APM 421/ MAT 1723 fall 2003

These notes summarize some basic facts about the divergence theorem. First, some definitions:

A **vector field** on $\mathbb{R}^d$ is a function $v : \mathbb{R}^d \to \mathbb{R}^d$. It can be written in components as $(v^1, \ldots, v^d)$, where each v^i is an ordinary scalar function $\mathbb{R}^d \to \mathbb{R}$.

If Ω is an open subset of $\mathbb{R}^d$, then a vector field on Ω is a function $\Omega \to \mathbb{R}^d$.

A vector field is C^k if all the component functions v^i are C^k, ie k times continuously differentiable.

The **divergence** of v is $\sum_{i=1}^d \partial v^i / \partial x_i$. It is written in various ways, including $\operatorname{div} v$ and $\nabla \cdot v$. I am likely to use both notations interchangeably.

Theorem 1 (The Divergence Theorem) *Suppose that Ω is a bounded open subset of $\mathbb{R}^d$ with C^1 boundary, and that v is a C^1 vector field on Ω. Then*

$$\int_\Omega \nabla \cdot v \, dx = \int_{\partial\Omega} v \cdot \nu \, d\sigma$$

where $\nu(x)$ denotes the outer unit normal to $\partial\Omega$ at $x \in \partial\Omega$, and $d\sigma$ denotes integration with respect to $d-1$-dimensional "surface area" on $\partial\Omega$.

Briefly, $\partial\Omega$ is C^k if, for every point $x \in \partial\Omega$, one acn find a neighborhood of x in which $\partial\Omega$ can be represented locally as the graph of a C^k function. In particular, if $\partial\Omega$ is C^1, then there is a well-defined outer unit normal $\nu(x)$ at every point $x \in \partial\Omega$, and the map $x \in \partial\Omega \mapsto \nu(x)$ is continuous.

Versions of the theorem are true under considerably weaker hypotheses on v and Ω, but we will not pursue that here.

Remark 1: When $d = 1$, the basic case is when Ω is an open interval (a, b). Then $\partial\Omega$ is the set containing the two endpoints, $\{a, b\}$. The outer unit unit normal at $x = a$ is a vector (with one component) of length 1, pointing away from $\Omega = (a, b)$: this is the "vector" (-1). Similarly, the outer unit normal at $x = b$ is the "vector" (1). So in this case the divergence theorem reduces to

$$\int_a^b v'(x) dx = v(b) - v(a)$$

ie (part of) the fundamental theorem of calculus.

Now we list a number of consequences of the divergence theorem.

Corollary 1 *A version of the divergence theorem is still true (with analogous hypotheses) if we allow v to be a complex vector field, ie a function $v : \Omega \to \mathbb{C}^d$.*

Corollary 1 is proved by applying the divergence theorem to the real and imaginary parts of v respectively. In the same way, all the results I state below are valid both for real-valued and complex-valued functions.

Corollary 2 *Suppose that Ω is a bounded open subset of $\mathbb{R}^d$ with C^1 boundary, that w is a C^1 vector field on Ω, and that f is a C^1 function on Ω. Then*

$$\int_\Omega \nabla f \cdot v \, dx = -\int_\Omega f \nabla \cdot v + \int_{\partial\Omega} f v \cdot \nu \, d\sigma.$$

This follows by applying the divergence theorem to $v = fw$ and rewriting. Similar to Remark 1, in 1 dimensions this becomes the familiar integration by parts formula.

Corollary 3 *Suppose that Ω is a bounded open subset of $\mathbb{R}^d$ with C^1 boundary, and that f, g are C^1 functions on Ω. Then for every $i \in \{1, \dots, d\}$*

$$\int_\Omega \frac{\partial f}{\partial x_i} g \, dx = -\int_\Omega f \frac{\partial g}{\partial x_i} \, dx + \int_{\partial\Omega} f g \nu^i \, d\sigma.$$

where ν^i denotes the ith component of the outer unit normal ν.

This follows by applying the previous corollary with $w = (0, \dots, , 0, g, 0, \dots 0)$ (where g is in the "ith position").

Corollary 4 *Let f and g be C^1 functions with compact support, and let v, w be C^1 compactly supported vector fields in $\mathbb{R}^d$. Then the following hold:*

$$\int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i} g \, dx = -\int_{\mathbb{R}^d} f \frac{\partial g}{\partial x_i} \, dx,$$

$$\int_{\mathbb{R}^d} \nabla f \cdot w \, dx = -\int_{\mathbb{R}^d} f \nabla \cdot w$$

$$\int_{\mathbb{R}^d} \nabla \cdot v \, dx = 0.$$

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This follows by applying the earlier corollaries on the domain say $B_R(x) := \{x \in \mathbb{R}^d \ : \ |x| < R\}$ for R so large that $f = g = 0$ and $v = w = 0$ in the set $\{x \in \mathbb{R}^d \ : \ |x| > R/2\}$.

Remark 2: We have deduced corollaries 2, 3 from the theorem 1. Conversely, Theorem 1 can easily be derived from corollary 2 or 3. So these should all be seen as equivalent statements of the divergence theorem.

Remark 3: The starting point of the proofs of the all above results is 1-dimensional integration by parts. As a number of students observed, using this to prove Corollary 4 is at the level of a homework exercise (eg, for this class, some problem in assignment 1, if you did not know or did not feel like quoting the divergence theorem.)

Proofs of the theorem with the correct boundary term included are a bit harder but still can be based on the 1-dimensional case of integration by parts.

1. A first proof goes along these lines:

1a. prove Corollary 3 by using 1-dim integration by parts in the x_i variable, with all other variables fixed, then integrating in the transverse directions and keeping careful track of the boundary terms. When these are rewritten as a surface integral over $\partial\Omega$ one obtains the term $\int_{\partial\Omega} f g \nu^i d\sigma$.

1b. use Corollary 3 to prove the other results. This is almost immediate, as remarked above.

2. A second strategy for proving the divergence theorem is to

2a. first use 1-dim integration by parts to prove prove Corollary 4. This is easier than trying to prove Corollary 3 directly, because one doesn't have to worry about those pesky boundary terms

2b. Given an open set Ω with C^1 boundary, apply the second identity in Corollary 4 with f replaced by χ_ε, where χ_ε is a sequence in C_0^∞ such that $\chi_\varepsilon(x) \searrow \chi(x) := \left\{ \begin{array}{ll} 1 & \text{if } x \in \Omega, \\ 0 & \text{otherwise} \end{array} \right.$, as $\varepsilon \to 0$. By carefully taking limits as $\varepsilon \to 0$ one obtains Theorem 1.