Classifying Subgroups of $\mathrm{PSL}_2(\mathbb{R})$ by Hyperbolic Isometries

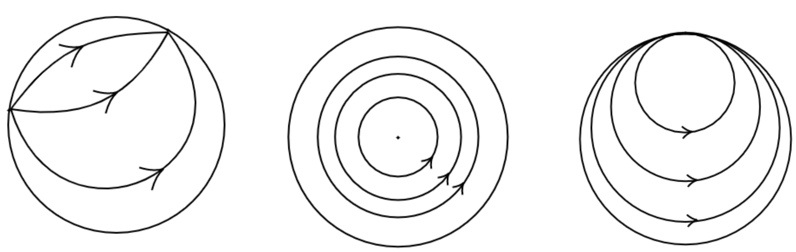
Siyan He

Dec 2020

1 Classification of Hyperbolic Isometries

We can realize elements of $\mathrm{PSL}_2(\mathbb{R})$ as orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2$. Specifically, a matrix $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}^2$ by the Möbius transformation $\frac{az+b}{cz+d}$, which sends the upper-half plane to itself. Notice that A and $-A$ would give the same Möbius transformation, so this action is well defined.

This allows us to study the behaviour elements of $\mathrm{PSL}_2(\mathbb{R})$ by examining hyperbolic isometries. The following figure summarizes the three types of hyperbolic isometries



(a) A hyperbolic isometry has two fixed points on the boundary. Points move from one fixed points to the other. Hyperbolic isometries correspond to $A \in \mathrm{PSL}_2(\mathbb{R})$ where $|\mathrm{tr}(A)| > 2$

(b) An elliptic isometry has one fixed point in the interior. Points move around the fixed point. Elliptic isometries correspond to $A \in \mathrm{PSL}_2(\mathbb{R})$ where $|\mathrm{tr}(A)| = 2$

(c) A parabolic isometry has one fixed point on the boundary. Points move around the fixed point. Parabolic isometries correspond to $A \in \mathrm{PSL}_2(\mathbb{R})$ where $|\mathrm{tr}(A)| < 2$

Figure 1: Classification of hyperbolic isometries

We will now use elements in $\mathrm{PSL}_2(\mathbb{R})$ and their corresponding isometries interchangeably. We will now examine the subgroups generated by two hyperbolic, two parabolic, and two elliptic elements.

2 Hyperbolic and Parabolic Case

If we have two hyperbolic or parabolic elements a and b, we can use the Ping-pong lemma to show that some power of them generate a free group.

Theorem 2.1 (Ping-pong Lemma). *Suppose a and b generate a group G that acts on a set X. If X has disjoint nonempty subsets X_a and X_b such that $a^k(X_b) \subset X_a$ and $b^k(X_a) \subset X_b$ for all nonzero k, then $G \cong \mathbb{F}_2$.*

When we have two hyperbolic elements a and b, each of them has two fixed points. Let X_a be the union of some neighborhoods of fixed points of a, and let X_b be the union of some neighborhoods of fixed points of b. Choose the neighborhoods small enough so that they are disjoint.

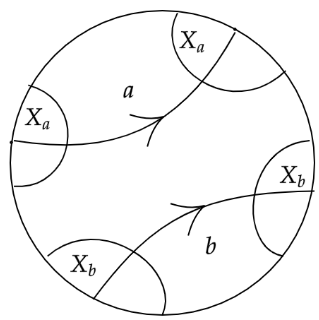


Figure 2: Applying ping-pong lemma to two hyperbolic elements

Since hyperbolic isometries pushes all points towards one endpoint on the boundary, $a^k(X_b) \subset X_a$ for k large enough and $b^k(X_a) \subset X_a$ for k large enough. And $\bar{a}$ and $\bar{b}$ push all points towards the other two endpoints, so $a^{-k}(X_b) \subset X_a$ and $b^{-k}(X_a) \subset X_a$ for k large enough. Hence we can apply the ping-pong lemma and conclude that $\langle a^k, b^k \rangle = F_2$ for large enough k.

The argument for parabolic elements is similar. If we have two parabolic elements a and b, each of them has one fixed point on the boundary. Then both a and $\bar{a}$ push all points on the disk towards the fixed point for a and both b and $\bar{b}$ push all points on the disk towards the fixed point for b. Hence we can take a neighborhood of the fixed point of a to be X_a and the neighborhood of the fixed point of b to be X_b. Then a^k, b^k satisfy the criterion for the ping-pong lemma for large enough k.

