sp(W). \end{array}$$

Here, however, there are two small differences, which turn out to fit together nicely. First of all, we would like the space V to be split, since we want to consider the representation $R_r(W)$. This means that, for quadratic spaces V_1 and V_2 in the same Witt tower, we should use $-V_2$, the space V_2 with the negative of its given quadratic form, in place of V_2 in the construction of the seesaw. Second, in the arguments above, we have used the fact that irreducible admissible representations of $O(V)$ are isomorphic to their contragradients. In general, a basic result of [46] must be used. We will only state the cases needed here. For the full statement, the reader should consult [46], pp.91 –92.

Theorem 1.6. *(i) Let V be a quadratic space over F. For any $\tau \in \mathrm{Irr}(O(V))$, $\tau \simeq \tau^\vee$.*
(ii) Let W be a symplectic space over F. Let $\delta \in GL(W)$ be such that

$$< \delta x, \delta y > = - < x, y >,$$

for all x and $y \in W$. For any representation π of $Sp(W)$, let $\pi^\delta(g) = \pi(\delta g \delta^{-1})$. The for any $\pi \in \mathrm{Irr}(Sp(W))$, $\pi^\delta \simeq \pi^\vee$.

(iii) The automorphism $g \mapsto Ad(\delta)(g) = \delta g \delta^{-1}$ of $Sp(W)$ has a unique lift to an automorphism

$$\widetilde{Ad(\delta)} : Mp(W) \longrightarrow Mp(W)$$

which restricts to the map $z \mapsto z^{-1}$ on the central $\mathbb{C}^{\times}$. For any representation π of $Mp(W)$, let $\pi^{\delta}(g) = \pi(\widetilde{Ad(\delta)}(g))$. Then, for any $\pi \in \mathrm{Irr}(Mp(W))$, such that the character of π is a locally integrable function[1]*, $\pi^{\delta} \simeq \pi^{\vee}$.*

We will refer to the involution $\pi \mapsto \pi^{\delta}$ as the MVW involution on the category $\mathcal{A}(G)$, where $G = O(V)$, $Sp(W)$ or $Mp(W)$. In the $O(V)$ case, take $\pi^{\delta} = \pi$. Note that this involution preserves composition series, so that it does not carry arbitrary representations to their contragradients. See Lemma I.4.6 for the uniqueness of the lift of $Ad(\delta)$ to $Mp(W)$. **Remark 1.7.** The analogous involution for a representation π of $GL_n(F)$ is given by $\pi^{\delta}(g) = \pi({}^t g^{-1})$. For an irreducible admissible representation $\pi \in \mathrm{Irr}(GL_n(F))$, $\pi^{\delta} \simeq \pi^{\vee}$. This fact is due to Kazhdan and Gelfand [30].

We now determine the action of the MVW involution on the Weil representation. The element $\delta \in GL(W)$ induces an automorphism

$$\delta : (w,t) \mapsto (\delta(w), -t)$$

of the Heisenberg group $H(W)$. Clearly, $\rho_{\psi} \circ \delta \simeq \rho_{\psi}^{\vee}$. It is then easily checked that

$$\omega_{\psi}^{\delta} \simeq \omega_{\psi}^{\vee}.$$

For a quadratic space V and a symplectic space W, let $\mathbb{W} = V \otimes W$, as usual. We can take $\delta_{\mathbb{W}} = 1_V \otimes \delta_W$, so that, conjugation by $\delta_{\mathbb{W}}$ preserves the groups in the dual pair $(O(V), Sp(W))$ and induces the automorphism $id \times Ad(\delta_W)$ on them. If $\dim_F V$ is odd, the automorphism $\widetilde{Ad(\delta_{\mathbb{W}})}$ of $Mp(\mathbb{W})$ has the same properties. For the moment, let us denote by $S_{V,W,\psi}$ the restriction of the Weil representation ω_{ψ} of $Mp(\mathbb{W})$ to $O(V) \times G(W)$, where $G(W) = Sp(W)$, if $\dim_F V$ is even, and $G_n = Mp(W)$, if $\dim_F V$ is odd. For $\tau \in \mathrm{Irr}(O(V))$ and $\pi \in \mathrm{Irr}(G(W))$, suppose that there is a nonzero $O(V) \times G(W)$ equivariant map

$$S_{V,W,\psi} \longrightarrow \tau \otimes \pi.$$

Applying the MVW involution, we obtain

$$S_{V,W,\psi}^{\delta} \longrightarrow \tau^{\delta} \otimes \pi^{\delta} \simeq \tau \otimes \pi^{\vee}.$$

[1] We will assume that this condition is always satisfied!

Here we could write either τ or $\tau^\vee$. But, it is easy to check that

$$S^{\delta}_{V,W,\psi} \simeq S_{V,W,-\psi} \simeq S_{-V,W,\psi},$$

where $-V$ is the space V with the negative of its original quadratic form. (Here, for the middle term, the reader who is sensitive about central characters should refer to Remark II.4.1.) Note that $O(-V) = O(V)$, canonically. In the next result, we temporarily included the quadratic space V, which determines the 'source' of the representation τ, in the notation for the theta correspondence. Thus we write $\Theta(\tau, V, W)$ for the space which has been denoted by $\Theta(\tau, W)$.

Lemma 1.8. *Suppose that $\tau \in \mathrm{Irr}(O(V))$ and that $\pi \in \mathrm{Irr}(G(W))$ is an irreducible quotient of $\Theta(\tau, V, W)$. Then, viewing $\tau \in \mathrm{Irr}(O(-V))$, $\pi^\vee$ is an irreducible quotient of $\Theta(\tau, -V, W)$.*

In fact, the same argument proves a little more. For $a \in F^\times$, let $\delta_a \in GL(W)$ satisfy $< \delta_a x, \delta_a y >= a < x, y >$, for all x and $y \in W$, i.e., let δ_a be the symplectic similitude of scale a. The automorphism $Ad(\delta_1)$ of $Sp(W)$ has a unique lift to an automorphism $\widetilde{Ad(\delta_a)}$ of $Mp(W)$, restricting to the identity map on the central $\mathbb{C}^\times$.

Proposition 1.9. *Suppose that $\tau \in \mathrm{Irr}(O(V))$ and that $\pi \in \mathrm{Irr}(G(W))$ is an irreducible quotient of $\Theta(\tau, V, W)$. Then, viewing $\tau \in \mathrm{Irr}(O(aV))$, π^{δ_a} is an irreducible quotient of $\Theta(\tau, aV, W)$. Here, aV is the space V with the original quadratic form q replaced by aq.*

In effect, this result describes how $\theta(\tau, W)$ changes as we view $\tau \in \mathrm{Irr}(O(V))$ as an element of $\mathrm{Irr}(O(aV))$, for varying a, noting that the groups $O(aV)$ all coincide with $O(V)$.

Returning to the orthogonal analogue of Lemma 1.4, the role of the seesaw can now be pictured in the following way.

$$\begin{array}{ccccc}
R_r(V) & O(V) & & G(W) \times G(W) & \pi \otimes \pi^\vee \\
\downarrow & \uparrow & \times & \uparrow & \downarrow \\
\tau_1 \otimes \tau_2 & O(V_1) \times O(-V_2) & & Sp(W) & \mathbb{1}_W.
\end{array}$$

Here, note that the space $V = V_{r,r} = V_1 + (-V_2)$ is split, with $r = \frac{1}{2}(\dim_F V_1 + \dim_F V_2)$. Also note that, if $\dim_F V_1$ and $\dim_F V_2$ are odd, the pullback of $\pi \otimes \pi^\vee$ to the diagonal $G(W)$ descends to $Sp(W)$ and has the trivial representation as quotient.

We leave it as an exercise for the reader to formulate and prove the analogue of Lemma 1.4. We simply observe that, if V_1 and V_2 are in a fixed Witt tower, and if τ_1 is a quotient of $\Theta(\pi, V_1)$ and τ_2 is a quotient of $\Theta(\pi, V_2)$, then there is an equivariant surjection

$$S_1 \otimes S_2^{\delta} \longrightarrow \tau_1 \otimes \pi \otimes \tau_2 \otimes \pi^{\vee} \longrightarrow \tau_1 \otimes \tau_2 \otimes \mathbb{1}_W.$$

The rest of the proof continues as before.

Again, given the fact that $\Theta(\tau, W)$ is either zero or has finite length, the arguments of this section prove:

Proposition 1.10. *Let $m = \dim_F V$. Then, for any $\tau \in \mathrm{Irr}(O(V))$, and for any symplectic space W,*

$$\Theta(\tau, W) \neq 0 \iff \mathrm{Hom}_{O(V)\times O(V)}(R_m(W), \tau \otimes \tau) \neq 0.$$

IV.2. Restrictions of Degenerate principal series. As we have seen in the previous section, there is a close connection between occurrence in the theta correspondence and the occurrence of certain quotients of the representation $R_n(V)$ of G_n or $R_m(W)$ of $O(V_{m,m})$, upon restriction to certain subgroups. This connection is a consequence of the seesaw dual pair. For example, Proposition 1.5 asserts that, for an irreducible $\pi \in \mathrm{Irr}(G_n)$,

$$\Theta(\pi, V) \neq 0 \iff \mathrm{Hom}_{G_n\times G_n}(R_{2n}(V), \pi \otimes \pi) \neq 0.$$

On the other hand, as explained in section III.5, the representation $R_n(V)$ is a submodule of the degenerate principal series representation $I_{P_n}^{G_n}(\chi_V^{\psi}|\det|^{\frac{m}{2}-\frac{n+1}{2}})$. This suggests that one should consider the restriction of this whole induced representation to $G_n \times G_n$. The study of such restrictions turns out to be very fruitful. In some sense, the relation between $R_n(V)$ and the degenerate principal series is a local analogue of the relation between theta integrals and Eisenstein series in the global theory. This global relationship is the the Siegel-Weil formula; it will be described in the second part of these notes.

To slightly simplify notation, in the symplectic case and for $s \in \mathbb{C}$, we consider the family of degenerate principal series representations of G_n:

$$I_n(s, \chi) = I_{P_n}^{G_n}(\chi^{\psi}|\det|^s),$$

where χ^{ψ} is defined in Proposition II.4.3. Similarly, in the orthogonal case, we consider the family of degenerate principal series representations of $O(V_{r,r})$:

$$I_r(s, \chi) = I_{Q_r}^{O(V_{r,r})}(\chi|\det|^s).$$

in both cases, the induction is normalized and the character χ of $F^\times$ need not be quadratic. Since these two families of representations will almost never occur together in our discussions, the similarity in our notation for them should not cause difficulties. Note that, in the orthogonal case, if η is a quadratic character of $F^\times$ and if $\tilde{\eta} = \eta \circ \nu$ is it composition with the spinor norm $\nu : O(V) \to F^\times/F^{\times,2}$, as in section V.2, then, by Lemma V.2.2,

$$I_r(s, \chi) \otimes \tilde{\eta} = I_r(s, \chi \cdot \eta).$$

In order to obtain information about the restriction of these degenerate principal series to subgroups, we must consider the double coset spaces

$$P_n \backslash Sp(W_n)/(Sp(W_{n_1}) \times Sp(W_{n_2})),$$

where $W_n = W_{n_1} + W_{n_2}$, and

$$Q_r \backslash O(V)/(O(V_{r_1}) \times O(V_{r_2})),$$

where $V = V_{r,r} = V_{r_1} + (-V_{r_2})$, with $r = \frac{1}{2}(m_1 + m_2)$, $m_i = \dim_F V_{r_i}$ and V_{r_1} and V_{r_2} in a fixed Witt tower. To do the analysis, we view $P_n \backslash Sp(W_n) \simeq \Omega(W)$ (resp. $Q_r \backslash O(V_{r,r}) \simeq \Omega(V)$) as the space of maximal isotropic subspaces of $W = W_n$ (resp. $V = V_{r,r}$), and we want to calculate the orbits of $Sp(W_{n_1}) \times Sp(W_{n_2})$ (resp. $O(V_{r_1}) \times O(V_{r_2})$) in this space.

Proposition 2.1. *(i) Suppose that $n_1 \geq n_2$. Then the $Sp(W_{n_1}) \times Sp(W_{n_2})$ orbits in $\Omega(W)$ are parameterized by t with $0 \leq t \leq n_2 = \min(n_1, n_2)$, where the t-th orbit is*

$$\mathcal{O}_t = \{Y \in \Omega(W) \mid \dim(Y \cap W_{n_1}) = t + n_1 - n_2, \text{ and } \dim(Y \cap W_{n_2}) = t\}.$$

(ii) Suppose that $m_1 \geq m_2$. Since V_{r_1} and V_{r_2} are in the same Witt tower, this is equivalent to $r_1 \geq r_2$. Then the $O(V_{r_1}) \times O(V_{r_2})$ orbits in $\Omega(V)$ are parameterized by t with $0 \leq t \leq r_2$. The t-th orbit is

$$\mathcal{O}_t = \{Y \in \Omega(V) \mid \dim(Y \cap V_{r_2}) = t, \text{ and } \dim(Y \cap V_{r_1}) = r_1 - r_2 + t\}.$$

(iii) For the symplectic groups, exclude the case $n_1 = n_2$ and $t = 0$. Then, the stabilizer St_t in $Sp(W_{n_1}) \times Sp(W_{n_2})$ of a point in the orbit $\mathcal{O}_t$ is contained, up to conjugacy, in the maximal parabolic subgroup $P_{t+n_1-n_2} \times P_t$. Moreover, St_t contains the full unipotent radical of this parabolic subgroup.
(iv) For the orthogonal groups, exclude the case $r_1 = r_2$ and $t = 0$. Then the

stabilizer St_t in $O(V_{r_1})\times O(V_{r_2})$ of a point in $\mathcal{O}_t$ is contained, up to conjugacy, in the product of maximal parabolic subgroups $Q_{t+r_1-r_2}\times Q_t$, and St_t contains the full unipotent radical of this parabolic subgroup.
(v) In the symplectic case with $n_1 = n_2$, $\mathcal{O}_0 \simeq Sp(W_{n_1})$ with the action of $(g_1,g_2)\in Sp(W_{n_1})\times Sp(W_{n_1})$ given by $x\mapsto g_1xg_2^{-1}$.
(vi) In the orthogonal case with $r_1=r_2$, $\mathcal{O}_0\simeq O(V_{r_1})$ with the action of $(g_1,g_2)\in O(V_{r_1})\times O(V_{r_1})$ given by $x\mapsto g_1xg_2^{-1}$.

