MAT382 Project

Elnaz Hessami Pilehrood

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Definition: A group G is residually finite if for every nontrivial element g of G there is a normal subgroup N of finite index in G so that g is not in N.

This means that there exists a finite group H and a homomorphism $\phi : G \to H$ so that $\phi(g) \neq 1$.

Examples:

1. Finite groups are residually finite since we can take N to be the trivial subgroup.

2. Free groups

3. Finitely generated abelian groups are residually finite. This is because every finitely generated abelian group G can be written as

$$G \cong \mathbb{Z}^r \times Z_{n_1} \times Z_{n_2} \times ... \times Z_{n_s},$$

where $\mathbb{Z}^r = \mathbb{Z} \times \mathbb{Z} \times ... \times \mathbb{Z}$ is the direct product of r copies of the group $\mathbb{Z}$. The groups Z_{n_i} are finite, and $\mathbb{Z}$ is a free group, and the direct product of residually finite groups is residually finite.

4. $GL_n(\mathbb{Z})$ is residually finite.

Proof: Consider a non-identity $n \times n$ matrix $A \in GL_n(\mathbb{Z})$. Since A is non-identity, $A - I$ has some nonzero entries. Let $p \in \mathbb{Z}$ be a prime such that none of the nonzero entries in $A - I$ are divisible by p. Now consider the homomorphism $\phi : GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/p\mathbb{Z})$ that reduces the entries in each matrix modulo p. Then we have that $\phi(A) \neq I$.

Properties:

1. Subgroups of residually finite groups are residually finite. This is because we can restrict the homomorphism to a finite group to the subgroup.

2. If G is a group with a finite index subgroup H that is residually finite, then G is residually finite as well.

Proof: Take any element $g \in G$, if $g \notin H$, then we can take the intersection of all the conjugates of H to get a normal subgroup of finite index that does not contain g. If $g \in H$, then H has a finite index subgroup H' such that $g \notin H'$, and moreover, $[G : H'] = [G : H][H : H']$, and since both $[G : H]$ and $[H : H']$ are finite, H' is a finite index subgroup of G as well. Now we can take the intersection of all the conjugates of H' to get a finite index normal subgroup of G that does not contain g.

3. Direct products of residually finite groups is residually finite.

Proof: Let G be the direct product $G = \prod_{i\in I} G_i$, where G_i are residually finite groups. Let $g = (g_i)_{i\in I}$ be a non-identity element of G. Since $g \neq 1$, there must be some $k \in I$ such that $g_k \neq 1$. Since G_k is residually finite, there exists a finite group H, and a homomorphism $\phi : G_k \to H$ such that $\phi(g_k) \neq 1$. Let $\pi_k : G \to G_k$ be the projection homomorphism onto G_k, and define the homomorphism $\psi : G \to H$ by $\psi = \phi \circ \pi_k$. We have that

$$\psi(g) \;=\; \phi(\pi_k(g)) \;=\; \phi(g_k) \neq 1,$$

so G is residually finite.

4. If G is a finitely generated residually finite group then the group $\mathrm{Aut}(G)$ is residually finite.

Proof: Consider any $\alpha \in \mathrm{Aut}(G)$ such that $\alpha \neq Id_G$. Since α is not the identity map, there exists an element $g_0 \in G$ such that $\alpha(g_0) \neq g_0$. Since G is residually finite, and $\alpha(g_0) \cdot g_0^{-1} \neq 1$, there exists a finite group K and a homomorphism $\phi : G \to K$ where $\phi(\alpha(g_0) \cdot g_0^{-1}) \neq 1$, that is $\phi(\alpha(g_0)) \neq \phi(g_0)$. Now let

$$H \;=\; \bigcap_{\psi\in \mathrm{Hom(G,K)}} \mathrm{Ker}(\psi).$$

H is an intersection of normal subgroups of G, so it is a normal subgroup of G. Now let $\beta \in \mathrm{Aut}(G)$, and consider the image of H under β :

$$\beta(H) \;=\; \beta\Big(\bigcap_{\psi\in \mathrm{Hom(G,K)}} \mathrm{Ker}(\psi) \Big) \;=\; \bigcap_{\psi\in \mathrm{Hom(G,K)}} \beta(\mathrm{Ker}(\psi)).$$

If $g \in \mathrm{Ker}(\psi)$, then $\psi(g) = 1$, so $\psi(\beta^{-1}(\beta(g))) = 1$, so $\beta(g) \in \mathrm{Ker}(\psi \circ \beta^{-1})$. Moreover, there is a bijection from $\mathrm{Hom(G,K)}$ to itself given by $\psi \to \psi \circ \beta^{-1}$, so

$$\beta(H) = \bigcap_{\psi \in \mathrm{Hom(G,K)}} \mathrm{Ker}(\psi \circ \beta^{-1}) = \bigcap_{\psi \in \mathrm{Hom(G,K)}} \mathrm{Ker}(\psi) = \mathrm{H}.$$

Therefore, we can define a homomorphism $\phi' : \mathrm{Aut}(G) \to \mathrm{Aut}(G/H)$ given by $\phi'(\beta) = \beta'$ where

$$\beta'(gH) = \beta(g)\ H.$$

Then we have that $\phi'(\alpha) = \alpha'$ where

$$\alpha'(g_0 H) = \alpha(g_0)H.$$

If $\alpha(g_0)H = g_0 H$, then that would mean that $\alpha(g_0)\cdot g_0^{-1} \in H$, which means $\alpha(g_0)\cdot g_0^{-1}$ is in the kernel of every homomorphism from G to K, which is a contradiction since $\phi(\alpha(g_0) \cdot g_0^{-1}) \neq 1$. Therefore, $\alpha(g_0)H \neq g_0 H$, which means $\alpha' \neq Id_{G/H}$. Therefore, the map ϕ' does not send α to the identity.