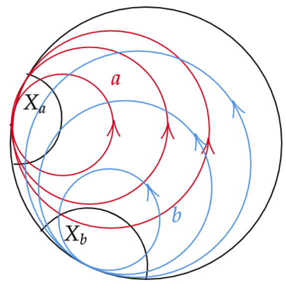


Figure 3: Applying ping-pong lemma to two parabolic elements

3 The Elliptic Case

The ping-pong lemma argument will not work for elliptic isometries because elliptic isometries do not push all points into a neighborhood. We will instead use the following theorem:

Theorem 3.1. *Suppose that a group G acts without inversions on a tree T in such a way that G acts freely and transitively on edges. Choose one edge e of T and say that the stabilizers of its vertices are H_1 and H_2. Then*

$$G \cong H_1 * H_2.$$

We will focus on the case where a corresponds to a rotation by $2\pi/m$ and b corresponds to a rotation by $2\pi/n$, and the two fixed points are sufficiently far apart. Let v_1 and v_2 be the fixed points of a and b respectively, and let e be the geodesic connecting them.

Clearly in this case $\langle a, b \rangle$ is not free, as both a and b have finite order. We can construct a tree: let $T = \{g \cdot e \mid g \in \langle a, b \rangle\}$, i.e. the orbit of e. Note that a has order m and b has order n. So an edge in T is of the form $a^{e_1} b^{f_1} a^{e_2} b^{f_2} \dots a^{e_k} b^{f_k} \cdot e$, where $1 \leq e_i \leq m-1$ and $1 \leq f_i \leq n-1$, and e_1 and f_k can be 0. We can see that T is a tree provided that all images of e are disjoint.

A sufficient condition is that v_1 and v_2 are far apart, so that the angle bisector of e and $a \cdot e$ and the angle bisector of e and $b \cdot e$ do not intersect. We can compute the critical distance between v_1 and v_2.

We may do the calculation in the upper half plane. Moreover, up to isometry, we may assume that v_2 is i in the upper half plane and v_1 lies on the the imaginary axis. (See figure 5b). Then a Euclidean geometry calculation gives the Euclidean distance between v_1 and v_2 (in the upper half plane) is

$$\frac{(1 + \cos(\pi/n))(\sin(\pi/m))}{(1 - \cos(\pi/n))(\sin(\pi/n))}$$

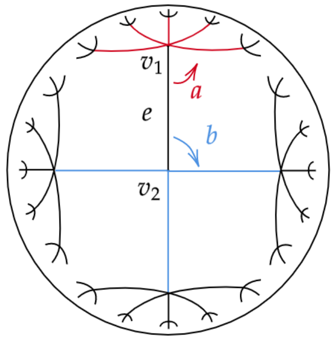


Figure 4: Construction of T with two hyperbolic elements

. Since $v_2 = i$, v_1 is on the imaginary axis, the hyperbolic distance is simply

$$\log\left(\frac{(1+\cos(\pi/n))(\sin(\pi/m))}{(1-\cos(\pi/n))(\sin(\pi/n))}\right)$$

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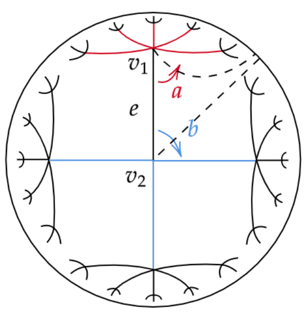
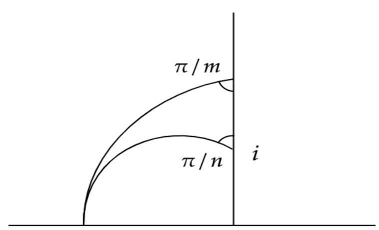


(a) Construction of T with two hyperbolic elements



(b) The two angle bisectors in the upper half plane model

Figure 5: Angle bisectors in the Poincare disk and the upper half plane

Assuming the minimal distance condition is satisfied, then T is a tree. We see that G acts without inversion on T because for any $g \in G$, gv_1 is odd distance away from v_2, so it is impossible for $gv_1 = v_2$. The action is edge transitive on T since T is the orbit of an edge.

The action is free because if $ge = e$, then g fixes v_1 and v_2 since it is acting without inversion. An orientation preserving isometry that fixes two points must be trivial.