Proof. We will only consider the symplectic case, and, for convenience, we write W_1 for W_{n_1} and W_2 for W_{n_2}. For $Y\in\Omega(W)$, let $Y_1=Y\cap W_1$ and $Y_2=Y\cap Y_2$. Let t_1 and t_2 be the dimensions of these isotropic subspaces. Let $pr_i : W\to W_i$ be the projection on the i th summand. Then

$$W_i\supset (Y_i)^\perp\supset pr_i(Y)\supset Y_i,$$

Considering dimensions, we have

$$2n_1-t_1\ge n-t_2\ge t_1$$

and

$$2n_2-t_2\ge n-t_1\ge t_2.$$

Adding these inequalities, we get

$$2n-t_1-t_2\ge 2n-t_1-t_2\ge t_1+t_2,$$

so that we must have $2n_1-t_1=n-t_2$ and $2n_2-t_2=n-t_1$. This shows that

$$(Y_i)^\perp = pr_i(Y)$$

$$\dim(Y_1)^\perp/Y_1=\dim(Y_2)^\perp/Y_2=n-t_1-t_2,$$

and

$$t_1-t_2=n_1-n_2.$$

Set $t=t_2$, so that $t_1=t+n_1-n_2$. Define an anti-isometry

$$\phi_Y : (Y_1)^\perp/Y_1\longrightarrow (Y_2)^\perp/Y_2,$$

by

$$pr_2(y)+Y_2=\phi_Y(pr_1(y)+Y_1),$$

for all $y\in Y$. Note that, for y and $y'\in Y$,

$$0=<y,y'>=<y_1,y_1'>+<y_2,y_2'>;$$

this shows that ϕ_Y is an anti-isometry. Conversely, given isotropic subspaces $Y_1 \subset W_1$ and $Y_2 \subset W_2$ of dimensions $t + n_1 - n_2$ and t, respectively, and an anti-isometry $\phi : (Y_1)^\perp/Y_1 \longrightarrow (Y_2)^\perp/Y_2$, we can define an isotropic subspace $Y \subset W$ by requiring that $pr_i(Y) = Y_i^\perp$ and that Y contains all of the subsets $u + Y_1 + \phi(u) + Y_2$, for $u \in Y_1^\perp$. Thus there is a bijection

$$\mathcal{O}_t \simeq \{(Y_1, Y_2, \phi)\}.$$

The action of $(g_1, g_2) \in Sp(W_1) \times Sp(W_2)$ on the right hand side is given by

$$(Y_1, Y_2, \phi) \mapsto (Y_1 g_1, Y_2 g_2, g_1^{-1} \circ \phi \circ g_2),$$

and it is not difficult to see that this action is transitive, so that $\mathcal{O}_t$ is indeed a single orbit. Moreover, the stabilizer of the point (Y_1, Y_2, ϕ) is the subgroup of the $P_{Y_1} \times P_{Y_2}$ consisting of the pairs (g_1, g_2) such that $g_1^{-1} \circ \phi \circ g_2 = \phi$. Here, of course, it is only the image $\bar{g}_1$ of $g_1 \in P_{Y_1}$ in $Sp(Y_1^\perp/Y_1)$ (resp. the image $\bar{g}_2$ of $g_2 \in P_{Y_2}$ in $Sp(Y_2^\perp/Y_2)$) which matters.

The proof in the orthogonal case is analogous, with the slight variation that we can have $r_1 + r_2 < r$. This is left as an exercise for the reader. □

It will be useful to have a slightly more precise description of the stabilizer St_t. Let W_0 be a sympectic space of dimension $2n_2 - 2t$, and write

$$W_1 = X_1 + W_0 + Y_1, \qquad \text{and} \qquad W_2 = X_2 + W_0 + Y_2,$$

where $X_1 \in \Omega(W_1)$ and $X_2 \in \Omega(W_2)$ are isotropic subspaces complementary to $Y_1^\perp = W_0 + Y_1$ and $Y_2^\perp = W_0 + Y_2$ respectively. A Levi factor of P_{Y_i} is then $GL(X_i) \times Sp(W_0)$. Let N_i be the unipotent radical of P_{Y_i}. Choose an element $\delta = \delta_{W_0}$, as in Theorem 3.4 above. Note that δ is then an anti-isometry from W_0 to itself.

Lemma 2.2. *The stabilizer St_t in (iii) of Proposition 2.1 is the subgroup*

$$St_t = \Big(GL(X_1) \times GL(X_2) \times \{\ (g_0, (g_0^\delta)) \mid g_0 \in Sp(W_0)\ \}\Big) \ltimes \big(N_1 \times N_2\big)$$

of the parabolic subgroup

$$P_{Y_1} \times P_{Y_2} = \big(GL(X_1) \times GL(X_2) \times Sp(W_0) \times Sp(W_0)\big) \ltimes \big(N_1 \times N_2\big).$$

Note that the case $n_1 = n_2$ and $t = 0$ is included in this description.

The orbit structure described in Proposition 2.1 gives rise to a filtration on the restriction to $G_{n_1} \times G_{n_2}$ of the induced representation $I_n(s,\chi)$. Note that the closure $\overline{\mathcal{O}_t}$ of the t-th orbit is

$$\overline{\mathcal{O}_t} = \coprod_{t' \geq t} \mathcal{O}_{t'}.$$

For convenience, we will only consider the symplectic case. We retain the notation of Proposition 2.1, so that, for example, $n_1 \geq n_2$.

Proposition 2.3. *Let* $I_n(s,\chi)^{(t)} \subset I_n(s,\chi)$ *be the subspace of functions whose restriction to the closure of* $\mathcal{O}_t$ *vanishes. Then*

$$I_n(s,\chi) = I_n(s,\chi)^{(n_2)} \supset \cdots \supset I_n(s,\chi)^{(0)},$$

is a decreasing filtration on $I_n(s,\chi)$ *which is stable under the action of* $G_{n_1} \times G_{n_2}$. *Let*

$$Q_t(s,\chi) := I_n(s,\chi)^{(t)}/I_n(s,\chi)^{(t-1)}.$$

Then

$$Q_t(s,\chi) \simeq I^{G_{n_1}\times G_{n_2}}_{P_{t+n_1-n_2}\times P_t}\big(?\otimes ?? \otimes S(G_{n_2-t})\big).$$

In particular,

$$Q_0(s,\chi) \simeq I^{G_{n_1}\times G_{n_2}}_{P_{n_1-n_2}\times G_{n_2}}\big(?\otimes ?? \otimes S(G_{n_2})\big).$$

Here $?$ and $??$ are characters which are easily determined.

Notice that the action of the group $G_r \times G_r$ on $S(G_r)$ which occurs in Proposition 2.3 is:

$$r(g_1,g_2)\varphi(x) = \varphi((g_2^\delta)^{-1} x g_1).$$

In fact, if we set $St = \{(g, g^\delta) \mid g \in G_r\}$, then

$$Ind_{St}^{G_r\times G_r}(1\!\!1) = \{f : G_r \times G_r \longrightarrow \mathbb{C} \mid f(h(g_1,g_2)) = f((g_1,g_2)),\ \forall h \in St\ \}.$$

Setting $\varphi(x) = f((x,1))$, we have

$$r(g_1,g_2)\varphi(x) = f((x,1)(g_1,g_2)) = f((g_2^\delta, g_2)((g_2^\delta)^{-1}xg_1, 1)) = \varphi((g_2^\delta)^{-1}xg_1),$$

as claimed.

IV.3. Some Proofs: Theorems III.6.1 and III.6.2 We now have enough machinery to begin the proofs of some of the results of Chapter III. These proof are mostly take from [32], with occassional refinements or improvements taken from Chapter 3 of [46].

We begin with Theorem III.6.2. As in Theorem III.6.2, consider a supercuspidal representation $\tau \in \mathrm{Irr}_{sc}(O(V))$, and let $n_0 = n(\tau)$ be the first occurrence index for τ for the symplectic Witt tower $\{W_n \mid n \geq 0\}$. Write $V = V_r$, and let $\omega_{n,r}$ denote that Weil representation of the dual pair $(O(V), G_n)$, as in Chapter III.

We first show that $\Theta(\tau, W_{n_0})$ is a supercuspidal representation. If not, then there is some $i > 0$ such that the Jacquet functor $R_{P_i}(\Theta(\tau, W_{n_0})) \neq 0$, where P_i is a maximal parabolic subgroup of G_{n_0}. Thus, we obtain a surjection

$$\lambda : R_i(\omega_{n_0,r}) \longrightarrow \tau \otimes R_i(\Theta(\tau, W_{n_0})).$$

Consider the filtration of $J_i = R_i(\omega_{n_0,r})$ described in Theorem III.8.1. Since τ is cuspidal and all of the filtration steps below the top are induced, as representations of $O(V_r)$, the restriction of λ to $J_i^{(1)}$ must be zero. Therefore λ induces a surjection:

$$J_{i0} = \omega_{n_0-i,r} \longrightarrow \tau \otimes R_i(\Theta(\tau, W_{n_0})),$$

equivariant for the action of $O(V_r) \times G_{n_0-i}$. But this contradicts the fact that $n_0 = n(\tau)$ is the first occurrence index for τ. Hence, $\Theta(\tau, W_{n_0})$ is supercuspidal.

Next suppose that $n_1 \geq n_2$ and that there are supercuspidal representations $\pi_1 \in \mathrm{Irr}_{sc}(G_{n_1})$ and $\pi_2 \in \mathrm{Irr}_{sc}(G_{n_2})$ and surjections

$$\lambda_1 : S_1 \longrightarrow \tau \otimes \pi_1 \qquad \text{and} \qquad \lambda_2 : S_2 \longrightarrow \tau \otimes \pi_2,$$

where we write S_1 (resp. S_2) for the space of the Weil representation $\omega_{n_1,r}$ (resp. $o_{n_2,r}$). Let $n = n_1 + n_2$. By (i) of Lemma 1.2, we obtain a nonzero element of $\mathrm{Hom}_{G_{n_1}\times G_{n_2}}(R_n(V), \pi_1 \otimes \pi_2)$. Recall that supercuspidal representations are projectives in the category of smooth representations, [10]. Therefore, since $R_n(V) \subset I_n(s_0, \chi_V)$, where $s_0 = \frac{m}{2} - \frac{n+1}{2}$, we have

$$\mathrm{Hom}_{G_{n_1}\times G_{n_2}}(R_n(V), \pi_1 \otimes \pi_2) \neq 0 \iff \mathrm{Hom}_{G_{n_1}\times G_{n_2}}(I_n(s_0, \chi), \pi_1 \otimes \pi_2) \neq 0.$$

Now we can apply the results of section IV.2, specifically, Proposition 2.3.

Proposition 3.1. *Suppose that* $\pi_1 \in \mathrm{Irr}_{cs}(G_{n_1})$ *and* $\pi_2 \in \mathrm{Irr}_{sc}(G_{n_2})$ *are supercuspidal representations. Then*

$$\mathrm{Hom}_{G_{n_1}\times G_{n_2}}(I_n(s, \chi), \pi_1 \otimes \pi_2) \neq 0$$

if and only if $G_1 = G_2$ *and* $\pi_1 = \pi_2$. *Moreover, in that case*

$$\dim \mathrm{Hom}_{G_{n_1}\times G_{n_1}}(I_n(s,\chi), \pi_1 \otimes \pi_1) = 1.$$

Proof. If $n_1 > n_2$, then all of the steps in the filtration of Proposition 2.3 are induced, as representations of G_{n_1} , so that there can be no supercuspidal constituents in that case. If $n_1 = n_2$, then only the open orbit $\mathcal{O}_0$ can support supercuspidal representations. It is well known that the only supercuspidal constituents of the representation of $G_{n_1} \times G_{n_1}$ on $S(G_{n_1})$ for the action $(g_1, g_2) : x \mapsto g_1 x g_2^{-1}$, have the form $\pi \otimes \pi^{\vee}$. Note that this is just the decomposition of the regular representation. The action of the MVW involution then gives the desired result. □

Thus, $\Theta(\tau, W_{n_0})$ is irreducible and $\Theta(\tau, W_{n_0}) = \theta(\tau, W_{n_0})$ is the only supercuspidal constituent of any of the $\Theta(\tau, W_n)$'s.

Corollary. *Let* $\tau \in \mathrm{Irr}_c(O(V))$. *Suppose that* $\pi_1 \in \mathrm{Irr}_c(G_1)$ *and* $\pi_2 \in \mathrm{Irr}_c(G_2)$ *are supercuspidal representations such that* π_1 *is a quotient of* $\Theta(\tau, W_1)$ *and* π_2 *is a quotient of* $\Theta(\tau, W_2)$. *Then* $G_1 = G_2$ *and* $\pi_1 = \pi_2$. *In particular,* $n_1 = n_2 = n(\tau)$ *and*

$$\pi_1 = \pi_2 = \theta(\tau, W_{n(\tau)}) = \Theta(\tau, W_{n(\tau)})$$

is the cuspidal first occurrence of τ *in the symplectic Witt tower.*

Still under the assumption that τ is supercuspidal, we consider $\Theta(\tau, W_n)$ for $n > n_0$.

