LIMIT SETS OF WEIL-PETERSSON GEODESICS WITH NONMINIMAL ENDING LAMINATIONS

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Abstract. In this paper we construct examples of Weil-Petersson geodesics with nonminimal ending laminations which have 1–dimensional limit sets in the Thurston compactification of Teichmüller space.

Contents

1. Introduction

A number of authors have studied the limiting behavior of Teichmüller geodesics in relation to the Thurston compactification of Teichmüller space, [Mas82, Ker80] [Len08, LM10, LLR13, CMW14, BLMR16a, LMR16]. This work has highlighted the delicate relationship between the vertical foliation of the quadratic differential defining the geodesic and the limit set in the Thurston boundary.

The *ending lamination* of a Weil-Petersson (WP) geodesic ray was introduced by Brock, Masur and Minsky in [BMM10] and in some sense serves as a rough analogue of the vertical foliation of the quadratic differential defining a geodesic for the Teichmüller metric. Ending laminations have been used to study the behavior of WP geodesics [BMM10, BMM11, Mod15, Mod16, BLMR16b] and dynamics of the WP geodesic flow on moduli spaces [BMM11, BM15, Ham15]. In this paper, complementing our work in [BLMR16b], we provide examples of WP geodesic rays with non minimal, and hence nonuniquely ergodic, ending laminations whose limit sets in the Thurston compactification of Teichmüller space is larger than a single point.

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2. Preliminaries

Notation 2.1. Let $K \geq 1$, $C \geq 0$, and let X be any set. For two functions $f, g : X \to [0, \infty)$ we write $f \asymp_{K,C} g$ if $\frac{1}{K} g(x) - C \leq f(x) \leq K g(x) + C$ for all $x \in X$. Similarly, we write $f \overset{*}{\asymp}_K g$ if $\frac{1}{K} g(x) \leq f(x) \leq K g(x)$ for all $x \in X$, and $f \overset{+}{\asymp}_C g$ if $g(x) - C \leq f(x) \leq g(x) + C$ for all $x \in X$. Moreover, $f \overset{*}{\preceq}_K g$ means that $f(x) \leq K g(x)$ for all $x \in X$ and $f \overset{*}{\preceq} g$ means that $f(x) \leq g(x) + C$ for all $x \in X$. We drop K, C from the notation when the constants are understood from the context.

Teichmüller space. Given a finite type surface S, we denote its Teichmüller space by $\mathrm{Teich}(S)$. The points in $\mathrm{Teich}(S)$ are isotopy classes of (finite type) Riemann surface structures on S. When the Euler characteristic $\chi(S) < 0$, we also view $X \in \mathrm{Teich}(S)$ as an isotopy class of complete, finite area, hyperbolic metric on S. In this case, given a homotopy class of closed curve α and $X \in \mathrm{Teich}(S)$, we write $\ell_\alpha(X)$ for the length of the $X-$ geodesic representative of α. We also let $w_\alpha(X)$ denote the maximal width of a collar neighborhood about α.

Weil-Petersson metric. When $\chi(S) < 0$, the Weil-Petersson (WP) metric is a negatively curved, incomplete, geodesically convex, Riemannian metric on $\mathrm{Teich}(S)$. Its completion, $\overline{\mathrm{Teich}(S)}$, is a stratified CAT(0) space, with a stratum $\mathcal{S}(\sigma)$ for each (possibly empty isotopy class of) multicurve σ, consisting of appropriately marked Riemann surfaces pinched precisely along σ. The stratum $\mathcal{S}(\sigma)$ is totally geodesic and isometric to the product of the Teichmüller spaces of the connected components of $S \backslash \sigma$ with their WP metric. The completion of $\mathcal{S}(\sigma)$ is the union of all strata $\mathcal{S}(\sigma')$ for which $\sigma \subset \sigma'$; see [Mas76]. The stratification has the so called *non-refraction property*: the interior of a geodesic segment with end points in two strata $\mathcal{S}(\sigma_1)$ and $\mathcal{S}(\sigma_2)$ lies in the stratum $\mathcal{S}(\sigma_1 \cap \sigma_2)$; see [DW03, Wol08].

Curve complexes, markings, and projections. We refer the reader to [MM99, MM00] for more complete definitions of the objects described in this subsection. In this paper we denote the curve complex of a subsurface Y by $\mathcal{C}(Y)$. The set of vertices of $\mathcal{C}(Y)$, denoted by $\mathcal{C}_0(Y)$, is the set of curves on Y (more precisely, the set of isotopy classes of essential simple closed curves on Y). A (partial) marking μ on S consists of a pants decomposition, $\mathrm{base}(\mu)$, and a transversal for (some) all curves in $\mathrm{base}(\mu)$. For a curve or marking μ, we denote the subsurface projection of μ to the subsurface Y by $\pi_Y(\mu)$ (see [MM00]), and for two markings/curves μ, μ' define

$$d_Y(\mu, \mu') := \mathrm{diam}_{\mathcal{C}(Y)} \Big(\pi_Y(\mu) \cup \pi_Y(\mu') \Big). \tag{2.1}$$

If Y is an annulus with core curve α, we also write $d_\alpha(\mu, \mu') = d_Y(\mu, \mu')$.

There exists a constant $L_S > 0$, called the *Bers constant*, depending on S, such that for any $X \in \mathrm{Teich}(S)$ there is a pants decomposition such that every curve in the pants decomposition has hyperbolic length at most L_S with respect to X; see e.g. [Bus10]. Such a pants decomposition is called a *Bers pants decomposition* for X. A *Bers curve* for X is a curve α for which $\ell_\alpha(X) \leq L_S$. A *Bers marking* for X is a marking μ such that $\mathrm{base}(\mu)$ is a Bers pants decomposition for X.

Given a point $X \in \mathrm{Teich}(S)$ and a curve α, the projection of X to α, $\pi_\alpha(X)$, is the collection of all arcs in the annular cover corresponding to α which are the lifts of the X–geodesic curves that are orthogonal to the X–geodesic representative of α. Distance in α between points of $\mathrm{Teich}(S)$ and/or curves/markings is defined as the diameter of the union of their projections (as with the case of two curves or markings). This is often called the *relative twisting*, and for $\alpha, \delta \in \mathcal{C}_0(S)$ and $X \in \mathrm{Teich}(S)$, we write

$$\mathrm{tw}_\alpha(\delta, X) = d_\alpha(\delta, X) := d_\alpha(\delta, \pi_\alpha(X)).$$

If α has bounded length and μ is a bounded length marking for X, then

$$\mathrm{tw}_\alpha(\delta, X) \stackrel{+}{\asymp} d_\alpha(\delta, \mu),$$

where the additive error depends on the bounds on the lengths of α and μ (including the lengths of the transverse curves defining the transversals of μ), but not on the length of δ.