Moreover, Since G is finitely generated and K is a finite group, $\mathrm{Hom(G,K)}$ is finite, and because $\mathrm{Ker}(\psi)$ has finite index in G, H is a finite intersection of finite index subgroups, so it has finite index itself. Therefore, G/H is a finite group, and thus $\mathrm{Aut}(G/H)$ is finite as well. Thus, we have a homomorphism from $\mathrm{Aut}(G)$ to the finite group $\mathrm{Aut}(G/H)$ which does not send α to the identity. Thus, $\mathrm{Aut}(G)$ is residually finite.

5. Finitely presented residually finite groups have solvable word problem.

On a Theorem of Peter Scott by Priyam Patel:

The diagrams below are taken from this paper.

Theorem 1: Let Σ be a compact surface of negative Euler characteristic. There exists a hyperbolic metric on Σ so that for any $\alpha \in \pi_1(\Sigma) - \{id\}$, there exists a subgroup H' of $\pi_1(\Sigma)$ such that $\alpha \notin H'$. The index of H' is bounded by $32.3\ l$, where l is the length of the unique geodesic representative of α.

Theorem 2: Let Σ be a closed surface or a compact surface with boundary of negative Euler characteristic. Then there exists a hyperbolic metric on Σ so that any closed geodesic of length l lifts to an embedded loop in a finite cover whose index is bounded by $16.2\ l$.

To prove this theorem:

→ tessellate the hyperbolic plane by regular, right angled pentagons

→ induces a tessellation on Σ and any cover of Σ

→ for a compact subsurface of a tessellated surface, estimate the area of the smallest closed, convex union of pentagons containing it

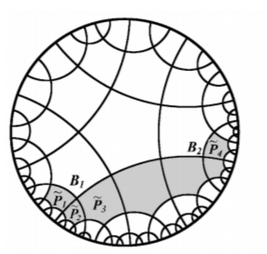
Now consider a hyperbolic surface Σ tessellated by pentagons, and the closed geodesic α in Σ.
Let S be the union of all pentagons P in Σ such that

$$P \cap \alpha \neq \varnothing.$$

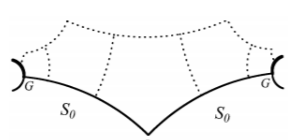
Now we will add pentagons along ∂S to make S a convex set.

We have non-convexity when 3 pentagons form a $\frac{3\pi}{2}$ angle at a vertex of ∂S.

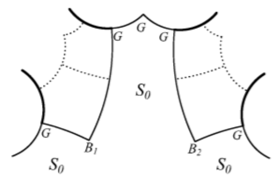
Let $\frac{\pi}{2}$ vertices be good corners (G) and $\frac{3\pi}{2}$ vertices be bad corners (B). Two bad corners can't appear consecutively on the boundary. This is because if we had two consecutive bad corners like in the diagram below, pentagons P_1 and P_4 at these bad vertices would both be intersecting the geodesic α, but none of the white pentagons between them would, which is a contradiction.



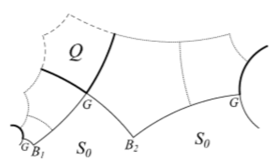
Extend the geodesic edges of the good corners and add the pentagons that intersect S between these two geodesic segments



If we have two bad corners separated by more than one good corner, then by adding the pentagons above no new not-convex areas are formed.

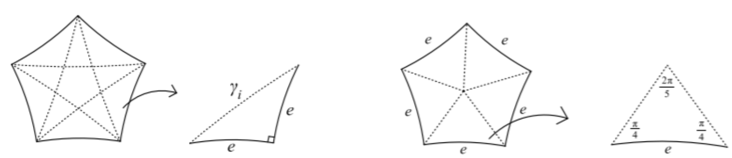


However, if we have two bad corners separated by only one good corner, then by adding the pentagons above we create a new not-convex region, which can be fixed by adding one pentagon.



Now we have obtained a convex set Y of pentagons containing S.

Now we find the length d_0 of the diameter of each pentagon:



Using the hyperbolic Law of Cosines we get that

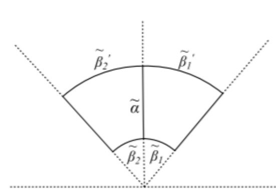
$$e = \cosh^{-1}(1 + 2\cos(\frac{2\pi}{5}))$$

$$d_0 = \cosh^{-1}((\cosh e)^2) \sim 1.167.$$

Now we estimate the area of Y:

Let Z be the union of the points x such that $d(x, \alpha) \leq 2d_0$.

Every pentagon in Y either intersects α, and thus is contained in S, or it intersects S. This means every point in Y is at most two pentagons away from α. Thus $\sup_{y \in Y}\{d(y, \alpha)\} \leq 2d_0$, so $Y \subset Z$. Now lift Z and α to $\mathbb{H}$:



$Area(Z) \ = \ 2l \sinh(2d_0)$ and $Area(Y) \leq Area(Z)$ so

$$Area(Y) \ \leq \ 2\, l \ \sinh(2d_0).$$

Each pentagon has area $\frac{\pi}{2}$, so if Y consists of k pentagons, then

$$Area(Y) \ = \ k\frac{\pi}{2} \ \leq \ 2\, l \ \sinh(2d_0)$$

$$k \ < \ 16.2\, l.$$