Proposition. *For* $n = n_0 + t$ *with* $t > 0$,
(i) $\Theta(\tau, W_n)$ *is irreducible, and*
(ii) for each i *with* $1 \le i \le t$,

$$R_{P_i}(\Theta(\tau, W_n)) = \alpha_{i0} \otimes \Theta(\tau, W_{n-i}).$$

In particular, $R_{P_i}(\Theta(\tau, W_n))$ *is nonzero and irreducible for* i *in this range.*

Proof. Suppose, by induction, that $\Theta(\tau, W_{n_1})$ is irreducible for $n_0 \le n_1 < n$, and that,

$$R_{P_j}(\Theta(\tau, W_{n_1})) = (\text{character}) \otimes \Theta(\tau, W_{n_1 - i}),$$

for $1 \le j \le n_1 - n_0$. Since τ is a supercuspidal and via the filtration of Theorem III.8.1, the surjection

$$R_{P_i}(\omega_{n,r}) \longrightarrow \tau \otimes R_{P_i}(\Theta(\tau, W_n))$$

yields a surjection

$$\alpha_{i0} \otimes \omega_{n-i,r} \longrightarrow \tau \otimes R_{P_i}(\Theta(\tau, W_n)),$$

as a representation of $O(V_r) \times GL_i(F) \times G_{n-i}$. This surjection factors through a surjection

$$\tau \otimes \Theta(\tau, W_{n-i})\tau \otimes R_{P_i}(\Theta(\tau, W_n)).$$

By the irreducibility part of our induction hypothesis, we conclude that, for all i,

$$R_{P_i}(\Theta(\tau, W_n)) = \begin{cases} \alpha_{i0} \otimes \Theta(\tau, W_{n-i}) & \text{or} \\ 0. \end{cases}$$

Of course, for $i > n - n_0$, $R_{P_i}(\Theta(\tau, W_n)) = 0$. Next observe that the rank 2 parabolic $P_i \cap P_j$ has Levi factor $M_i \cap M_j$, and that $M_i \cap M_j$ is also the Levi factor of the maximal parabolics $M_i \cap P_j$ in M_i and $P_i \cap M_j$ in M_j. Note that

$$M_i \cap M_j = \begin{cases} GL_j(F) \times GL_{i-j}(F) \times G_{n-i} & \text{if } j < i \\ GL_i(F) \times GL_{j-i}(F) \times G_{n-j} & \text{if } j > i. \end{cases}$$

Also, the Jacquet functor can be computed in stages:

$$R_{P_i \cap P_j} = R_{M_i \cap P_j} \circ R_{P_i} = R_{P_i \cap M_j} \circ R_{P_j}.$$

Suppose that, $R_{P_i}(\Theta(\tau, W_n)) = 0$ for some i with $1 \le i \le t$. Then, for all $j \neq i$,

$$R_{P_i \cap M_j} \circ R_{P_j}(\Theta(\tau, W_n)) = R_{P_j \cap M_i} \circ R_{P_i}(\Theta(\tau, W_n)) = 0.$$

Since

$$R_{P_i \cap M_j}(\alpha_{i0} \otimes \Theta(\tau, W_{n-i})) = \begin{cases} (\text{character}) \otimes \Theta(\tau, W_{n-i}) & \text{if } j < i \\ (\text{character}) \otimes \Theta(\tau, W_{n-j}) & \text{if } j > i, \end{cases}$$

this implies that $R_{P_j}(\Theta(\tau, W_n)) = 0$ for all j. But this would imply that $\Theta(\tau, W_n)$ is cuspidal, contradicting the uniqueness of the cuspidal constituent $\theta(\tau, W_{n_0})$, which we have proved above. This proves (ii) for $\Theta(\tau, W_n)$.

Next, if

$$0 \longrightarrow A \longrightarrow \Theta(\tau, W_n) \longrightarrow B \longrightarrow 0$$

is an exact sequence, the argument just given implies that, if $R_{P_i}(A) = 0$ for some i with $1 \le i \le t$, then $R_{P_j}(A) = 0$ for all j, and similarly for B. Since $\Theta(\tau, W_n)$

can have no cuspidal constituent, we conclude, by the exactness of R_{P_i} and the irreducibility part of the induction hypothesis, that either A or B must be 0, i.e., that $\Theta(\tau, W_n)$ is irreducible. $\square$

This completes the proof of Theorem III.6.2. The proof of Theorem III.6.1 is identical.

To be continued ...

CHAPTER V

In this Chapter, we will consider the behavior of the first occurrence indices associated to *pairs* of Witt towers. The basic idea is to look for relations among the first occurrence indices of a fixed representation π in the pair of Witt towers determined by a quadratic character χ. Most of what we discuss will appear in a joint paper with Steve Rallis, [38]. In the case of dual pairs of unitary groups over nonarchimedean local fields, the analogous dichotomy and conservation principles were proposed and partially proved in [22].

V.1. First occurrence indices for pairs of orthogonal Witt towers. First consider Witt towers of orthogonal groups. The isomorphism class of a quadratic space V over F is determined by (i) $\dim_F V$, (ii) the quadratic character $\chi_V(x) = (x, (-1)^{\frac{m(m-1)}{2}})_F$, and (iii) the Hasse invariant $\epsilon(V)$. The parity of $\dim_F V$ and the character χ_V are the same for V and for $V + V_{r,r}$, and hence are constant on a Witt tower. For a fixed quadratic character χ of $F^\times$, the even dimensional quadratic spaces V with $\chi_V = \chi$ make up two Witt towers. Similarly, the odd dimensional quadratic spaces with $\chi_V = \chi$ make up two Witt towers. Here is a more concrete description.

Example 1.1: Suppose that m is even and that $\chi = \chi_V$ is trivial. The pair of Witt towers in this case are the split tower

$$\mathcal{V}_{split} = \mathcal{V}_\chi^+ = \{V_{r,r} \mid r \geq 0\},$$

and the quaternionic tower

$$\mathcal{V}_{quat} = \mathcal{V}_\chi^- = \{V_0 + V_{r,r} \mid r \geq 0\},$$

where V_0 is the anisotropic space of dimension 4. If B is the unique division quaternion algebra over F, we may take $V_0 = B$ with quadratic form $q(x) = \nu(x)$, where $\nu : B \to F$ is the reduced norm. This quadratic form has matrix

$$\begin{pmatrix} 1 & & & \\ & a & & \\ & & b & \\ & & & -ab \end{pmatrix},$$

where a and $b \in F^\times$ with $(a, b)_F = -1$. Note that there are no spaces of dimensions 0 or 2 in the quaternionic tower.

Example 1.2: Suppose that m is even and that χ is nontrivial. Then the two towers are

$$\mathcal{V}_\chi^\pm = \{V_0^\pm + V_{r,r} \mid r \geq 0\},$$

where $V_0^{\pm}$ is a 2 dimensional quadratic space with character χ. We label these spaces as follows. Let E be the quadratic extension of F with $NE^{\times} = \ker(\chi)$, and let $V_0^+ = E$ with $q(x) = N_{E/F}(x)$. For $a \in F^{\times}$ with $\chi(a) = -1$, let $V_0^- = E$ with quadratic form $q(x) = a \cdot N_{E/F}(x)$.

Example 1.3: Suppose that m is odd, and let $\chi(x) = (x,\kappa)_F$, for $\kappa \in F^{\times}$. Let $V_0^+ = F$ be the one dimensional quadratic space with $q(x) = \kappa \cdot x^2$. Let $V_0^- = \{x \in B \mid tr(x) = 0\}$ be the vector space of trace 0 elements in the division quaternion algebra B, and let $q(x) = \kappa \cdot \nu(x)$. It is easy to check that $\chi_{V_0^{\pm}} = \chi$. The pair of odd dimensional Witt towers determined by χ is then

$$\mathcal{V}_{\chi}^{\pm} = \{V_0^{\pm} + V_{r,r} \mid r \geq 0\}.$$

Remark 1.4. In some sense, the Witt towers in a pair, are distinguished by their Hasse invariants. We are using the Hasse invariant defined by

$$\epsilon(V) = \prod_{i<j}(a_i, a_j)_F$$

if the quadratic form on V is given by $q(x) = a_1x_1^2 + \dots a_mx_m^2$, for a suitable choice of basis, see Lemma I.4.2. This version of the Hasse invariant is not constant on a Witt tower. Rather, we have

$$\epsilon(V_r) = \epsilon(V_0)(-1, \det(V_0))_F^r(-1,-1)^{\frac{r(r-1)}{2}}.$$

We could renormalize the Hasse invariant to fix this, but prefer to assign a ± 1 to a tower in a pair via the explicit description above.

We now fix a quadratic character χ and a pair of Witt towers $\mathcal{V}_{\chi}^{\pm}$ of even or odd dimensional quadratic spaces. As before, let G_n be $Sp(W_n)$ in the even dimensional case and $Mp(W_n)$ in the odd dimensional case. For any $\pi \in \mathrm{Irr}(G_n)$, we consider the *pair* of first occurrence indices $r_{\chi}^{\pm}(\pi)$, which describe the points at which π first occurs in the theta correspondence with spaces in the towers $\mathcal{V}_{\chi}^{\pm}$.

There is a good deal of evidence for a 'conservation principle' linking the first occurrence of a given representation in the two towers in a pair.

Conjecture 1.5. (Conservation Conjecture) *Fix χ and the parity, as above. For any $\pi \in \mathrm{Irr}(G_n)$,*

$$r_{\chi}^{+}(\pi) + r_{\chi}^{-}(\pi) = 2n = \dim_F W.$$

Note that this principle gives a quantitative version of the assertion that if π occurs 'early' for one tower in a pair, then π occurs late for the other! The conjecture could be rephrased in terms of the dimensions of the quadratic spaces in place of the Witt indices. Since the dimensions of the anisotropic parts of the spaces in a pair of Witt towers always add up to 4, we could write the conjectured relation as:

Conjecture 1.5. (bis).

$$m_\chi^+(\pi) + m_\chi^-(\pi) = 4n + 4.$$

Several special cases of this conjecture should be noted.

First, consider the case of a pair of Witt towers $\mathcal{V}_\chi^\pm$ of odd dimensional spaces. The Conjecture implies that the largest possible pair of first occurrence dimensions is

$$\{m_\chi^+(\pi), m_\chi^-(\pi)\} = \{2n+1, 2n+3\},$$

i.e.,

$$\min(m_\chi^+(\pi), m_\chi^-(\pi)) \leq 2n+1.$$

This yields an important special case of the previous conjecture.

Conjecture 1.6. (Dichotomy Conjecture) *Fix a quadratic character* χ . *Then for any* $\pi \in \mathrm{Irr}(Mp(W_n))^{gen} = \mathrm{Irr}(G_n)$, *there is a unique quadratic space* V^+ *or* V^- *with* $\dim_F V = 2n+1$ *and* $\chi_V = \chi$ *such that* $\Theta(\pi, V^\pm) \neq 0$.

Thus, for χ fixed, the representations of $Mp(W_n)$ are split up in the theta correspondence between the two quadratic spaces of dimension $2n+1$ character χ and the two possible values fof the Hasse invariant. This sort of dichotomy phenomena in quite important and is known to occur in several other contexts, [79],[74],[55], [18],[19], [65], [21], [85], [58]. The fundamental fact, which we have not had time to discuss here, is that such dichotomies are controlled by local root numbers! For this reason, they play an essential part in understanding the central values/derivatives of certain global L-function.

These conjectures are known to be true for 'almost all' representations π , in a sense which will be explained a little later. For example, we have the following results, which will appear in a forthcoming joint paper with Steve Rallis, [38].

Theorem 1.7. *For any representation* $\pi \in \mathrm{Irr}(G_n)$,

$$r_\chi^+(\pi) + r_\chi^-(\pi) \geq 2n.$$

Theorem 1.8. *For $\pi = 1\!\!1_{W_n}$ and for a pair of even dimensional Witt towers determined by χ, Conjecture 1.5 is true. More precisely, let χ_0 be the trivial quadratic character. Then*

$$r^+_{\chi_0}(1\!\!1_{W_n}) = 0 \qquad \textit{and} \qquad r^-_{\chi_0}(1\!\!1_{W_n}) = 2n.$$

In particular, the trivial representation of $Sp(W_n)$ first occurs in the quaternionic tower with the space V of dimension $4n+4$, i.e., when it is forced to occur by the stable range condition of Proposition III.4.3. If $\chi \neq \chi_0$, then

$$r^+_{\chi}(1\!\!1_{W_n}) = r^-_{\chi}(1\!\!1_{W_n}) = n.$$

Note that this result to some extent completes the information about the occurrence of the trivial representation in the split tower given in section III.5. It would be interesting to have a precise description of $\Theta(1\!\!1_W, V)$ in all cases.

At the opposite extreme,

Theorem 1.9. *The conservation principle, Conjecture 1.5, and hence the dichotomy, Conjecture 1.6, hold for any supercupsidal representation $\pi \in \mathrm{Irr}_{sc}(G_n)$.*

V.2. Dichotomy for the theta correspondence from $O(V)$. Next, we consider the analogous phenomena in the other direction, i.e., beginning with a representation $\tau \in \mathrm{Irr}(O(V_r))$. Here, of course, there is only one Witt tower of symplectic spaces. On the other hand, the orthogonal group has a nontrivial group of characters, all of order 2, and it turns out that twisting by these characters now plays the role previously played by the invariants (Galois cohomology) of the 'target' tower! Not all details of this case have been checked, so that the reader may wish to view this section as slightly speculative.

For a quadratic space V over F, let $\mathrm{sgn} = \mathrm{sgn}_V$ be the sign or determinant character of $O(V)$. Also, let

$$\nu : SO(V) \longrightarrow F^\times/F^{\times,2}$$

be the spinor norm. The image of the spinor norm is described in many places, for example [eichler],[kneser],[omeara],[kitaoka].

Proposition 2.1. *(i) If $\dim_F V = 1$, then $\nu(SO(V)) = 1$.*
(ii) If $\dim_F V = 2$, and if $\chi_V = \chi_0$ is trivial, then $\nu(SO(V)) = F^\times/F^{\times,2}$.

(iii) If $\dim_F V = 2$ *and* $\chi_V \neq \chi_0$ *, then* V *is a multiple of the norm form of a quadratic extension* E/F *, and* $\nu(SO(V)) = NE^\times/F^{\times,2}$ *.*
(iv) If $\dim_F V \geq 3$ *, then* $\nu(SO(V)) = F^\times/F^{\times,2}$ *.*

Here is another useful fact. Recall that Q_k is the maximal parabolic subgroup of $O(V_r)$ with Levi factor M_k isomorphic to $GL_k(F) \times O(V_{r-k})$.

Lemma 2.2. *The restriction of the spinor norm* ν *to the Levi factor* M_k *is the character* $\overline{\det} \otimes \nu$ *, where* $\overline{\det} : GL_k(F) \to F^\times/F^{\times,2}$ *is the homomorphism induced by the determinant.*

By composition with the spinor norm, any quadratic character χ of $F^\times$ defines a quadratic character $\tilde{\chi} = \chi \circ \nu$ of $SO(V)$. We choose an extension of this character to $O(V)$ and also denoted it by $\tilde{\chi}$. The two characters of $O(V)$ which restrict to $\tilde{\chi}$ on $SO(V)$ are then $\tilde{\chi}$ and $\mathrm{sgn} \cdot \tilde{\chi}$.