The Thurston compactification. The *Thurston boundary of the Teichmüller space* is the space of projective classes of measured laminations $\mathcal{PML}(S)$; see [FLP79]. A sequence of points $X_k \in \mathrm{Teich}(S)$ converge to $[\bar{\lambda}]$, the projective class of a measured lamination $\bar{\lambda} \in \mathcal{ML}(S)$, if for a sequence $\{u_k\}_k$ we have that

$$\lim_{k\to\infty} u_k \ell_\delta(X_k) = \mathrm{i}(\delta, \bar{\lambda}), \tag{2.2}$$

for every $\delta \in \mathcal{C}_0(S)$. We call $\{u_k\}_k$ a *scaling sequence* for $\{X_k\}_k$. Moreover, $X_k \to [\bar{\lambda}]$ if (2.2) holds for some sequence $\{u_k\}_k$ and an (appropriately chosen) finite set of curves $\delta_1, \ldots, \delta_k$. For us it suffices to note that, for any given curve α, the finite set of curves $\delta_1, \ldots, \delta_k$ can all be chosen to have nonzero intersection number with α.

Ending lamination. The (forward) ending lamination of a WP geodesic ray $r : [0, b) \to \mathrm{Teich}(S)$, denoted by ν^+, is a lamination or (partial) marking: when r is extendable to b in $\overline{\mathrm{Teich}(S)}$, ν^+ is a Bers marking at $r(b)$. Otherwise, it is the union of the supports of limits in $\mathcal{PML}(S)$ of Bers curves at distinct times $t_i \to b$, together with pinching curves; see [BMM10] for more detail.

2.1. **Bounded length WP geodesic segments.** To study the behavior of the geodesic rays constructed in this paper we need the geodesic limit theorem [Mod15, Theorem 4.2] which is the strengthened of Wolpert's geodesic limit theorem from [Wol03] as follows:

For a multi-curve σ on a surface S let $\mathrm{tw}(\sigma)$ be the subgroup of $\mathrm{Mod}(S)$ generated by positive Dehn twists about curves in σ.

Theorem 2.2. (Geodesic limit)
Given $T > 0$, let $\zeta_n : [0, T] \to \mathrm{Teich}(S)$ be a sequence of WP geodesic segments. After possibly passing to a subsequence, there exist a partition of the interval $[0, T]$ by $0 = t_0 < t_1 < \ldots < t_{k+1} = T$, possibly empty multicurves σ_l, $l = 0, \ldots, k+1$ and the possibly empty multicurve $\hat{\tau}$, with $\sigma_l \cap \sigma_{l+1} = \hat{\tau}$ for $l = 0, \ldots, k$, and a piecewise geodesic segment

$$\hat{\zeta} : [0, T] \to \overline{\mathrm{Teich}(S)}$$

such that

1. *$\hat{\zeta}(t_l) \in \mathcal{S}(\sigma_l)$ for each $l = 0, \ldots, k+1$,*
2. *$\hat{\zeta}((t_l, t_{l+1})) \subset \mathcal{S}(\hat{\tau})$ for each $l = 0, \ldots, k$,*
3. *there exist elements of mapping class group ψ_n which are either identity or unbounded, and elements $\mathcal{T}_{l,n} \in \mathrm{tw}(\sigma_l \setminus \hat{\tau})$ such that for any $\gamma \in \sigma_l - \hat{\tau}$ the power of the positive Dehn twist D_γ about γ is unbounded, and we have*

$$\lim_{n \to \infty} \psi_n(\zeta_n(t))(t) = \hat{\zeta}(t)$$

for all $t \in [t_0, t_1]$. Moreover, for each $l = 1, \ldots, k$ let

$$\varphi_{l,n} = \mathcal{T}_{l,n} \circ \ldots \circ \mathcal{T}_{1,n}, \tag{2.3}$$

then we have

$$\lim_{n \to \infty} \varphi_{l,n}(\zeta_n(t)) = \hat{\zeta}(t)$$

for all $t \in [t_l, t_{l+1}]$.

We also need the following consequence of this theorem which is [Mod15, Corollary 4.10]. Denote a Bers marking at a point $X \in \mathrm{Teich}(S)$ by $\mu(X)$.

Theorem 2.3. *Given ϵ_0, T positive and $\epsilon \in (0, \epsilon_0]$, there is an $N \in \mathbb{N}$ with the following property. Suppose that $\zeta \colon [a, b] \to \mathrm{Teich}(S)$ is a WP geodesic segment of length at most T such that $\sup_{t \in [a,b]} \ell_\alpha(\zeta(t)) \geq \epsilon_0$ and $d_\alpha(\mu(\zeta(a)), \mu(\zeta(b))) > N$. Then, we have*

$$\inf_{t \in [a,b]} \ell_\alpha(\zeta(t)) \leq \epsilon.$$

3. Geodesics with nonminimal ending laminations

First, let us briefly sketch our construction of Weil-Petersson geodesic rays with nonminimal, nonuniquely ergodic ending laminations that do not accumulate on a single point. The basic idea is similar to Lenzhen's construction for Teichmüller geodesics in [Len08]. Let S be the closed, genus 2 surface and let $\alpha \subset S$ be a separating simple closed curve cutting S into two one-holed tori that we denote by S_0 and S_1. The stratum $\mathcal{S}(\alpha)$ is isometric to a product of Teichmüller spaces of once-punctured tori, i.e., $\mathcal{S}(\alpha) \cong \mathrm{Teich}(S_0) \times \mathrm{Teich}(S_1)$.

We carefully choose sequences of curves $\{\gamma_i^h\}_i \subset \mathcal{C}(S_h)$, $h = 0, 1$ which form quasi-geodesics and limit to minimal filling laminations λ_h, $h = 0, 1$.

Using the fact that $\mathrm{Teich}(S_h)$ with the WP metric is quasi-isometric to $\mathcal{C}(S_h)$, and that it has negative curvature bounded away from 0 we construct geodesic rays $\hat{r}^h$ in $\mathrm{Teich}(S_h)$ which have forward ending laminations λ_h, $h = 0, 1$.