Recall that, for $\tau \in \mathrm{Irr}(O(V))$, $n(\tau)$ is the smallest n such that $\Theta(\tau, W_n) \neq 0$. Here is the analogue of Conjecture 1.5.

Conjecture 2.3. *For any quadratic character* χ *of* $F^\times$ *, and for any* $\tau \in \mathrm{Irr}(O(V))$ *,*

$$n(\tilde{\chi} \otimes \tau) + n(\mathrm{sgn} \cdot \tilde{\chi} \otimes \tau) = \dim_F V.$$

Thus, the role of the *pair of orthogonal Witt towers* determined by χ the previous section is played by the *pair of twists* $\tilde{\chi} \otimes \tau$ and $\mathrm{sgn} \cdot \tilde{\chi} \otimes \tau$.

Again, this conjecture is known for almost all representations, as will be explained below.

V.3. Quotients of degenerate principal series.

Recall that Proposition IV.1.5, asserts provides a criterion for the occurrence of a representation $\pi \in \mathrm{Irr}(G_n)$, with $G_n = Sp(W_n)$ (resp. $Mp(W_n)$) in the theta correspondence with a quadratic space V with $\dim_F V$ even (resp. odd). Let G_{2n} be $Sp(W_2 n)$ (resp. $Mp(W_{2n})$) and identify W_{2n} with the "doubled" symplectic space $W_n + W_n^-$, where W_n^- is the space W_n with the negative of its original symplectic form. We thus obtain a homomorphism

$$G_n \times G_n \longrightarrow G_{2n},$$

and we can consider the restriction of the representation $R_{2n}(V)$ of G_{2n} to $G_n \times G_n$. Then:

$$\Theta(\pi, V) \neq 0 \iff \mathrm{Hom}_{G_n \times G_n}(R_{2n}(V), \pi \otimes \pi^\vee) \neq 0.$$

Note that, $\pi^\vee$, the contragradient of π, occurs here due to the change made in the sign of the symplectic form in the second factor of the doubled space, which causes the MVW involution δ to be applies to the second copy of π in Proposition IV.1.5.

On the other hand, by Theorem III.5.1 and Definition III.5.2

$$R_{2n}(V) \hookrightarrow I_{2n}(s_0, \chi_V),$$

where $s_0 = \frac{m}{2} - \frac{2n+1}{2}$, so that it is natural to (i) consider quotients of the restriction of $I_{2n}(s, \chi)$ to $G_n \times G_n$, and (ii) study the position of the submodules $R_{2n}(V)$ in $I_{2n}(s_0, \chi)$, say for fixed χ. As a result, we obtain

Theorem 3.1. *For $\pi \in \mathrm{Irr}(G_n)$, suppose that*

$$\dim \mathrm{Hom}_{G_n \times G_n}(I_{2n}(s_0, \chi) = 1,$$

for all s_0 in the set:

$$Crit = \begin{cases} \{-n - \frac{1}{2}, -n + \frac{1}{2}, \dots, n - \frac{1}{2}, n + \frac{1}{2}\} & \text{for } G_{2n} = Sp(W_{2n}) \text{ and } \chi = 1, \\ \{-n + \frac{1}{2}, \dots, n - \frac{1}{2}\} & \text{for } G_{2n} = Sp(W_{2n}) \text{ and } \chi \neq 1, \\ \{-n, -n+1, \dots, n-1, n\} & \text{for } G_{2n} = Mp(W_{2n}). \end{cases}$$

Then

$$m_\chi^+(\pi) + m_\chi^-(\pi) = 4n + 4,$$

i.e., the conservation principle holds for π for the pair of Witt towers determined by χ.

Thus, the full conservation conjecture would follow from another conjecture.

Conjecture 3.2. *For all $\pi \in \mathrm{Irr}(G_n)$, for all χ and for all $s \in \mathbb{C}$,*

$$\dim \mathrm{Hom}_{G_n \times G_n}(I_{2n}(s, \chi), \pi \otimes \pi^\vee) = 1.$$

In fact, the criterion of Theorem 3.1 can be checked for a fairly large class of representations $\pi \in \mathrm{Irr}(G_n)$. To see this, recall the filtration of $I_{2n}(s, \chi)$ described in Proposition 2.3. Recall that, for $0 \le t \le n$, $I_{2n}(s,\chi)^{(t)} \subset I_n(s,\chi)$ is the subspace of functions whose restriction to the closure of the $G_n \times G_n$ orbit $\mathcal{O}_t$ in $P_{2n}\backslash G_{2n}$ vanishes. Then

$$I_{2n}(s,\chi) = I_{2n}(s,\chi)^{(n)} \supset \cdots \supset I_n(s,\chi)^{(0)},$$

is a decreasing filtration on $I_{2n}(s,\chi)$ which is stable under the action of $G_n \times G_n$. The quotients of successive steps in the filtration are

$$Q_t(s,\chi) := I_{2n}(s,\chi)^{(t)}/I_{2n}(s,\chi)^{(t-1)}.$$

Recall that

$$Q_t(s,\chi) \simeq I_{P_t\times P_t}^{G_n\times G_n}\big(\chi|\ |^{s+\frac{t}{2}} \otimes \chi|\ |^{s+\frac{t}{2}} \otimes S(G_{n-t})\big).$$

In particular, the bottom term in the filtration is the space

$$Q_0(s,\chi) \simeq S(G_n)$$

of functions supported in the open cell $P_{2n}\cdot\iota(G_n\times G_n)$. Here the action of G_n is given by $r(g_1,g_2)f(x) = f(g_1^{-1}xg_2)$.

Definition 3.3. *For a fixed character* χ, *an irreducible representation* $\pi \in \mathrm{Irr}(G_n)$ *is said to* **occur on the boundary at the point** s_0 *if*

$$\mathrm{Hom}_{G_n\times G_n}(Q_t(s_0,\chi), \pi\otimes\pi^\vee) \neq 0$$

for some $t > 0$.

Of course, every representation occurs uniquely on the open cell:

Lemma 3.4. *For every irreducible representation* $\pi \in \mathrm{Irr}(G_n)$,

$$\dim \mathrm{Hom}_{G_n\times G_n}(S(G_n), \pi\otimes\pi^\vee) = 1.$$

Proof. For any vectors $\xi \in \pi$ and $\xi^\vee \in \pi^\vee$, let

$$\phi(g) =< \pi^\vee(g)\xi^\vee, \xi >$$

be the associated matrix coefficient of $\pi^\vee$. For $f \in S(G_n)$, the integral

$$B(\xi^\vee,\xi,f) = \int_{G_n} \phi(g)\, f(g)\, dg$$

converges, and B defines a $G_n\times G_n$ invariant pairing

$$B : \pi^\vee \otimes \pi \otimes S(G_n) \longrightarrow \mathbb{C}.$$

Thus, B yields a nonzero $G_n \times G_n$ equivariant map

$$S(G_n) \longrightarrow \pi\otimes\pi^\vee,$$

as required. $\square$

Proposition 3.5. *Suppose that* $\pi \in \mathrm{Irr}(G_n)$ *does not occur on the boundary at* s_0 *. Then*

$$\dim \mathrm{Hom}_{G_n\times G_n}(I_{2n}(s_0,\chi), \pi\otimes\pi^\vee) = 1.$$

Proof. First note that if $\lambda : I_{2n}(s_0,\chi) \to \pi\otimes\pi^\vee$ is an equivariant map whose restriction to $S(G_n)$ is zero, then $\lambda = 0$. Indeed, if $I_{2n}(t)(s_0,\chi)$, with $t>0$, is the smallest step in the filtration on which λ is nonzero, then λ induces a nonzero map $Q_t(s_0,\chi) \to \pi\otimes\pi^\vee$, contrary to our hypothesis. In particular, any λ is determined by its restriction to $S(G_n)$.

It remains to show that there is a nonzero map. To do this, we consider the matrix coefficient ϕ, as in the proof of Lemma 3.4. For $\Phi(s_0)\in I_{2n}(s_0,\chi)$, let $\Phi(s)\in I_{2n}(s,\chi)$ be the unique section of the family of induced representations whose restriction to the fixed maximal compact (modulo center) open subgroup $K\subset G_n$ agrees with $\Phi(s_0)$. Then consider the *zeta integral*, [53],[54],

$$Z(s,\xi^\vee,\xi,\Phi) = \int_{Sp(W_n)} \Phi(\delta_0\iota(g,1),s)\,\phi(g)\,dg.$$

Here, in the metaplectic case, we arrange things so that the action of the central $C^\times$ is the same in $I_{2n}(s,\chi)$ and in π. Thus the integrand is invariant under $\mathbb{C}^\times$ and so defines a function on $G_n/C^\times = Sp(W_n)$. This integral converges for $\mathrm{Re}(s)$ sufficiently large and has a meromorphic analytic continuation to the whole s plane, [pnas], [psrallisbook]. Moreover, for a fixed point $s_0\in\mathbb{C}$, the order of the pole of $Z(s,\xi^\vee,\xi,\Phi)$ at s_0 is bounded, uniformly in the data $\xi^\vee$, ξ, and Φ. Let

$$Z(s,\xi^\vee,\xi,\Phi) = \frac{A_k(\xi^\vee,\xi,\Phi)}{(s-s_0)^k} + \cdots + \frac{A_1(\xi^\vee,\xi,\Phi)}{(s-s_0)} + A_0(\xi^\vee,\xi,\Phi) + O(s-s_0)$$

be the Laurent expansion at the point s_0, and suppose that A_k is the first nonzero term in this expansion, i.e., the highest order pole of the *family* of zeta integrals as the data $\xi^\vee$, ξ, and $\Phi(s_0)$ varies. It is not difficult to check that

$$A_k : \pi^\vee\otimes\pi\otimes I_{2n}(s_0,\chi) \longrightarrow \mathbb{C}$$

is then a $G_n\times G_n$ invariant pairing. Moreover, if $\Phi(s_0)$ has support in the open cell, then the zeta integral $Z(s,\xi^\vee,\xi,\Phi)$ is entire, for all ξ and $\xi^\vee$. In particular, if $k>0$, the restriction to A_k to $\pi^\vee\otimes\pi\otimes I^{(0)}_{2n}(s_0,\chi)$ is identically zero. But, since π does not occur on the boundary at s_0, this implies that A_k is identically zero! This proves that the zeta integral is holomorphic at s_0 for any choice of data. The constant term A_0 then defines a $G_n\times G_n$ invariant pairing

$$A_0 : \pi^\vee\otimes\pi\otimes I_{2n}(s_0,\chi) \longrightarrow \mathbb{C},$$

whose restriction to

$$\pi^\vee \otimes \pi \otimes I_{2n}^{(0)}(s_0,\chi) \simeq \pi^\vee \otimes \pi \otimes S(G_n)$$

coincides with B, defined in the proof of Lemma 3.4. Thus B has a unique extension A_0, and Proposition 3.5 is proved. □

Note that the proof just give shows quite a bit more.

Corollary 3.6. *If π does not occur on the boundary at s_0, then the zeta integral $Z(s,\xi^\vee,\xi,\Phi)$ is holomorphic at s_0 for any choice of data, and the map*

$$Z(s_0) : I_{2n}(s_0,\chi) \longrightarrow \pi \otimes \pi^\vee$$

defined by

$$\Big\langle Z(s_0,\Phi), \xi^\vee \otimes \xi \Big\rangle = Z(s,\xi^\vee,\xi,\Phi)|_{s=s_0}$$

spans the one dimensional space

$$\mathrm{Hom}_{G_n\times G_n}(I_{2n}(s_0,\chi), \pi\otimes\pi^\vee).$$

Very few representations of G_n can occur on the boundary at a fixed value s_0, or at a finite set of such values, like $Crit$, for that matter. For example:

Lemma. *(i) If $\pi \in \mathrm{Irr}_{sc}(G_n)$, then π does not occur on the boundary for any s.*
(ii) More generally, if $\pi \in \mathrm{Irr}(G_n)$ has Bernstein-Zelevinsky data of the form

$$[\pi] = [\ \sigma_1,\dots,\sigma_j,\pi_0\ ],$$

where $j>0$, and $\sigma_i \in \mathrm{Irr}_{sc}(GL_{r_i}(F))$ with $r_i > 1$ for all i, then π does not occur on the boundary for any s.

Of course, for a fixed value s_0, even more representations are excluded from the boundary. We leave it to the reader to formulate the precise set of such representations.

On the other hand, in the case $G_n = Sp(W_n)$, the trivial representation $1\!\!1_{W_n}$ does occur on the boundary for critical values of s.

Analogously, for a representation $\tau \in \mathrm{Irr}(O(V))$, and

$$R_r(W) \hookrightarrow I_r(s_0,\chi_0),$$

where $s_0 = n - \frac{r-1}{2}$, and χ_0 is the trivial character.

To be continued ...

CHAPTER VI

In this Chapter, we will discuss several consequences of the theory outlined so far. We begin by working out as much as possible about the theta correspondence for low dimensional examples, $(O(V), Sp(W))$ where $\dim_F V = 1,\ 2,\ 3$, or 4 and $\dim_F W = 1$. Actually, we formally include the cases $\dim_F V = 0$ and $\dim_F W = 0$. The goal is to see the extent of the information provided by the simple principles of first occurrence, presistence, stable range, and conservation. Of course, more work provides more detail, but for this we refer the reader to the literature. In passing, we identify the principal series representation of $Sp(W_n)$ or $Mp(W_n)$ which contains the representation $\theta(\mathrm{sgn}_V, W_n)$.

One rather striking phenomenon emerges, that is the existence of certain 'chains of supercuspidals', that is sequences of supercuspidal representations of an increasing sequence of symplectic and orthogonal groups. These sequences are constructed via the theta correspondence beginning with a single supercuspidal! The meaning of these chains is not completely understood, but, in any case, the supercuspidals which lie in them seem to be of a rather degenerate type, particularly as the dimension of the underlying group increases. Analogous chains occur for dual pairs of unitary groups, [22].

Finally, we discuss the behavior of unramified representations in the correspondence. On the one hand, these representations are important for the global theory. On the other hand, their behavior under the theta correspondence provides some basic clues about the relation between that correspondence and local Langlands functoriality. We conclude with some speculations about the L-packets involved in the chains of supercuspidals.

VI.1. Small examples. We now turn to several low dimensional examples and try to work out as explicitly as possible the consequences of the theory set up in Chapters III–V.

There are two 'simplest cases', depending on whether our orthogonal spaces have even or odd dimension. We begin with the even case. The explicit construction of (supercuspidal) representations of $SL_2(F)$ via the Weil representation was pioneered in [67], [71], [72], [68], [13]. For the metaplectic group $Mp(W_1)$, the reader should consult [63], [41], [42], [43], [16], [77], and [82].

Example 1.1. Consider the dual pair $(Sp(W_1), O(V_{1,1}))$, where we start with irreducible representation $\tau \in \mathrm{Irr}(O(V_{1,1}))$. The set $\mathrm{Irr}(O(V_{1,1}))$ is easy to describe:

$$\mathrm{Irr}(O(V_{1,1})) = \{ \, \mathbb{1}, \mathrm{sgn} \, \} \quad \cup \coprod_{\substack{\eta \neq 1 \\ \eta^2 = 1}} \{ \, \tau_\eta, \ \tau_\eta \otimes \mathrm{sgn} \, \} \quad \cup \coprod_{\substack{\eta \\ \eta^2 \neq 1}} \{ \, \tau_\eta \, \}.$$

Here η runs over characters of $F^\times$. We should think of $SO(V_{1,1})$ as the parabolic subgroup Q_1, the stabilizer of an isotropic line. In this very small case, the unipotent radical happens to be trivial, and $Q_1 = M_1 = GL_1(F) = F^\times$. The decomposition of $\mathrm{Irr}(O(V_{1,1}))$ just given is the Bernstein-Zelevinsky classification in this case. Every representation is a constituent of an induced representation from a character ($=$ supercuspidal!) of $F^\times$. The group $O(V_{1,1})$ has no supercuspidal representations (i.e., representations whose matrix coefficients are compactly supported modulo the center, or, equivalently, whose jacquet functor(s) vanish). For any representation $\tau \in \mathrm{Irr}(O(V_{1,1}))$, the conservation principle (known to hold in this example) asserts that

$$n(\tau) + n(\tau \otimes \mathrm{sgn}) = 2.$$

Thus,

$$\{ \, n(\tau), \ n(\tau \otimes \mathrm{sgn}) \, \} = \begin{cases} \{ \, 0, \ 2 \, \} & \text{if } \tau = \mathbb{1} \text{ or sgn}, \\ \{ \, 1, \ 1 \, \} & \text{otherwise.} \end{cases}$$

Here we note that the trivial representation, $\mathbb{1}$, should be viewed, formally, as the Weil representation for the dual pair $(Sp(W_0), O(V_{1,1}))$. This gives $n(\mathbb{1}) = 0$ and hence $n(\mathrm{sgn}) = 2$, so that the representation sgn first occurs in the theta correspondence with $Sp(W_2)$. By Theorem III.7.2, the τ_η's must correspond to constituents of the principal series representation $I_1(0, \eta)$ of $Sp(W_1)$. This representation is irreducible, except in the following cases:

(i) $\eta^2 = 1$, $\eta \neq 1$, in which case

$$I_1(0, \eta) = \theta(\tau_\eta, W_1) \oplus \theta(\tau_\eta \otimes \mathrm{sgn}, W_1),$$

(ii) $\eta = | \ |$, so that $I_1(0, \eta) = I_1(1, \chi_0)$, in which case, there is a nonsplit exact sequence:

$$0 \longrightarrow \mathrm{sp} \longrightarrow I_1(1, \chi_0) \longrightarrow \mathbb{1}_{W_1} \longrightarrow 0,$$

where sp is the special representation of $Sp(W_1) \simeq SL_2(F)$.

Since the trivial representation $\mathbb{1}_{W_1}$ occurs, formally, for the dual pair $(Sp(W_1), O(V_{0,0}))$ at the base of the split tower, we see that

$$\theta(\tau_{| \, |}, W_1) = \mathbb{1}_{W_1}$$

and that $\theta(\mathrm{sp}, V_{1,1}) = 0$, i.e., the special representation must first occur with $O(V_{2,2})$. Otherwise, we have, for η with $\eta^2 \neq 0$,

$$\theta(\tau_\eta, W_1) = I_1(0, \eta).$$

Note that

$$\theta(1\!\!1, W_1) = I_1(0, 1).$$

Note that the representation sgn is the only representation of $O(V_{1,1})$ which does not occur in the correspondence, and that no supercuspidal representations of $Sp(W_1)$ occur. These facts are easily checked directly, cf. [68].

Example 1.2. Consider the dual pair $(Sp(W_1), O(V_2^{\pm}))$ where $\dim_F V_2^{\pm} = 2$ and $\chi = \chi_{V_2^{\pm}} \neq 1$, [68], [13], [43]. Here, we must consider the pair of Witt towers determined by χ, cf. Example V.1.2. Let E/F be the quadratic extension determined by χ, and note that, in this case, $SO(V_2^{\pm}) \simeq E^1$, the kernel of the norm map $N_{E/F} : E^\times \to F^\times$. Every nontrivial representation $\tau \in \mathrm{Irr}(O(V_2^{\pm}))$ is supercuspidal, since there are no proper parabolic subgroups. Note that, on the one hand, $O(V_2^+) = O(V_2^-) = O(V_2)$, so that we can consider each representation $\tau \in \mathrm{Irr}(O(V_2))$ as coming in two types: $\tau^+ \in \mathrm{Irr}(O(V_2^+))$ and $\tau^- \in \mathrm{Irr}(O(V_2^-))$. Again we have

$$\{\ n(\tau),\ n(\tau \otimes \mathrm{sgn})\ \} = \begin{cases} \{\ 0,\ 2\ \} & \text{if } \tau = 1\!\!1_{V_2^{\pm}} \text{ or } \mathrm{sgn}_{V_2^{\pm}}, \\ \{\ 1,\ 1\ \} & \text{otherwise,} \end{cases}$$

with $n(1\!\!1_{V_2^{\pm}}) = 0$ and $n(\mathrm{sgn}_{V_2^{\pm}}) = 2$. In this case, we have

$$I_1(0, \chi) = \theta(1\!\!1_{V_2^+}, W_1) \oplus \theta(1\!\!1_{V_2^-}, W_1),$$

and

$$\theta(\tau^{\pm}, W_1) = \text{supercuspidal}$$

for τ not $1\!\!1_{V_2}$ or sgn_{V_2}. Note that, if $a \in F^\times$ with $\chi(a) = -1$, and if $\delta_a \in GSp(W_1)$ is a similitude of scale a, cf. Lemma IV.3.7, then

$$\theta(\tau^-, W_1) = \theta(\tau^+, W_1)^{\delta_a}.$$

Note that, again, the special representation does not occur.

Exercise: Prove that the representations $\theta(\tau^+, W_1)$ and $\theta(\tau^-, W_1)$ are not isomorphic by using a seesaw argument like that of the paragraph following Lemma III.3.7.

Exercise: The only overlap between the correspondence just discussed and that of Example 1 is the decomposition:

$$I_1(0, \chi) = \theta(1\!\!1_{V_2^+}, W_1) \oplus \theta(1\!\!1_{V_2^-}, W_1) = \theta(\tau_\chi, W_1) \oplus \theta(\tau_\chi \otimes \mathrm{sgn}, W_1).$$

Here

$$I^{O(V_{1,1})}_{SO(V_{1,1})}(\chi) = \tau_\chi \oplus (\tau_\chi \otimes \mathrm{sgn}),$$

where the labeling of the pieces is arbitrary. Describe which piece of this induced representation corresponds to $\theta(\mathbb{1}_{V_2^+}, W_1)$ and which to $\theta(\mathbb{1}_{V_2^-}, W_1)$.

Exercise: Determine the overlaps among the images in $\mathrm{Irr}(Sp(W_1))$ of the $\mathrm{Irr}(O(V_2))$'s for V_2 's with dimension 2 and various $\chi_{V_2} \neq 1$. This will require the description of the nontrivial representations $\tau \in \mathrm{Irr}(O(V_2))$ in terms of characters of E^1. (For the answer, see the paper of Casselman, [casselmanbinary].

Example 1.3. Consider the theta correspondence for $Sp(W_1)$ and the pair of quadratic Witt towers with $\chi = \chi_0$, the trivial character. The conservation principle, which holds for all $\pi \in \mathrm{Irr}(Sp(W_1))$, implies that $m^+(\pi) + m^-(\pi) = 8$, where we are using the dimension version of Conjecture V.1.5(bis). By Example 1, we know precisely which π 's have $m^+(\pi) = 2$, so we can immediately write down the following table, where, for a nontrivial quadratic character χ, we write

$$I_1(0, \chi) = \pi_\chi^+ \oplus \pi_\chi^-.$$

$\pi \in \mathrm{Irr}(Sp(W_1))$	$m^+(\pi)$	$m^-(\pi)$
$I_1(s, \chi)$	2	6
$\mathbb{1}_{W_1}$	0	8
$\pi_\chi^\pm$	2	6
sp	4	4
supercuspidal	4	4

In the first line, $I_1(s, \chi)$ is an irreducible principal series representation.

Corollary. *Every supercuspidal representation $\pi \in \mathrm{Irr}_{sc}(Sp(W_1))$ occurs in the theta correspondence with $O(V_4^-)$, where V_4^- is the anisotropic quadratic space of dimension 4 defined by the quaternion norm form.*

Of course, every representation $\pi \in \mathrm{Irr}(Sp(W_1))$ occurs in the theta correspondence with $O(V_{2,2})$ via the stable range condition, Proposition III.4.3.

We note that $R_1(V_4^-) = \mathrm{sp}$, so that

$$\theta(\mathrm{sp}, V_4^-) = \mathbb{1}_{V_4^-}.$$

On the other hand, $\Theta(1\!\!1_{V_{2,2}}, W_1) = R_1(V_{2,2}) = I_1(1, \chi_0)$, so that, in this example, $\Theta(1\!\!1_{V_{2,2}}, W_1)$ is not irreducible, and we have

$$\theta(1\!\!1_{V_{2,2}}, W_1) = 1\!\!1_{W_1} \qquad \text{and} \qquad \theta(1\!\!1_{W_1}, V_{2,2}) = 1\!\!1_{V_{2,2}}.$$

Exercise. Determine $\theta(I_1(s,\chi), V_{2,2})$, $\theta(\mathrm{sp}, V_{2,2})$, and $\theta(\pi_\chi^\pm, V_{2,2})$.

Example 1.4. The theta correspondence for $Sp(W_1)$ and the pair of quadratic Witt towers with χ, a nontrivial character is quite parallel to that of Example 1.3. Again, the conservation principle, implies that, for all $\pi \in \mathrm{Irr}(Sp(W_1))$, $m^+(\pi) + m^-(\pi) = 8$. By Example 1.2, we know precisely which π's have $m^\pm(\pi) = 2$, so we can write down the following table:

$\pi \in \mathrm{Irr}(Sp(W_1))$	$m^+(\pi)$	$m^-(\pi)$
$I_1(s,\chi)$	4	4
$1\!\!1_{W_1}$	4	4
π_χ^+	2	6
π_χ^-	6	2
$\pi_\eta^\pm$	?	?
sp	4	4
$\pi = \theta(\tau^+, W_1)$	2	6
$\pi = \theta(\tau^-, W_1)$	6	2
supercuspidal	4	4

In the first line, $I_1(s,\chi)$ is an irreducible principal series representation. The behavior of the $\pi_\eta^\pm$'s for $\eta^2 = 1$, $\eta \neq 1$ and $\eta \neq \chi$ depends on the overlaps mentioned in the third Exercise in Example 1.3. Finally, the last three lines describe what happens for supercuspidals, where, in the last line π is assumed to be a supercuspidal other than one of the $\theta(\tau^\pm, W_1)$'s coming from the $O(V_2^\pm)$'s with $\chi_{V_2^\pm} = \chi$.

We now turn to the case of odd dimensional quadratic spaces.

Example 1.5. Consider the dual pair $(Mp(W_n), O(1))$, where the one dimensional quadratic space is F with quadratic form $a \cdot x^2$. Let $\chi_V(x) = (x,a)_F$. Of course, we then have

$$\theta(1\!\!1_V, W_n) = \omega_{\psi_a}^+ \qquad \text{and} \qquad \theta(\mathrm{sgn}_V, W_n) = \omega_{\psi_a}^-,$$

the even and odd parts of the Weil representation of $Mp(W_n)$ determined by the additive character ψ_a. Since

$$n(1\!\!1_V) + n(\mathrm{sgn}_V) = 1$$

and $n(\mathbb{1}_V)=0$, as usual, we see that $\theta(\mathrm{sgn}_V, W_1)$ is a supercuspidal representation of $Mp(W_1)$. For general n, we can apply Proposition III.4.5 to obtain

$$\theta(\mathbb{1}_V, W_n)=\Theta(\mathbb{1}_V, W_n)=R_n(V)\subset I_n\Big(-\frac{n-1}{2},\chi_V\Big),$$

and

$$\theta(\mathrm{sgn}_V, W_n)=\Theta(\mathrm{sgn}_V, W_n)\subset I_{P_{n-1}}^{Mp(W_n)}(\chi_V|\ |^{??}\otimes\theta(\mathrm{sgn}_V, W_1)).$$

Notice that, for $Mp(W_1)$ and for any character χ of $F^\times$ such that $\chi^2=1$, there is an exact sequence

$$0\longrightarrow \mathrm{sp}_\chi\longrightarrow I_1(\frac{1}{2},\chi)\longrightarrow \omega_{\psi_a}^+\longrightarrow 0,$$

where $R_1(V)=\omega_{\psi_a}^+$. We will call the representation sp_χ the special representation associated to $\chi=\chi_V$. For convenience, we will also write

$$\theta_\chi^+=\theta(\mathbb{1}_V, W_1)=\omega_{\psi_a}^+\qquad\text{and}\qquad\theta_\chi^-=\theta(\mathrm{sgn}_V, W_1)=\omega_{\psi_a}^-.$$

Note that the representations $\theta_\chi^\pm$ and sp_χ for different trivial quadratic characters χ of $F^\times$ are all distinct. The example just discussed accounts for all of the reducibility of the principal series of $Mp(W_1)$, and so the set $\mathrm{Irr}(Mp(W_1))$ consists of the (isomorphism classes of the) θ_χ^+'s, sp_χ's, irreducible principal series, and supercuspidals (including the θ_χ^-'s).