Next, we consider the geodesic $\hat{r} = (\hat{r}^0, \hat{r}^1)$ in $\overline{\mathrm{Teich}(S)}$, and construct a geodesic ray r which fellow travels $\hat{r}$. We estimate the length of an arbitrary curve along r using estimates from [CRS08]. From the conditions we imposed on our sequences of curves, we will see that most of the length of the curve comes from its intersection with curves γ_i^0 and γ_i^1, and so lengths are eventually well–approximated by intersection numbers with linear combinations of measure $\bar{\lambda}_0$ and $\bar{\lambda}_1$ on λ_0 and λ_1, respectively. Consequently, this geodesic ray accumulates on a 1–simplex with vertices $[\bar{\lambda}_0]$ and $[\bar{\lambda}_1]$ in the Thurston boundary. Analyzing a pair of particular sequences of times, we see that the endpoints of the simplex are in the limit set, and so by connectivity, the limit set consists of the entire 1–simplex.

3.1. **Continued fraction expansions and geodesics in** $\mathrm{Teich}(S_{1,1})$**.** Let λ_h be a minimal, irrational lamination on S_h. This lamination is the straightening of a foliation of the flat square torus, and we assume for convenience that the slope of the leaves of this foliation is greater than 1. The reciprocal of this slope is an irrational number less than 1 which we denote by x_h, and we write its continued fraction expansion as

$$x_h = [0; e_0^h, e_1^h, \ldots], \tag{3.1}$$

(the first coefficient is zero since $x_h < 1$). We assume in all that follows that $e_i^h \geq 4$ for all i and for $h = 0, 1$.

Next, let $\frac{p_i^h}{q_i^h} = [0; e_0^h, e_1^h, \ldots, e_{i-1}^h]$ be the i^{th} *convergent* with finite continued fraction expansion as shown, obtained by truncating that of x_h. Let γ_i^h be the simple closed curve on the torus whose slope is the reciprocal, $\frac{q_i^h}{p_i^h}$. Note that γ_0^h is the curve whose reciprocal slope is 0 (that is, γ_0^h is the vertical curve) and we let γ_{-1}^h denote the horizontal curve, by convention.

The *Farey graph* is the graph with vertices corresponding to $\mathbb{Q} \cup \{\infty\}$ and edges between $\frac{p}{q}$ and $\frac{r}{s}$ whenever $|ps - rq| = 1$. Identifying a simple closed curve on the (flat, square) torus with the reciprocal of its slope identifies the curve graph $\mathcal{C}(S_h)$ with Farey graph, and we confuse the two via this identification whenever it is convenient. Our assumption that $e_i^h \geq 4$ ensures that the sequence of curves $\{\gamma_i^h\}_i$ (or equivalently, the sequence of convergents $\{\frac{p_i^h}{q_i^h}\}_i$) is a geodesic; see [Ser85, Min96]. Our index convention leads to

$$\gamma_{i+1}^h = D_{\gamma_i^h}^{\pm e_i^h}(\gamma_{i-1}^h), \tag{3.2}$$

with the sign determined by the parity of i.

The curve graph $\mathcal{C}(S_h)$—or equivalently the Farey graph—naturally embeds into the Weil-Petersson completion of $\mathrm{Teich}(S_h)$ in such a way that the vertex corresponding to the curve γ is sent to the point in which γ has been pinched, and so that edges between adjacent vertices are sent to WP geodesics. Furthermore, this is a quasi-isometry; see [Bro03]. The usual identification of $\overline{\mathrm{Teich}(S_h)}$ with a subset of the compactified upper half-plane provides the standard embedding of the Farey graph into $\overline{\mathbb{H}^2}$, with vertex set $\mathbb{Q} \cup \{\infty\} \subset \mathbb{R} \cup \{\infty\} = S^1_\infty$.

For each $i \geq 0$, let $X_i^h \in \overline{\mathrm{Teich}(S_h)}$ denote the point at which γ_i^h is pinched and $[X_i^h, X_i^{h+1}]$ the geodesic between consecutive points. These geodesic segments are the images of the geodesics $[\gamma_i^h, \gamma_{i+1}^h]$ in the $\mathcal{C}(S_h)$, and since the concatenation of the latter set of segments is a geodesic in $\mathcal{C}(S_h)$, the image is a quasi-geodesic in $\overline{\mathrm{Teich}(S_h)}$. Since $\mathrm{Mod}(S_h)$ acts transitively on these geodesics, they all have the same length, which we denote by $D = d_{\mathrm{WP}}(X_i^h, X_{i+1}^h) > 0$. Note that $\gamma_0^0 = \gamma_0^1$ is the curve corresponding to the rational number 0, and for convenience we let X_{-1}^h denote the midpoint of the geodesic segment between (the image of) γ_0^h and γ_{-1}^h which has distance $\frac{D}{2}$ to X_0^h (note that $X_{-1}^h = \sqrt{-1}$ in the upper half plane).

Let $\hat{r}_c^h$ be the unit speed parameterization of the concatenation of segments $[X_i^h, X_{i+1}^h]$, for $h = 0, 1$. The set of geodesic rays starting at X_{-1}^h and passing through the geodesic segment $[X_i^h, X_{i+1}^h]$ forms a nested sequence, indexed by i. Strict negative curvature of WP metric on $\mathrm{Teich}(S_{1,1})$ (see [Wol03]) guarantees the existence of a unique ray in the intersection of all these sets, which we denote $\hat{r}^h$ which fellow travels $\hat{r}_c^h$.

Lemma 3.1. *There exists a sequence $\{K_i\}_{i=1}^\infty$ so that if $e_i^h > K_i$ for all $i \geq 0$, and if $\{t_i^h\}$ is the sequence of times for which $d_{\mathrm{WP}}(\hat{r}^h(t_i^h), X_i^h)$ is minimized, then*

(1) $d_{\mathrm{WP}}(\hat{r}_c^h(s), \hat{r}^h(s)) \leq \frac{D}{4}$ for all $s > 0$,

(2) $d_{\mathrm{WP}}(\hat{r}^h(t_i^h), X_i^h) \leq \frac{D}{2^{i+6}} \to 0$ as $i \to \infty$, and

(3) $\left|t_i^h - \left(\frac{D}{2} + iD\right)\right| < \frac{D}{8}$ (in particular, $\{t_i^h\}_i$ is increasing).