Example 1.6. Similarly, suppose that V is the *anisotropic* quadratic space with $\dim_F V=3$ and with $\chi_V=\chi$, [63], [82], [41], [42], [43]. Then $n(\mathbb{1}_V)=0$ and $n(\mathrm{sgn}_V)=2$. For any other irreducible representation $\tau\in\mathrm{Irr}(V)$, we have $\{n(\tau), n(\tau\otimes\mathrm{sgn})\}=\{1,2\}$. Thus, for each irreducible representation τ_1 of $SO(V)$, precisely one of the two extensions of τ_1 to $O(V)=SO(V)\times\mu_2$ yields a supercuspidal representation of $Mp(W_1)$, while the other yields a supercuspidal representation of $Mp(W_2)$. We will label these two extensions by τ_χ and $\tau_\chi\otimes\mathrm{sgn}$. This labeling depends on the choice of χ in a rather subtle way, [82], even though the group $SO(V)$ depends only on the quadratic form on V up to scaling. Thus, with this labeling, we have

$$\theta(\tau_\chi, W_1)\in\mathrm{Irr}_{sc}(Mp(W_1)),$$

and

$$\theta(\tau_\chi\otimes\mathrm{sgn}, W_2)\in\mathrm{Irr}_{sc}(Mp(W_2)).$$

Finally, note that $R_1(V) \subset I_1(\frac{1}{2}, \chi_V)$ is precisely the special representation sp_χ defined in Example 1.5, so that

$$\theta(1\!\!1_V, W_1) = \mathrm{sp}_\chi.$$

Example 1.7. For a fixed quadratic character χ, consider the theta correspondence for $Mp(W_1)$ with the pair of odd dimensional Witt towers determined by χ, cf. Example V.1.3. For any $\pi \in \mathrm{Irr}(Mp(W_1))$ the possibilities for $\{m_\chi^+(\pi), m_\chi^-(\pi)\}$ are $\{1,7\}$, $\{3,5\}$ and $\{5,3\}$. From Example 1.5 and 1.6, we see that:

$\pi \in \mathrm{Irr}(Mp(W_1))$	$m_\chi^+(\pi)$	$m_\chi^-(\pi)$	
θ_χ^+	1	7	analogue of trivial rep.
θ_η^+	3	5	$\eta \neq \chi$
θ_χ^-	1	7	a supercuspidal (!)
θ_η^-	?	?	$\eta \neq \chi$
$I_1(s,\mu)$	3	5	irred. pr. series
sp_χ	5	3	
sp_η	3	5	$\eta \neq \chi$
supercuspidal	5	3	if $\pi = \theta(\tau_\chi, W_1)$
supercuspidal	3	5	otherwise

Note that we know that $m_\chi^+(I_1(s,\mu)) = 3$ because the first occurrence pair for this representation must be $\{3,5\}$ or $\{5,3\}$, but we know that only supercuspidal or special representations have $m_\chi^-(\pi) = 3$. (Or look at the Bernstein-Zelevinsky data...). The same comment holds for sp_η, with $\eta \neq \chi$.

Again, we have written down only those relations which follow directly from the conservation principal, etc. There are many additional details to fill in. For example, it is not difficult to determine the images of θ_χ^+ and θ_χ^- in the $+$ tower. A more delicate point is to determine whether or not θ_η^- can have the form $\theta(\tau_\chi, W_1)$ where τ_χ is as in Example 1.6.

A detailed study of the supercuspidals of $Mp(W_1)$ occuring in the correspondence with various $O(3)$'s can be found in [63], [77], and [manderone], [42], [43].

Problem: Give a reasonable description of the correspondence for $Mp(W_2)$ and the pair of Witt towers determined by χ. In particular, work out what happens with the two $O(5)$'s!!

VI.2. Chains of supercuspidals. The combination of the conservation/dichotomy principles and the fact that the first occurrence of s supercuspidal representation in some Witt tower is again supercuspidal yields a rather curious phenomenon, the existence of chains of supercuspidals. In general, supercuspidal representations are relatively difficult to construct, and are usually obtained via indiction from compact open subgroups. We do not attempt to cite the extensive literature in this area, but simply mention the articles [49], [48], and [8], which discuss the case of symplectic groups. Such constructions require rather more delicate techniques than we have employed in these notes. Nonetheless, it turns out that one can use the theta correspondence to generate infinite families of supercuspidals (on larger and larger groups) beginning with one such representation! Such chains were first observed for unitary groups in [22].

Fix a quadratic character χ and the associated pair of Witt towers of even dimensional quadratic spaces. We will slightly change our usual notation and let $V_m^{\pm}$ be the space of *dimension* m in the $\pm$ tower. We will sometime drop the superscript $\pm$, when we are not concerned with about a particular tower. For a fixed dimension $m = 2n$ and a supercuspidal representation $\tau \in \mathrm{Irr}_{sc}(O(V_m^+))$, recall that $n(\tau) + n(\tau \otimes \mathrm{sgn}) = m$. Here the choice of the $+$ tower was simply made for convenience. Suppose that for $\tau = \tau_0$ we have $n(\tau) = n(\tau \otimes \mathrm{sgn}) = n$, and let $\pi_0 = \theta(\tau_0, W_n) \in \mathrm{Irr}_{sc}(Sp(W_n))$ be the corresponding supercuspidal representation. Applied to π_0, Theorem V.1.9 yields $m_\chi^+(\pi_0) + m_\chi^-(\pi_0) = 4n + 4$. Since, by construction, $m_\chi^+(\pi_0) = 2n$, we must have $m_\chi^-(\pi_0) = 2n + 4$. Let $\tau_1 = \theta(\pi_0, V_{2n+4}^-) \in \mathrm{Irr}_{sc}(O(V_{2n+4}^-))$ be the corresponding supercuspidal representation. Again, by Theorem V.2.4, we have $n(\tau_1) + n(\tau_1 \otimes \mathrm{sgn}) = 2n + 4$, and, by construction, $n(\tau_1) = n$. Thus, $n(\tau_1 \otimes \mathrm{sgn}) = n+4$, and we obtain a supercuspidal representation

$$\theta(\tau_1 \otimes \mathrm{sgn}, W_{n+4}) = \pi_1 \in \mathrm{Irr}_{sc}(Sp(W_{n+4})).$$

Continuing in this fashion, we obtain a **chain of supercuspidals:**

$$\tau_0,\ \pi_0,\ \tau_1,\ \pi_1,\ \tau_2,\ \pi_2, \ldots$$

which are representations of the groups:

$$O(V_{2n}^+),\ Sp(W_n),\ O(V_{2n+4}^-),\ Sp(W_{n+4}),\ O(V_{2n+16}^+),\ Sp(W_{n+12}), \ldots\ .$$

Note that we are using Theorem III.6.1 and Theorem III.6.2 repeatedly and that the sequence of dimensions is determined by the conservation principles, Theorem V.1.9 and Theorem V.2.4.

The simplest example of such a chain begins with $\chi \neq 1$, and an irreducible two dimensional representation $\tau_0 \in \mathrm{Irr}_{sc}(O(V_2^+))$. Such a representation is attached to any character η of E^1, for which $\eta^2 \neq 1$. Here E/F is the quadratic extension of F associated to the character χ. The resulting chain of supercuspidals is

$$\tau_0,\ \pi_0,\ \tau_1,\ \pi_1,\ \tau_2,\ \pi_2, \dots$$

which are representations of the groups:

$$O(V_2^+),\ Sp(W_1),\ O(V_6^-),\ Sp(W_5),\ O(V_{18}^+),\ Sp(W_{13}), \dots\ .$$

The representation π_0 occurred in Example 1.2 above. Note that we could have taken $\tau^- \in \mathrm{Irr}_{sc}(O(V_2^-))$ as starting point, and we would then obtain a chain of supercuspidals for the opposite orthogonal group at each step. These two chains are simply related by the involution δ_a, for $\chi(a) = -1$ for the symplectic groups and by the identification of $\tau \in \mathrm{Irr}_{sc}(O(V_m))$ with $\tau^\pm \in \mathrm{Irr}_{sc}(O(V_m^\pm))$ for the orthogonal groups $O(V_m) = O(V_m^\pm)$.

Similarly, if $\chi = 1$, we can construct a chain of supercuspidals for the groups

$$O(V_4^-),\ Sp(W_2),\ O(V_8^+),\ Sp(W_6),\ O(V_{20}^-),\ Sp(W_{14}), \dots$$

where τ_0 is an irreducible (finite dimensional) representation of the compact orthogonal group $O(V_4^-)$ which does not occur in the theta correspondence with $Sp(W_1)$, i.e., for which $n(\tau_0) = n(\tau_0 \otimes \mathrm{sgn}) = 2$.

A second type of chain arises from any supercuspidal representation $\pi = \pi_0 \in \mathrm{Irr}_{sc}(Sp(W_n))$ for which

$$\{m_\chi^+(\pi), m_\chi^-(\pi)\} = \{2n+2, 2n+2\}.$$

We leave it as an exercise for the reader to determine the sequence of groups involved here and to show that this type of chain is distinct from the previous one.

Problem: Characterize the supercuspidal representations occuring at the base of such chains intrinsically.

Finally, we consider the analogous phenomenon in the case of odd dimensional Witt towers. Again, fix a character χ and the associated pair of Witt towers.

Suppose that $\tau_0 \in \mathrm{Irr}_{sc}(V_{2n+1}^\epsilon))$, $\epsilon = \pm$, $\dim_F V_{2n+1}^\epsilon = 2n+1$, is a supercuspidal representation the odd orthogonal group $O(V_{2n+1}^\epsilon)$, for which

$$n(\tau_0) = n, \qquad \text{and} \qquad n(\tau_0 \otimes \mathrm{sgn}) = n+1.$$

We then obtain a chain of supercuspidals:

$$\tau_0,\ \pi_0,\ \tau_1,\ \pi_1,\ \tau_2,\ \pi_2,\ \dots \quad ,$$

on the groups

$$O(V^{\epsilon}_{2n+1}),\ Sp(W_n),\ O(V^{-\epsilon}_{2n+3}),\ Sp(W_{n+3}),\ O(V^{\epsilon}_{2n+13}),\ Sp(W_{n+10}),\ \dots \quad .$$

These dimensions are determined by a repeated application of the conservation principle. On the other hand, if we were to take $\theta(\tau_0 \otimes \mathrm{sgn}, W_{n+1})$, and proceed, we would get a chain of supercuspidals

$$\tau_0 \otimes \mathrm{sgn},\ \pi'_0,\ \tau'_1,\ \pi'_1,\ \tau'_2,\ \pi'_2,\ \dots \quad ,$$

for the groups

$$O(V^{\epsilon}_{2n+1}),\ Sp(W_{n+1}),\ O(V^{-\epsilon}_{2n+7}),\ Sp(W_{n+6}),\ O(V^{\epsilon}_{2n+21}), Sp(W_{n+15}),\ \dots \quad .$$

For example, consider $\dim_F V_1^+ = 1$, so that the *only* representations of $O(V_1^+)$ are $1\!\!1_{V_1^+}$ and $\mathrm{sgn}_{V_1^+}$. Since $n = 0$ in this case, we get the representations are

$$1\!\!1_{V_1^+},\ \theta^+_\chi,\ 1\!\!1_{V_3^-},\ \pi_1,\ \tau_2,\ \pi_2,\ \dots \quad ,$$

of the groups

$$O(V_1^+),\ Mp(W_0),\ O(V_3^-),\ Mp(W_3),\ O(V_{13}^+),\ Mp(W_{10}), \dots \quad .$$

If we start with $\mathrm{sgn}_{V_1^+}$ we get representations

$$\mathrm{sgn}_{V_0},\ \theta^-_\chi,\ \tau'_1,\ \pi'_1,\ \tau'_2,\ \pi'_2,\ \dots \quad ,$$

of the groups

$$O(V_1^+),\ Sp(W_1),\ O(V_7^-),\ Sp(W_6),\ O(V_{21}^+),\ Sp(W_{15}),\ \dots \quad .$$

These rather peculiar supercuspidal representations do not seem to have received much attention.

VI.3. Unramified representations For the moment, we will restrict our discussion to the case of even dimensional quadratic spaces. Recall that $\mathcal{O}_F$ is the ring of integers in F and $\mathcal{P} = \mathcal{O}_F \cdot \varpi$ is the maximal ideal, with uniformizer ϖ. Also, $q = |\mathcal{O}_F/\mathcal{P}|$, and the valuation ord and absolute value $|\ |$ are normalized so that

$\mathrm{ord}(\varpi) = 1$ and $|\varpi| = q^{-1}$. We assume that the additive character ψ is chosen to be trivial on $\mathcal{O}_F$ and nontrivial on $\mathcal{P}^{-1}$.

Recall that, for a p-adic reductive group G, a representation $\pi \in \mathrm{Irr}(G)$ is unramified if it has a nonzero vector invariant under a (good hyperspecial) maximal compact subgroup $K \subset G$, [12], [15]. A basic result of Howe [24], [26], is that, in the theta correspondence for unramified dual pairs (G, H), [46], unramified representations are matched with unramified representations. This fact, together with the standard description of unramified representations in terms of their Satake parameters and the known behavior of Bernstein-Zelevinsky data, Theorems III.7.1 and 7.2, yields an explicit description of the matching of unramified representations in the theta correspondence. This description plays an essential role in the global theory. The unramified correspondence can be best expressed in terms of the Langlands L-groups and a suitable L-homomorphism. This observation is due to Rallis, [59]. However, the full theta correspondence is more subtle, and its relationship to Langlands functoriality is not yet fully understood.

For an even dimensional quadratic space V, the dual pair $(O(V), Sp(W))$ is unramified if and only if (i) the residue characteristic for F is not 2, and (ii) χ_V is unramified, and $V = V_r^+$, i.e., V lies in the $+$ Witt tower, cf. Examples V.1.1 and V.1.2. Thus, either $\chi_V = 1$ and $V = V_{r,r}$ is split, or χ_V is asociated to an unramified quadratic extension E/F and $V = V_0 + V_{r,r}$ where $V_0 = E$ with quadratic form given by the norm $q(x) = N_{E/F}(x)$.