Proof. Observe that the geodesic $\hat{r}^h$ passes through the segments $[X_{i-2}^h, X_{i-1}^h]$ and $[X_{i+1}^h, X_{i+2}^h]$. For any small $\eta > 0$, any path between a point on the first segment and a point on the second segment that lies outside the η–neighborhood of X_i^h can be projected to the boundary of that neighborhood and its length can be bounded below by a function of η and e_i^h. On the other hand, the distance between any point on $[X_{i-2}^h, X_{i-1}^h]$ and any point on $[X_{i+1}^h, X_{i+2}^h]$ is at most $3D$ (by following edges of the Farey graph). Therefore, we can guarantee that $\hat{r}^h$ passes as close as we like to X_i^h by taking e_i^h sufficiently large.

In particular, for all $i \geq 0$, we choose $K_i > 0$ such that if $e_i^h > K_i$, then we have

$$d_{\mathrm{WP}}(\hat{r}^h(t_i^h), X_i^h) < \tfrac{D}{2^{i+6}}.$$

Then assuming (as we will from here on out) that $e_i^h > K_i$ for all $i \geq 0$, we observe that part (2) of the lemma holds.

By the triangle inequality, it follows that for all $i \geq 0$

$$\begin{aligned} \Big| |t_i^h - t_{i+1}^h| - D \Big| &= \Big| d_{\mathrm{WP}}(\hat{r}^h(t_i^h), \hat{r}^h(t_{i+1}^h)) - d_{\mathrm{WP}}(X_i^h, X_{i+1}^h) \Big| \\ &\leq \tfrac{D}{2^{i+6}} + \tfrac{D}{2^{i+7}} < \tfrac{D}{2^{i+5}}. \end{aligned} \tag{3.3}$$

A similar (simpler) argument proves $|t_0^h - \frac{D}{2}| < \frac{D}{2^6}$.

We claim that for all $i \geq 1$, t_{i+1}^h must lie between t_i^h and t_{i+2}^h. If not, and say $t_{i+1} > t_i, t_{i+2}$, then $t_{i+1} - t_i > 0$ and $t_{i+1} - t_{i+2} > 0$, and applying inequality (3.3) to i and $i+1$, we see that

$$|t_i^h - t_{i+2}^h| = |t_{i+1}^h - t_{i+2}^h - D + (D - (t_{i+1}^h - t_i^h))| \leq \tfrac{D}{2^{i+5}} + \tfrac{D}{2^{i+6}}.$$

Hence by the triangle inequality (as above)

$$\begin{aligned} d_{\mathrm{WP}}(X_i^h, X_{i+2}^h) &\leq d_{\mathrm{WP}}(\hat{r}^h(t_i^h), \hat{r}^h(t_{i+2}^h)) + \tfrac{D}{2^{i+6}} + \tfrac{D}{2^{i+8}} \\ &\leq |t_i^h - t_{i+2}^h| + \tfrac{D}{2^{i+6}} + \tfrac{D}{2^{i+8}} < \tfrac{D}{4}. \end{aligned}$$

On the other hand, [BM07, Lemma 3.2] implies that since γ_i^h and γ_{i+2}^h are not adjacent in $\mathcal{C}(S_h)$, we must have $d_{\mathrm{WP}}(X_i^h, X_{i+2}^h) > D$, which is a contradiction. A similar argument produces a contradiction if $t_{i+1}^h < t_i^h, t_{i+2}^h$, and thus $\{t_i^h\}$ is an increasing sequence.

From (3.3) (and the inequality $|t_0^h - \frac{D}{2}| < \frac{D}{2^6}$), we have

$$\begin{aligned} \left| t_i^h - \left(\tfrac{D}{2} + iD\right) \right| &= \left| t_0^h + \sum_{j=1}^{i} t_j^h - t_{j-1}^h - \tfrac{D}{2} - iD \right| \\ &\leq \left| t_0^h - \tfrac{D}{2} \right| + \sum_{j=1}^{i} \left| t_j^h - t_{j-1}^h - D \right| \leq \tfrac{D}{2^6} + \sum_{j=1}^{i} \tfrac{D}{2^{j+4}} < \tfrac{D}{8}. \end{aligned}$$

This proves part (3).

Finally, we note that part (1) follows from (3.3), parts (2) and (3), and convexity of distance between two geodesics in a CAT(0) space. To see this, first note that for all $i \geq 0$

$$\begin{aligned} d_{\mathrm{WP}}(\hat{r}^h(\tfrac{D}{2} + iD), \hat{r}_c^h(\tfrac{D}{2} + iD)) &\leq d_{\mathrm{WP}}(\hat{r}^h(\tfrac{D}{2} + iD), \hat{r}^h(t_i^h)) + d_{\mathrm{WP}}(\hat{r}^h(t_i^h), X_i^h) \\ &\leq |t_i^h - (\tfrac{D}{2} + iD)| + \tfrac{D}{2^{i+6}} < \frac{D}{8} + \frac{D}{2^{i+6}} < \tfrac{D}{4}. \end{aligned}$$

Thus, for all $i \geq 0$, convexity of the distance between geodesics implies

$$d_{\mathrm{WP}}(\hat{r}^h(s), \hat{r}_c^h(s)) < \tfrac{D}{4}, \quad \text{for all } s \in \left[\tfrac{D}{2} + iD, \tfrac{D}{2} + (i+1)D\right]$$

(and for $s \in [0, \frac{D}{2}]$). This proves (1) for all $s \geq 0$, completing the proof. $\square$

3.2. **Sequences of times.** Throughout the following, we will always assume that for each $h = 0, 1$, the sequence $\{e_i^h\}_i$ is chosen so that $e_i^h > K_i$ from Lemma 3.1, and we write $\hat{r}^h, \hat{r}_c^h$ to denote the associated geodesics/quasi-geodesics. We keep the same parameterization for $\hat{r}^0$ and $\hat{r}_c^0$ as above, but adjust the parameterization of $\hat{r}^1$ and $\hat{r}_c^1$ by precomposing with the maps $t \mapsto t - \frac{D}{2}$. This does not make sense for $t \in [0, \frac{D}{2})$, so we define $\hat{r}^1$ and $\hat{r}_c^1$ to be constant on this interval.