For a symplectic space $W = W_n$, let $\mathbf{e}$ be a standard symplectic basis, as in Chapter I.4, so that $Sp(W_n) \hookrightarrow GL_{2n}(F)$ (cf. material after Lemma I.4.2). Let $K = K_{\mathbf{e}} = Sp(W_n) \cap GL_{2n}(\mathcal{O}_F)$, so that K is the maximal compact open subgroup of $Sp(W_n)$ which preserves the lattice

$$L_W = \mathcal{O}_F \cdot \mathbf{e} = \mathcal{O}_F e_1 + \cdots + \mathcal{O}_F e_n'.$$

This lattice is self dual in the sense that

$$L_W = \{\ w \in W \mid \psi(< w, x >) = 1, \forall x \in L_W\ \}.$$

For the quadratic space

$$V = \begin{cases} V_{r,r} & \text{if } \chi_V = 1, \\ V_0 + V_{r,r} & \text{if } \chi_V \neq 1, \end{cases}$$

fix a standard basis $v_1, \ldots, v_r, v_1', \ldots, v_r'$ in $V_{r,r}$ and let

$$L_{r,r} = \mathcal{O}_F \cdot v_1 + \cdots + \mathcal{O}_F \cdot v_r'$$

be the lattice they span. If $\chi_V \neq 1$, so that $V_0 = E$, let $L_0 = \mathcal{O}_E$ be the ring of integers in E. Then let

$$L_V = \begin{cases} L_{r,r} & \text{if } \chi_V = 1, \\ L_0 + L_{r,r} & \text{if } \chi_V \neq 1. \end{cases}$$

Again, L_V is self dual:

$$L_V = \{\ v \in V \mid \psi((w,x)) = 1, \forall x \in L_V\ \}.$$

Let $B = B_W = P_{(1,\ldots,1)}$ be the minimal parabolic subgroup of $Sp(W)$ which preserves the isotropic generated by $e'_1, \ldots, e'_n$, as Chapter III.2. There is an Iwasawa decompsition $Sp(W) = BK_W$. For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of *unramified* characters of $F^\times$, the induced representation $I_B^{Sp(W)}(\lambda)$ has a unique K fixed line, spanned by the function f_λ whose restriction to K is 1. If $I_B^{Sp(W)}(\lambda)$ is irreducible, we let

$$\pi(\lambda) = I_B^{Sp(W)}(\lambda).$$

In general, $I_B^{Sp(W)}(\lambda)$ has a unique constituent which has a K fixed vector, and we write $\pi(\lambda)$ for this constituent.

Analogously, let $Q = Q_V = Q_{\mathbf{1}} = Q_{(1,\ldots,1)}$ be the minimal parabolic subgroup of $O(V)$ determined by the basis chosen above, cf. Chapter III.2, and note that the Levi factor of Q is isomorphic to

$$M_Q \simeq \underbrace{GL_1(F) \times \cdots \times GL_1(F)}_{r \text{ times}} \times \begin{cases} 1 & \text{if } \chi_V = 1, \\ O(V_0) & \text{if } \chi_V \neq 1. \end{cases}$$

We have an Iwasawa decomposition $O(V) = QK_V$. Note that, when $\chi_V \neq 1$, the subgroup $O(V_0)$ of M_Q is compact and lies in K_V. For a sequence of unramified characters $\mu = (\mu_1, \ldots, \mu_r)$, let $I_Q^{O(V)}(\mu)$ be the induced representation. Here, in the case $\chi_V \neq 1$, we extend μ to a character of M_Q, trivial in $O(V_0)$. Again, $I_Q^{O(V)}(\mu)$ has a unique K_V invariant line, spanned by the function f_μ which takes the value 1 on K_V. We write $\tau(\mu)$ for the unique constituent of $I_Q^{O(V)}(\mu)$ containing a nonzero K_V fixed vector.

A basic fact is the following, [15],[14].

Proposition 3.1. *Every irreducible admissible representation of $Sp(W)$ (resp. $O(V)$) which has a nonzero K_W (resp. K_V) fixed vector is isomorphic to one of the $\pi(\lambda)$'s (resp. $\tau(\mu)$'s). Two representations $\pi(\lambda)$ and $\pi(\lambda')$ (resp. $\tau(\mu)$*

and $\tau(\mu')$ *) are isomorphic if and only if* λ *and* λ' *(resp.* μ *and* μ' *) lie in the same orbit under the action of the Weyl group, i.e., agree up to permutations and inversions of components.*

Returning to the theta correspondence,

$$\begin{array}{rcl} \mathrm{Howe}_\psi(O(V);Sp(W)) & \overset{\sim}{\longleftrightarrow} & Howe_\psi(Sp(W);O(V)) \\ \tau & \mapsto & \theta(\tau,W) \\ \theta(\pi,V) & \leftarrow\!\shortmid & \pi. \end{array}$$

we have

$$\begin{array}{rcl} \mathrm{Howe}_\psi(O(V);Sp(W))^{\mathrm{ur}} & \overset{\sim}{\longleftrightarrow} & Howe_\psi(Sp(W);O(V))^{\mathrm{ur}} \\ \tau & \mapsto & \theta(\tau,W) \\ \theta(\pi,V) & \leftarrow\!\shortmid & \pi. \end{array}$$

where the superscript denote the subset of $\mathrm{Howe}_\psi(O(V);Sp(W))$ (resp. $Howe_\psi(Sp(W);O(V))$) of representations which have nonzero K_V (resp. K_W) fixed vectors.

Let $V = V_r$ with $\dim_F V = m$, and let $W = W_n$. The next result gives an explicit description of the unramified part of the correspondence. It was proved by Rallis, [59], by a direct calculation.

Proposition 3.2. *(i) Suppose that* $n \geq r$*, then, with the notation above*

$$\theta(\tau(\mu), W_n) = \pi(\theta(\mu)),$$

where

$$\theta(\mu) = (\chi_V\mu_1, \dots, \chi_V\mu_r, \chi_V|\ |^{\frac{m}{2}-n}, \dots, \chi_V|\ |^{\frac{m}{2}-r-1}).$$

(ii) Suppose that $r \geq n$*. Then*

$$\theta(\pi(\lambda), V_r) = \tau(\theta(\lambda)),$$

where

$$\theta(\lambda) = (\chi_V\lambda_1, \dots, \chi_V\lambda_n, |\ |^{\frac{m}{2}-n}, \dots, |\ |^{\frac{1}{2}\dim V_0}),$$

where $\dim_F V_0 = 2$ *if* $\chi_V \neq 1$ *and* $\dim_F V_0 = 0$ *if* $\chi_V = 1$.

VI.4. L-groups and functoriality. The unramified correspondence can be described in a more striking way in terms of the Langlands L-group and Satake parameters, [12], [15], [59], [37]. For convenience, we will write $G = Sp(W_n)$ and $H = O(V_r)$. Write $m = \dim_F V = 2t$. Let W_F be the Weil group of F. Recall that, the group W_F sits in the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_F & \longrightarrow & \mathrm{Gal}(\bar{F}/F) & \longrightarrow & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) & \longrightarrow & 1 \\ & & || & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & I_F & \longrightarrow & W_F & \longrightarrow & \mathbb{Z} & \longrightarrow & 1, \end{array}$$

where I_F is the inertia group. Let Fr_q be the topological generator $x \mapsto x^q$ of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, and let $\phi_q \in W_F$ be an element which maps to Fr_q. Also,

$$W_F^{\mathrm{ab}} \xrightarrow{\sim} F^\times,$$

where W_F^{ab} is the abelianization of W_F. Thus, characters of W_F can be identified with characters of $F^\times$.

The L-group of G is then

$${}^L G = SO(2n+1, \mathbb{C}) \times W_F.$$

The standard definition of L-group only applies to a *connected* reductive group, [12], and thus some modified definition must be used for the orthogonal group $O(V)$. Following [1], we take the L-group of the (algebraically) *disconnected* group $H = O(V)$ to be

$${}^L H = O(m, \mathbb{C}) \times W_F,$$

where we note that the standard L-group ${}^L SO(V)$ is isomorphic to the subgroup

$${}^L SO(V) = \{(h, w) \in {}^L H \mid \det h = \chi_V(w)\}.$$

Here χ_V is viewed as a character of W_F. This variation on the definition of [1] is taken from [18].

The Satake parameter of an unramified representation of the p-adic points of a connected reductive group is a certain semisimple conjugacy class in the L-group, [15].

For $G = Sp(W_n)$, let $\hat{G} = SO(2n+1, \mathbb{C})$ be the special orthogonal group over $\mathbb{C}$ of the quadratic form defined by

$$\begin{pmatrix} 0 & 0 & 1_n \\ 0 & 1 & 0 \\ 1_n & 0 & 0 \end{pmatrix}.$$

The Satake parameter of the unramified representation $\pi(\lambda)$ is the conjugacy class of the element

$$\begin{pmatrix} \lambda(\varpi) & & \\ & 1 & \\ & & \lambda(\varpi)^{-1} \end{pmatrix} \times \phi_q \in SO(2n+1,\mathbb{C}) \times W_F.$$

Here,

$$\lambda(\varpi) = \mathrm{diag}(\lambda_1(\varpi),\dots,\lambda_n(\varpi)) = \begin{pmatrix} \lambda_1(\varpi) & & \\ & \ddots & \\ & & \lambda_n(\varpi) \end{pmatrix} \in GL_n(\mathbb{C}).$$

Note that this element lies in the diagonal torus $\hat{T}$ in $SO(2n+1,\mathbb{C})$. The action of the Weyl group of G on the principal series parameter λ coincides with the action of the Weyl group of $\hat{G}$ on the Satake parameter in ${}^LT$. In fact, it is better to associate to λ a $\hat{G}$ conjugacy class of homomorphisms

$$\varphi_\lambda : W_F \longrightarrow {}^LG,$$

defined by

$$\varphi_\lambda(w) = \begin{pmatrix} \lambda(w) & & \\ & 1 & \\ & & \lambda(w)^{-1} \end{pmatrix} \times w.$$

In this form, φ_λ can be defined for any (possibly ramified) principal series parameter λ. When λ is unramified, the homomorphism is determined by the Satake parameter, i.e., by $\varphi_\lambda(\phi_q)$. Here we are using the reciprocity law of local class field theory which asserts that the image of ϕ_q in $F^\times$ is ϖ, and that the image of I_F is $\mathcal{O}_F^\times$. Moreover, any conjugacy class of homomorphisms $\varphi : W_F \longrightarrow {}^LG,$, whose projection to W_F is the identity map, and whose projection to $\hat{G}$ is trivial on I_F and has image consisting of semisimple elements, is conjugate, via $\hat{G}$ to one of the φ_λ's. Thus:

Proposition 4.1. *There is a bijection between:*

$$\left\{ \begin{array}{c} \hat{G} \textit{ conjugacy classes of homomorphisms } \varphi : W_F \longrightarrow {}^LG \textit{ such that} \\ \textit{(i)} \quad \varphi(w) = \varphi(w)_1 \times w, \\ \textit{(ii)} \quad \varphi(w)_1 = 1 \textit{ for all } w \in I_F, \\ \textit{(iii) for all } w \in W_F,\ \varphi(w)_1 \textit{ is a semisimple element of } \hat{G}, \end{array} \right\}$$

and

$$\mathrm{Irr}(G)^{\mathrm{ur}},$$

the set of isomorphism classes of irreducible admissible representations of G *which have a nonzero* K *fixed vector.*

The orthogonal group case is slightly more subtle, due to the disconnectedness. and to the existence of two towers (two forms of the quadratic space). We take the group $\hat{H} = O(2t, \mathbb{C})$ to be the orthogonal group over $\mathbb{C}$ of the quadratic form defined by

$$\begin{pmatrix} 0 & 1_t \\ 1_t & 0 \end{pmatrix}.$$

If $\chi_V = 1$ so that $m = 2r = 2t$, the Satake parameter of the unramified representation $\tau(\mu)$ is the conjugacy class of the element

$$\begin{pmatrix} \mu(\varpi) & \\ & \mu(\varpi)^{-1} \end{pmatrix} \times \phi_q \in SO(2t, \mathbb{C}) \times W_F.$$

Note that this parameter actually lies in

$${}^L T = \hat{T} \times W_F \subset {}^L SO(V) = SO(2t, \mathbb{C}) \times W_F \subset LO(V).$$

Here $\hat{T}$ is the diagonal torus in $SO(2t, \mathbb{C})$. However, the equivalences for the parameter μ consist of *all* permutations and inversions of components, and these correspond to the action of the Weyl group for $O(2r, \mathbb{C})$ rather than for $SO(2r, \mathbb{C})$. Recall that the Weyl group for $SO(2t, \mathbb{C})$ (resp. $O(2t, \mathbb{C})$) is the quotient $N(\hat{T})/\hat{T}$, where $N(\hat{T})$ is the normalizer in $SO(2t, \mathbb{C})$ (resp. $O(2t, \mathbb{C})$) of the diagonal torus $\hat{T}$. In the case of $SO(2t, \mathbb{C})$, the Weyl group acts by permutations and inversions *of an even number* of coordinates of the torus, while in the $O(2t, \mathbb{C})$ case, permutations and any number of inversions are allowed. We will give an example in a moment. Again, it is better to consider the homomorphism $\varphi_\mu : W_F \to {}^L SO(V)$, defined as in the symplectic case. The analogue of Proposition 4.1 holds, where the homomorphisms $\varphi : W_F \to {}^L H$ is required to have image in ${}^L SO(V)$, while the conjugacy is taken with respect to $\hat{H} = O(2t, \mathbb{C})$.

Next suppose that $\chi_V \neq 1$, so that $\dim_F V = 2r + 2$ and $t = r + 1$. In this case, let E/F be the quadratic extension associated to χ_V. If we view χ_V as a character of W_F, and the kernel of χ_V is the Weil group W_E, i.e., the intersection of W_F with $\mathrm{Gal}(\bar{F}/E) \subset \mathrm{Gal}(\bar{F}/F)$.