With this new parameterization, the sequences $\{t_i^1\}$ must be shifted by $\frac{D}{2}$, so that parts (1) and (2) of Lemma 3.1 remain valid. The conclusion in part (3) of the lemma then becomes

$$|t_i^0 - \left(\tfrac{D}{2} + iD\right)| < \tfrac{D}{8} \quad \text{and} \quad |t_i^1 - (i+1)D| < \tfrac{D}{8}. \tag{3.4}$$

Identifying $\mathcal{S}(\alpha) = \mathrm{Teich}(S_0) \times \mathrm{Teich}(S_1)$, we set

$$\hat{r} = (\hat{r}^0, \hat{r}^1) \colon [0, \infty) \to \mathcal{S}(\alpha) \subset \overline{\mathrm{Teich}(S)}.$$

Notation 3.2. (Relabeling sequences) To simplify some statements and avoid duplication in some of the arguments that follow, we make the following notational convention. For $h = 0, 1$ and $i \geq 0$, set

$$\begin{aligned} e_{2i+h} &= e_i^h \\ \gamma_{2i+h} &= \gamma_i^h \\ t_{2i+h} &= t_i^h \\ X_{2i+h} &= X_i^h. \end{aligned}$$

We will use the index k for these sequences, and write $\{e_k\}$, $\{\gamma_k\}$, $\{t_k\}$, and $\{X_k\}$. We also let $\bar{k} \in \{0, 1\}$ to denote the residue of k modulo 2, and $i = i(k)$ for the floor of $k/2$. Thus, when we need it, can write $e_k = e_i^{\bar{k}}$, etc. As an abuse of notation, we say things like "$\hat{r}(t_k)$ is close to X_k", though what we really mean is that $\hat{r}^{\bar{k}}(t_k)$ is close to X_k. We also view γ_k as a curve on both S and $S \setminus \alpha$, rather than just a curve on $S_{\bar{k}} \subset S \setminus \alpha \subset S$. Finally, the following sequence of times will also be useful for us

$$t_k' = \frac{t_k + t_{k+1}}{2}.$$

Proposition 3.3. *For all $k \geq 0$, $t_{k+1} - t_k > \frac{D}{4}$. In particular, $\{t_k\}_k$ is increasing. Consequently, $t_k' - t_k > \frac{D}{8}$ and $t_{k+1} - t_k' > \frac{D}{8}$.*

Proof. For $k = 2i$ (even), (3.4) implies

$$t_{k+1} - t_k = t_i^1 - t_i^0 > \left((i+1)D - \tfrac{D}{8}\right) - \left(\tfrac{D}{2} + iD + \tfrac{D}{8}\right) = \tfrac{D}{2} - \tfrac{D}{4} = \tfrac{D}{4}.$$

A similar computation verifies the claim for k odd.

The last sentence follows from the first, and the fact that t_k' is the average of t_k and t_{k+1}. □

Figure 1 provides a useful illustration of the relationship between $\{t_k\}$, $\{t_k'\}$, $\{\gamma_k\}$, and $\{X_k\}$.

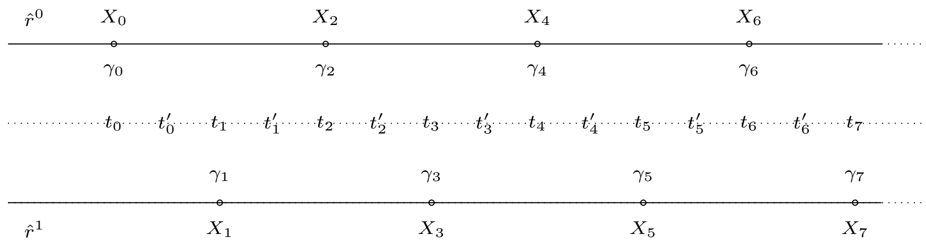

FIGURE 1. The times t_k and t'_k are "spaced out" by at least $\frac{D}{8}$. The former are the times when $\hat{r} = (\hat{r}^0, \hat{r}^1)$ is closest to the points $\{X_k\}$: the curve γ_k is very short at time t_k.

Lemma 3.4. *There exists $C > 0$, so that for all $k \geq 2$, we have*

$$\ell_{\gamma_k}(\hat{r}(t'_{k-2})), \ell_{\gamma_k}(\hat{r}(t'_{k+1})) \leq C.$$

Consequently, $\ell_{\gamma_k}(\hat{r}(t)) \leq C$ for all $t \in [t'_{k-2}, t'_{k+1}]$

Proof. We prove the bound on $\ell_{\gamma_k}(\hat{r}(t'_{k-2}))$. The proof of the other bound is similar.

According to part (2) of Lemma 3.1, for k sufficiently large

$$d_{\mathrm{WP}}(\hat{r}^{\bar{k}}(t_{k-2}), X_{k-2}), d_{\mathrm{WP}}(\hat{r}^{\bar{k}}(t_k), X_k) \leq \tfrac{D}{2^6}.$$

By convexity of distance between geodesics, it follows that there is a point $Y_k \in [X_{k-2}, X_k]$ such that

$$d_{\mathrm{WP}}(\hat{r}^{\bar{k}}(t'_{k-2}), Y_k) \leq \tfrac{D}{2^6}.$$

On the other hand, by Proposition 3.3, we have

$$t_{k-2} + \tfrac{D}{8} < t'_{k-2} < t_k - \tfrac{3D}{8}.$$

Therefore, by the triangle inequality, we see that

$$\begin{aligned} d_{\mathrm{WP}}(Y_k, X_{k-2}) \; &\geq \; (t'_{k-2} - t_{k-2}) - d_{\mathrm{WP}}(\hat{r}^{\bar{k}}(t'_{k-2}), Y_k) - d_{\mathrm{WP}}(X_{k-2}, \hat{r}^{\bar{k}}(t_{k-2})) \\ &> \; \tfrac{D}{8} - \tfrac{D}{2^6} - \tfrac{D}{2^6} > \tfrac{D}{16} \end{aligned}$$

Similar computations show that

$$d_{\mathrm{WP}}(Y_k, X_k) \geq \tfrac{D}{4}.$$

So, Y_k is further than $\frac{D}{16}$ from the endpoints of $[X_{k-2}, X_k]$. On the other hand, for any point $Y \in [X_{k-2}, X_k]$, the closest point to Y in the boundary $\overline{\mathrm{Teich}(S_{\bar{k}})} \setminus \mathrm{Teich}(S_{\bar{k}})$ of the Teichmüller space of the torus $\mathrm{Teich}(S_{\bar{k}})$ is one of the endpoints X_{k-2} or X_k (cf. Lemma 3.1 and 3.2 of [BM07]). Therefore, the closed ball about Y_k of radius $\frac{D}{32}$ are compact. Indeed, all of these balls are contained in the $\mathrm{Mod}(S_{\bar{k}})$–orbit of a single compact subset of $\mathrm{Teich}(S_{\bar{k}})$. It follows that the length of γ_k (the curve pinched at X_k) is uniformly bounded in the $\frac{D}{32}$–neighborhood of Y_k, independent of k (and independent of the sequence $\{e_k\}$). Since $\hat{r}^{\bar{k}}(t'_{k-2})$ lies in this neighborhood, $\ell_{\gamma_k}(\hat{r}^{\bar{k}}(t'_{k-2}))$ is uniformly bounded, as required.

The proof of the bound on $\ell_{\gamma_k}(\hat{r}^{\bar{k}}(t'_{k+1}))$ is entirely analogous, using the geodesic segment $[X_k, X_{k+2}]$ in place of $[X_{k-2}, X_k]$. The very last statement follows from convexity of length-functions along WP geodesics [Wol03, §3.3]. $\square$

Corollary 3.5. *For all $k \geq 2$ and $j = k-1, k, k+1, k+2$, we have*

$$\ell_{\gamma_j}(\hat{r}(t'_k)) \leq C.$$

Proof. According to Lemma 3.4, the curve γ_j has length at most C on the interval $[t'_{j-2}, t'_{j+1}]$. The corollary thus follows from the fact that

$$\{t'_k\} = \bigcap_{j=k-1}^{k+2} [t'_{j-2}, t'_{j+1}].$$

$\square$

3.3. **Intersection number estimates.** We will require the following estimate for the intersection number of a curve δ and the curves γ_i^h in terms of the numbers e_i^h, $i \geq 0$.

Lemma 3.6. *Given $\delta \in \mathcal{C}_0(S)$ with $i(\delta, \alpha) \neq 0$, there exists $\kappa = \kappa(\delta) \geq 1$ so that for $h = 0, 1$ and all i sufficiently large we have*

$$\frac{1}{\kappa} e_{i-1}^h \leq i(\delta, \gamma_i^h) \leq \kappa I_h(i),$$

for $I_h(i) = \sum_J \prod_{j \in J} e_j^h$ where J runs over all subsets of $\{0, \ldots, i-1\}$ exactly once.

Proof. Suppose that $\delta \cap S_h$, $h = 0, 1$, consists of n_h geometric arcs with end points on $\alpha = \partial S_h$ (geometric arcs are proper arcs on the surface and homotopic geometric arcs are not identified). Let $[0; e_1^h, e_2^h, \ldots, e_{i-1}^h, \ldots]$ be a continued fraction expansion as in §3.1 and recall that the curve γ_i^h has slope reciprocal to $\frac{p_i^h}{q_i^h}$ the i^{th} convergent of the continued fraction expansion. Let τ_h be a geometric arc in $\delta \cap S_h$ with the largest intersection number with γ_i^h and let $\frac{a_h}{b_h}$ be the reciprocal of the slope of τ_h. Then, since $\gamma_i^h \subset S_h$, we have that $\mathrm{i}(\tau_h, \gamma_i^h) = |a_h q_i^h - b_h p_i^h|$.

The standard recursive formula for convergents of continued fraction expansions gives us $q_i^h = e_{i-1}^h q_{i-1}^h + q_{i-2}^h$ (see e.g. [Khi64, Theorem 1]; recall our index convention in §3.1), we also have that $q_0^h = 1$ and $q_1^h = e_0^h$. Then we can easily verify by induction on i that

$$q_i^h = \prod_{j=0}^{i-1} e_j^h + \sum_{J \subsetneq \{0,\ldots,i-1\}} \prod_{j \in J} e_j^h \tag{3.5}$$

where each subset J appears at most once in the sum. Now since $\lim_{i\to\infty} \frac{p_i^h}{q_i^h} = x_h$, where the irrational number x_h is the reciprocal of the slope of λ_h, we have

$$\lim_{i\to\infty} \frac{|a_h q_i^h - b_h p_i^h|}{q_i^h} = \lim_{i\to\infty} |a_h - (\frac{p_i^h}{q_i^h}) b_h| = |a_h - x_h b_h|.$$

Thus, for i sufficiently large

$$|a_h q_i^h - b_h p_i^h| \le 2|a_h - x_h b_h| q_i^h \le 2|a_h - x_h b_h| I_h(i).$$

Then since τ_h is a geometric arc with the largest intersection number with γ_i^h and since there are n_h geometric arcs in $\delta \cap S_h$ we have

$$\mathrm{i}(\delta, \gamma_i^h) \le 2n_h |a_h - x_h b_h| I_h(i). \tag{3.6}$$

Furthermore, by (3.5), $q_i^h \ge \prod_{j=0}^{i-1} e_j^h \ge e_{i-1}^h$. From this inequality and the above limit we deduce that the inequality

$$|a_h q_i^h - b_h p_i^h| \ge \frac{1}{2}|a_h - x_h b_h| q_i^h \ge \frac{1}{2}|a_h - x_h b_h| e_{i-1}^h \tag{3.7}$$

holds for all i sufficiently large.

Now from inequalities (3.6) and (3.7) we see that the inequalities of the lemma hold for $\kappa = \max\{2n_h|a_h - x_h b_h|, \frac{2}{|a_h - x_h b_h|} : h = 0, 1\}$. □

For any $k \in \mathbb{N}$, appealing to the Notation 3.2, let $I(k) = I_{\bar{k}}(i)$ where $k = 2i + \bar{k}$. The conclusion of Lemma 3.6 then becomes

$$\frac{1}{\kappa} e_{k-2} \le i(\delta, \gamma_k) \le \kappa I(k). \tag{3.8}$$

For the remainder of the paper, we assume that the sequence $\{e_k\}_k$ satisfies the additional growth condition

$$\lim_{k\to\infty} \frac{I(k)}{e_k} = 0, \text{ and } \lim_{k\to\infty} \frac{I(k+1)}{e_k} = 0. \tag{3.9}$$

This is possible since $I(k)$ depends only on $\{e_j\}_{j=0}^{k-2}$.

With this convention, we have the following corollary of Lemma 3.6.

Corollary 3.7. *For any curve $\delta \in \mathcal{C}_0(S)$ with $i(\delta, \alpha) \neq 0$ we have*

$$\lim_{k\to\infty} \frac{i(\delta, \gamma_k)}{e_k} = 0, \text{ and } \lim_{k\to\infty} \frac{i(\delta, \gamma_{k+1})}{e_k} = 0.$$

3.4. **Geodesics in** $\mathrm{Teich}(S)$ **and bounded length curves.** We begin by recalling [Mod15, Corollary 3.5] which we require to obtain some of the estimates in this section.