Let $w_\sigma \in W_F$ be an element which maps to the nontrivial element $\sigma \in \mathrm{Gal}(E/F)$. Recall that $w_\sigma^2 = \alpha \in W_E$, where the image a of α in $W_E^{\mathrm{ab}} \simeq E^\times$ satisfies $\chi_V(a) = -1$. Here, and below, Tate's article [73] can be consulted for details.

Before considering the general case, let us work out what happens when $\dim_F V = m = 2$, so that $SO(V) = E^1$, the kernel of the norm map $N_{E/F}$, and

$$O(V) = E^1 \rtimes \mathrm{Gal}(E/F).$$

By Hilbert's Theorem 90, we have an exact sequence:

$$1 \longrightarrow F^\times \longrightarrow E^\times \longrightarrow E^1 \longrightarrow 1,$$

where the surjection on the right end is given by $x \mapsto x/x^\sigma$. For any character η of E^1, let *tilde*η be its pullback to $E^\times$. Note that

$$\tilde{\eta}^\sigma(x) := \tilde{\eta}(x^\sigma) = \tilde{\eta}(x)^{-1}.$$

Viewing ${}^L SO(V)$ as a subgroup of ${}^L O(V)$, as above, we define $\varphi_\eta : W_F \to {}^L SO(V)$ by

$$\varphi_\eta : \begin{cases} w \mapsto \begin{pmatrix} \tilde{\eta}(w) & \\ & \tilde{\eta}(w)^{-1} \end{pmatrix} \times w & \text{for } w \in W_E, \\ w_\sigma \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \times w_\sigma & \end{cases}$$

We leave it to the reader to check that φ_η is well defined and that every homomorphism $\varphi : W_F \to {}^L SO(V)$ is conjugate by $SO(2, \mathbb{C})$ to one of this form. This proves the very elementary version of the local Langland parameterization:

Lemma 4.2. *For* $\dim_F V = 2$, *there is a bijection between*

$$\left\{ \begin{matrix} \varphi : W_F \longrightarrow {}^L SO(V) \subset O(2, \mathbb{C}) \times W_F \\ \textit{modulo conjugation by } SO(2, \mathbb{C}) \ , \end{matrix} \right\}$$

and

$$\mathrm{Irr}(SO(V)).$$

We also leave it to the reader to check the $\chi_V = 1$ case of the Lemma.

The irreducible representations of $O(V)$ are either (i) the two dimensional representations $I^{O(V)}_{SO(V)}(\eta)$ irreducibly induced from characters η of $SO(V) = E^1$ with $\eta^2 \neq 1$, or (ii) the one dimensional constituents η' and $\eta' \otimes \mathrm{sgn}_V$ of the reducible representation $I^{O(V)}_{SO(V)}(\eta)$, where $\eta^2 = 1$. Note that, $I^{O(V)}_{SO(V)}(\eta) = I^{O(V)}_{SO(V)}(\eta^{-1})$. Now the very elementary version of the local Langland parameterization involves L-packets!

Lemma 4.3. *For* $\dim_F V = 2$ *, there is a bijection between*

$$\left\{\begin{matrix} \varphi : W_F \longrightarrow {}^L SO(V) \subset O(2,\mathbb{C}) \times W_F \\ \textit{modulo conjugation by } O(2,\mathbb{C}) \ , \end{matrix}\right\}$$

and

$$\textit{L-packets of irreducible admissible representations of } O(V) \ .$$

The L-packets are

$$\Pi_{\varphi_\eta} = \{ \ I_{SO(V)}^{O(V)}(\eta) \ \} \qquad \textit{if } \eta^2 \neq 1 \ ,$$

and

$$\Pi_{\varphi_\eta} = \{ \ \eta', \ \eta' \otimes \mathrm{sgn}_V \ \} \qquad \textit{if } \eta^2 = 1 \ .$$

Here η *is a character of* E^1 *, if* $\chi_V \neq 1$ *and a character of* $F^\times$ *, if* $\chi_V = 1$ *.*

These examples motivated the definition of ${}^L O(V)$ given in [1].

There is another feature of the parameterization just described which should be pointed out. Recall that, if $\chi_V \neq 1$, there are two quadratic spaces of dimension 2, $V^\pm$, with opposite Hasse invariant. The orthogonal groups $O(V^+)$ and $O(V^-)$ are identical as groups, but, as we have seen, it is best to distinguish them, since they play separate roles in the theta correspondence. In particular, each irreducible representation τ of $O(V)$ should be viewed as a pair of representations τ^+ of $O(V^+)$ and τ^- of $O(V^-)$, see Example 1.2 above. On the other hand, when $\chi_V = 1$, there is only one binary quadratic space, $V_{1,1}$. Thus , each $\tau \in \mathrm{Irr}(O(V))$ comes in only one version.

It turns out that the parameterizations of Lemma 4.2 and 4.3 encode both the multiplicities due to the various forms of the quadratic spaces and the multiplicities due to the L-packets. For a parameter $\varphi : W_F \to {}^L SO(V)$, let

$$C_\varphi = \{ \ g \in O(2,\mathbb{C}) \mid g\varphi(w)g^{-1} = \varphi(w), \ \forall w \in W_F \ \}$$

be the centralizer in $\hat{H}$ of the image of φ. Let C_φ^0 be the identity component of C_φ, and let

$$A_\varphi = C_\varphi / C_\varphi^0$$

be the group of components.

Exercise 4.4: (i) Let

$$Z = \mathrm{Cent}_{SO(2,\mathbb{C})}({}^L SO(V))$$

be the centralizer in $SO(2,\mathbb{C})$ of ${}^L SO(V)$, and let Z^0 be its identity component. Note that $Z \subset C_\varphi$ for any parameter φ. Show that Z/Z^0 has order 2 if $\chi_V \neq 1$

and is trivial if $\chi_V = 1$.
(ii) Show that A_φ has order either 1, 2 or 4.
(iii) Find a nice matching of the group $\hat{A_\varphi}$ of characters of $\mathbb{A}_\varphi$ with the elements of the 'extended' L-packet attached to φ. Here 'extended' means that we take include distinct τ^+ and τ^-'s in the case $\chi_V \neq 1$. In this matching, the restriction of a character ξ of A_φ to Z/Z^0 should determine whether to associated τ is a τ^+ or a τ^-.
(iv) Define the analogue of A_φ for $SO(V)$, and check that it reflects the structure of the 'extended' L-packets.

This example provides (a small piece of) motivation for the very general conjectures of Langlands concerning the parametrization of $\mathrm{Irr}(G)$ for general connected reductive groups G over local fields, [12]. An introduction to the conjectured picture for $GL_n(F)$ can be found in [34] (and, of course, many other places). For the case of $SL_2(F)$, where L-packets first arise, see [69], [39].

As indicated above, the formalism used for the disconnected orthogonal groups is due to Adams, [1]. The idea that the groups $O(V^+)$ and $O(V^-)$ should be considered separately in the parameterization is consistent with the very general proposals of Vogan, [76], concerning the Langlands parameterization.

Some very interesting and more elaborate calculations involving the conjectural L-packets and A_φ's for general orthogonal groups are found in [18].

After this long digression on the Langlands parameterization, we now return to the unramified principal series for $O(V)$ when $\chi_V \neq 1$. Recall that $t = r + 1$, and that ${}^L O(V) = O(2t, \mathbb{C}) \times W_F$. Given an unramified principal series representation $\tau(\mu)$, the associated Satake parameter is

$$\begin{pmatrix} \mu(\varpi) & & \\ & 1_2 & \\ & & \mu(\varpi)^{-1} \end{pmatrix} \times \phi_q \in SO(2t, \mathbb{C}) \times W_F.$$

To define the associated homomorphism $\varphi_\mu : W_F \to {}^L SO(V)$, we use the procedure illustrated in the two dimensional case. For $\tilde{\eta}$ a character of $E^\times$ with $\tilde{\eta}^\sigma = \tilde{\eta}^{-1}$, let

$$\varphi_{\mu,\eta}:\begin{cases} w\mapsto\begin{pmatrix}\mu(w)&&&\\&\tilde{\eta}(w)&&\\&&\tilde{\eta}(w)^{-1}&\\&&&\mu(w)^{-1}\end{pmatrix}\times w & \text{for } w\in W_E,\\ w_\sigma\mapsto\begin{pmatrix}\mu(w_\sigma)&&&\\&&1&\\&1&&\\&&&\mu(w_\sigma)^{-1}\end{pmatrix}\times w_\sigma. \end{cases}$$

This parameter will be associated to the induced representation

$$I_Q^{O(V)}(\mu\otimes\tau_\eta).$$

This homomorphism makes sense for any r-tuple of characters μ and for any η. Taking μ unramified and $\eta=1$, we obtain the homomorphism φ_μ associated to the unramified representation $\tau(\mu)$.

Proposition 4.5. *There is a bijection between:*

$$\left\{\begin{array}{c} O(2t,\mathbb{C}) \textit{ conjugacy classes of homomorphisms } \varphi: W_F\longrightarrow {}^LSO(V) \textit{ such that}\\ \textit{(i)}\quad \varphi(w)=\varphi(w)_1\times w,\\ \textit{(ii)}\quad \varphi(w)_1=1 \textit{ for all } w\in I_F,\\ \textit{(iii) for all } w\in W_F,\ \varphi(w)_1 \textit{ is a semisimple element of } O(2t,\mathbb{C}), \end{array}\right\}$$

and

$$\mathrm{Irr}(O(V))^{\mathrm{ur}},$$

the set of isomorphism classes of irreducible admissible representations of $O(V)$ *which have a nonzero* K_V *fixed vector.*

Note that, if χ_V is ramified, so that $w_\sigma\in I_F$, the fact that $\det(\varphi_1(w_\sigma)) = \chi_V(w_\sigma) = -1$ implies that condition (ii) can never be satisfied, so there are no unramified parameters (as expected). Every unramified parameter φ is conjugate by $O(2t,\mathbb{C})$ to one of the φ_μ's. Also note that, in the conjectural Langlands parameterization, a conjugacy class of φ's should correspond to an L-packet of representations, but $\tau(\mu)$ will be the only representation in this finite set which has a K_V fixed vector.

Finally, we return to the theta correspondence, and we give a description of the unramified part of the correspondence for $G=Sp(W_n)$ and $H=O(V_m)$ in terms of the L-groups ${}^L G$ and ${}^L H$. To do this, it will be convenient to change coordinates in our L-groups. We write $\dim_F V = m = 2t$, as before.

Case 1. Assume that $1 \le t \le n$. We take $\hat{G} = SO(2n+1)$ to be the special orthogonal group over $\mathbb{C}$ of the quadratic form defined by

$$\begin{pmatrix} 0 & 1_n & 0 \\ 1_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and we take $\hat{H} = O(2t, \mathbb{C})$ to be the orthogonal group over $\mathbb{C}$ of the quadratic form defined by

$$\begin{pmatrix} 0 & 1_t \\ 1_t & 0 \end{pmatrix}.$$

Then define a homomorphism $C : {}^L H \to {}^L G$ by

$$C(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times 1) = \begin{pmatrix} a & & b & 0 & 0 \\ & 1_{n-t} & & 0 & 0 \\ c & & d & 0 & 0 \\ 0 & 0 & 0 & 1_{n-t} & \\ 0 & 0 & 0 & & \det(h) \end{pmatrix} \times 1,$$

where $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2t, \mathbb{C})$, and

$$C(1 \times w) = \begin{pmatrix} \chi_V(w) \cdot 1_t & & & & \\ & D(w) & & & \\ & & \chi_V(w) \cdot 1_t & & \\ & & & D(w)^{-1} & \\ & & & & 1 \end{pmatrix} \times w,$$

where

$$D(w) = \chi_V(w) \cdot \mathrm{diag}(|w|^{n-t}, |w|^{n-t-1}, \dots, |w|).$$

Case 2. Assume that $t \ge n+1$. We take $SO(2n+1, \mathbb{C})$ to be defined by the quadratic form

$$\begin{pmatrix} & & 1_n \\ & 1 & \\ 1_n & & \end{pmatrix},$$

and we take $O(2t, \mathbb{C})$ to be defined by the quadratic form

$$\begin{pmatrix} 0 & & 1_n & & & \\ & 1 & & & & \\ 1_n & & 0 & & & \\ & & & 0 & & 1_{t-n-1} \\ & & & & 1 & \\ & & & 1_{t-n-1} & & 0 \end{pmatrix}.$$

Then we define $C' : {}^L G \to {}^L H$ by

$$C'(g \times 1) = \begin{pmatrix} g & \\ & 1_{2t-2n-2} \end{pmatrix} \times 1,$$

and

$$C'(1 \times w) = \begin{pmatrix} \chi_V(w.1_{2n+1} & & & \\ & D'(w) & & \\ & & 1 & \\ & & & D'(w)^{-1} \end{pmatrix} \times w,$$

where

$$D'(w) = \mathrm{diag}(|w|^{n+1-t}, \ldots, |w|^{-1}).$$

Theorem. (Rallis, [59]) *The unramified part of the theta correspondence for the dual pair $G = Sp(W_n)$ and $H = O(V)$, where $\dim_F V = 2t$ and χ_V is unramified, can be described by a homomorphism of L-groups.*
(i) If $1 \leq t \leq n$, let $C : {}^L H \to {}^L G$ be the L-homomorphism defined above. If $\varphi : W_F \to {}^L H$ is an unramified parameter associated to the unramified representation $\tau(\varphi)$ of $O(V)$, then $\theta(\tau(\varphi), W_n)$ is the unramified representation of $G = Sp(W_n)$ associated to the unramified parameter $C \circ \varphi : W_F \to {}^L G$.
(ii) If $t \geq n+1$, let $C' : {}^L G \to {}^L H$ be the L-homomorphism defined above. If $\varphi : W_F \to {}^L G$ is an unramified parameter associated to the unramified representation $\pi(\varphi)$ of $G = Sp(W_n)$, then $\theta(\pi(\varphi), V)$ is the unramified representation of $H = O(V)$ associated to the unramified parameter $C' \circ \varphi : W_F \to {}^L H$.

This version of the the Theorem is taken from section7 of [37].

At the moment, the following list of references is nowhere near complete. For example, almost no references have been given for work on the archimedean local theta correspondence. Only a few sporadic works on the global correspondence are mentioned and the considerable literature involving global applications, especially to L-functions, cohomology of arithmetic groups and arithmetic is almost completely absent.

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