Lemma 3.8. *Given $c > 0$ let $l, a \in [0, c]$ with $l > a$. Suppose that for a curve $\beta \in \mathcal{C}_0(S)$ and points $X, X' \in \mathrm{Teich}(S)$ we have $\ell_\beta(X) \le l - a$ and $\ell_\beta(X') \ge l$, then*

$$d_{\mathrm{WP}}(X, X') \ge \frac{a}{\sqrt{\frac{2}{\pi} l + O(l^4)}},$$

where the constant of the O–notation depends only on c.

Lemma 3.9. *There is an $\epsilon > 0$ and a $C' > 0$ so that for all points Y in the ϵ–neighborhood of $\hat{r}^{\bar{k}}(t'_k)$ and all $j = k-1, k, k+1, k+2$, we have*

$$\ell_{\gamma_j}(Y) < C'.$$

Proof. Let C be the constant from Corollary 3.5 so that $\ell_{\gamma_j}(\hat{r}^{\bar{k}}(t'_k)) \leq C$ for $j = k-1, \ldots, k+2$. Let $a > 0, C' = a + C$ and $c = C' + a + 1$. Then

$$0 < a < C' < c \text{ and } \ell_{\gamma_j}(\hat{r}^{\bar{k}}(t'_k)) \leq C = C' - a,$$

for each $j = k-1, k, k+1, k+2$. Define

$$\epsilon = \frac{a}{\sqrt{\frac{2}{\pi}C' + O(C'^4)}}$$

to be as in Lemma 3.8 where the constant of the O–notation only depends on c. Now, if $\ell_{\gamma_j}(Y) \geq C'$ for a point Y in the ϵ–neighborhood of $\hat{r}^{\bar{k}}(t'_k)$ and a curve γ_j with $j = k-1, k, k+1, k+2$, then applying Lemma 3.8 we have $d_{\mathrm{WP}}(Y, \hat{r}^{\bar{k}}(t'_k)) \geq \epsilon$ which contradicts the fact that Y is in the ϵ–neighborhood of $\hat{r}^{\bar{k}}(t'_k)$. Therefore, $\ell_{\gamma_j}(Y) < C'$, for $j = k-1, k, k+1, k+2$, proving the lemma. □

Decreasing ϵ we may further assume that a point in the 2ϵ–neighborhood of $\mathcal{S}(\alpha)$ may only lie on a stratum corresponding to a (possibly empty) multicurve disjoint from α. This follows from the fact that there is a lower bound for the distance between any two disjoint strata of $\overline{\mathrm{Teich}(S)}$.

Now let $Z \in \mathrm{Teich}(S)$ be a point in the ϵ–neighborhood of $\hat{r}(0)$. Let then $[Z, \hat{r}(t'_k)]$ be the geodesic segment connecting Z to $\hat{r}(t'_k)$. The curves γ_k, γ_{k+1} have bounded length at $\hat{r}(t'_k)$, by Corollary 3.5, and the sequence of curves $\{\gamma_i^{\bar{k}}\}_i$, $\bar{k} = 0, 1$, is a quasi-geodesic in $\mathcal{C}(S_{\bar{k}})$ that converges to a point in the Gromov boundary of $\mathcal{C}(S_{\bar{k}})$. Moreover, $S \backslash \alpha$ is the union of S_0 and S_1. Then as in [Mod15, Lemma 8.1] we can show that after possibly passing to a subsequence $[Z, \hat{r}(t'_k)]$ converges to an infinite ray

$$r : [0, \infty) \to \mathrm{Teich}(S)$$

in the visual sphere of the Teichmüller space at Z. Also, note that the construction of r and the CAT(0) property of the WP metric imply the rays r and $\hat{r}$, ϵ–fellow travel.

The following are straightforward consequences of the results of this section.

Corollary 3.10. *For all $k \geq 2$ and $j = k-1, k, k+1, k+2$, we have*

$$\ell_{\gamma_j}(r(t'_k)) \leq C',$$

where $C' > 0$ is the constant from Lemma 3.9.

Proof. As noted above, the two geodesics rays r and $\hat{r}$, ϵ–fellow travel, and hence $d_{\mathrm{WP}}(\hat{r}(t'_k), r(t'_k)) < \epsilon$. Thus, by Lemma 3.9, for $j = k-1, k, k+1, k+2$, we have

$$\ell_{\gamma_j}(r(t'_k)) \leq C',$$

as desired. □

This, in turn, implies the following:

Corollary 3.11. *For all $k \geq 2$, $t \in [t'_k, t'_{k+1}]$, and $j = k, k+1, k+2$,*

$$\ell_{\gamma_j}(r(t)) \leq C'$$

where $C' > 0$ is the constant from Lemma 3.9.

Proof. By Corollary 3.10, we have $\ell_{\gamma_k}(r(t'_{k-2})) \leq C'$ and $\ell_{\gamma_k}(r(t'_{k+1})) \leq C'$. By convexity of length-functions [Wol03, §3.3], for all $t \in [t'_{k-2}, t'_{k+1}]$ we have

$$\ell_{\gamma_k}(r(t)) \leq C'.$$

Since $[t'_k, t'_{k+1}] \subset [t'_{k-2}, t'_{k+1}] \cap [t'_{k-1}, t'_{k+2}] \cap [t'_k, t'_{k+3}]$, the result follows. □

Proposition 3.12. *The ending lamination of r is the lamination $\lambda_0 \cup \alpha \cup \lambda_1$. Also, the length of α is bounded along r.*

Proof. By Corollary 3.10 the curves γ_k, γ_{k+1} have bounded length at $r(t'_k)$ for all k, hence by the definition of ending lamination, λ_0 and λ_1 are contained in the ending lamination of r. Moreover, note that r is the limit of geodesic segments with α pinched at their end points. The fact that the length of α is bounded along r then follows form the convergence of lengths of ending measures of WP geodesics [BMM10, Lemma 2.10]. This also tells us that α is contained in the ending lamination of r. Finally, we can conclude that the ending lamination of r is $\lambda_0 \cup \alpha \cup \lambda_1$. □

We now turn to estimates for twists about bounded length curves at $r(t'_k)$.

Lemma 3.13. *For any $\delta \in \mathcal{C}_0(S)$ with $i(\delta, \alpha) \neq 0$, there exists $c = c(\delta) > 0$ such that for all $t \in [t'_k, t'_{k+1}]$, we have*

$$\mathrm{tw}_{\gamma_k}(\delta, r(t)) \stackrel{+}{\asymp}_c e_k,$$

and

$$\mathrm{tw}_{\gamma_{k+1}}(\delta, r(t'_k)) \stackrel{+}{\asymp}_c 1$$

for all but finitely many k (namely, whenever $i(\gamma_k, \delta) \neq 0$ and $i(\gamma_{k+1}, \delta) \neq 0$, respectively).

Proof. By Corollary 3.11 we may choose a bounded length marking μ at $r(t)$ so that γ_k is in the base and γ_{k+2} projects to the transversal to γ_k. Recall that $\bar{k} \in \{0, 1\}$ is the residue of k modulo 2. Avoiding finitely many k, $i(\delta, \gamma_k) \neq 0$, so by the triangle inequality we have

$$\left| d_{\gamma_k}(\delta, \gamma_{k+2}) - d_{\gamma_k}(\gamma_{\bar{k}}, \gamma_{k+2}) \right| \leq d_{\gamma_k}(\delta, \gamma_{\bar{k}}). \tag{3.10}$$

Since $\mu|_{S_{\bar{k}}}$ is a uniformly bounded length marking at $r(t)$, we have uniform errors (depending on δ, but not on k) in the following coarse equations. First,

$$\mathrm{tw}_{\gamma_k}(\delta, r(t)) \overset{+}{\asymp} d_{\gamma_k}(\delta, \mu) = d_{\gamma_k}(\delta, \gamma_{k+2}).$$

Since the curves $\{\gamma_{\bar{k}+2i}\}_i$ are the vertices of a geodesic in $\mathcal{C}(S_{\bar{k}})$, and the projection distance between γ_{k-2} and γ_{k+2} to γ_k is e_k, we have

$$d_{\gamma_k}(\gamma_{\bar{k}}, \gamma_{k+2}) \overset{+}{\asymp} d_{\gamma_k}(\gamma_{k-2}, \gamma_{k+2}) = e_k.$$

Moreover, since we are allowing our error $c = c(\delta)$ to depend on δ, we can combine these coarse equations with inequality (3.10) and deduce

$$\mathrm{tw}_{\gamma_k}(\delta, r(t)) \overset{+}{\asymp} e_k.$$

This proves the first statement of the lemma.

For the second statement, we note that by Corollary 3.10, we may choose our bounded length marking μ at $r(t'_k)$ so that γ_{k+1} is a base curve and γ_{k-1} projects to a transversal for γ_{k+1}. We may carry out computations similar to the above, except now we use the fact that $\gamma_{\overline{k+1}}$ and γ_{k-1} precede γ_{k+1} in the $\mathcal{C}(S_{\overline{k+1}})$–geodesic, and hence

$$d_{\gamma_{k+1}}(\gamma_{\overline{k+1}}, \gamma_{k-1}) \overset{+}{\asymp} 1,$$

as required. This completes the proof of the lemma. □

3.5. **Estimates for the separating curve.** We will eventually impose additional growth conditions on our sequence $\{e_k\}$ to control the length and twisting about the separating curve α. The next two lemmas are used to determine those conditions.

Lemma 3.14. *There exists a function $f_1 : [0, \infty) \to \mathbb{R}^+$, so that for any geodesic ray r constructed as above (from sequences $\{e^h_i\}_i$, $h = 0, 1$, beginning at Z) we have $\ell_\alpha(r(T)) \geq f_1(T)$ for all $T \in [0, \infty)$.*

Proof. Suppose on the contrary that such a function does not exist. Then there is a sequence of geodesic rays r_n starting at Z, coming from sequences $\{e^h_i(n)\}_i$, as above, and a $T > 0$, so that $\lim_{n\to\infty} \ell_\alpha(r_n(T)) = 0$. By the choice of Z and ϵ_1, the distance of r_n and all strata of Teichmüller space except the ones corresponding to the multi-curves disjoint from α is at least ϵ_1. Now note that $\ell_\alpha(r_n(0)) = \ell_\alpha(Z) > 0$, then since $\lim_{n\to\infty} \ell_\alpha(r_n(T)) = 0$ for any $t > T$ by convexity of the α length-function (see [Wol03, §3.3]) we have that $\ell_\alpha(r_n(t)) \leq \ell_\alpha(r_n(T))$ for all n sufficiently large. Then, since $\lim_{n\to\infty} \ell_\alpha(r_n(T)) = 0$ we get $\lim_{n\to\infty} \ell_\alpha(r_n(t)) = 0$ as well.

Now fix a $T' > T$ and apply Theorem 2.2 to the sequence of geodesic segments $r_n|_{[0,T']}$ and let the partition $0 = t_0 < t_1 < \ldots < t_{k+1} = T'$ and piecewise geodesic segment $\hat{\zeta}$ be from the theorem. Moreover, let multi-curves σ_l, $l = 0, \ldots, k+1$, $\hat{\tau}$ and elements of mapping class group $\psi_n, \mathcal{T}_{i,n}$, $l = 0, \ldots, k$ and $n \in \mathbb{N}$, be from the theorem. Also, as in the theorem let

$\varphi_{l,n} = \mathcal{T}_{l,n} \circ \ldots \circ \mathcal{T}_{1,n} \circ \psi_n$. Since all the geodesic segments $r_n|_{[0,T]}$ start at the point $Z \in \text{Teich}(S)$, $\psi_n \equiv I$, moreover $\hat{\tau} = \emptyset$ and $\sigma_0 = \emptyset$.

For each $0 \leq l \leq k+1$ and $n \geq 1$, $\mathcal{T}_{l,n}$ is the composition of powers of positive Dehn twists about curves in σ_l, but since r_n has distance at least ϵ_1 from all completion strata except strata of multi-curves disjoint from α, σ_l consists of possibly the curve α and a number of curves disjoint from α. Therefore, $\varphi_{l,n}(\alpha) = \alpha$, and $\ell_\alpha(\varphi_{l,n}(r_n(t))) = \ell_\alpha(r_n(t))$ for all $t \in [t_l, t_{l+1}]$. Then by the fact that $\lim_{n\to\infty} \ell_\alpha(r_n(t)) = 0$ for all $t \in [T, T']$, we have $\lim_{n\to\infty} \ell_\alpha(\varphi_{l,n}(r_n(t))) = 0$. Thus, by part (3) of the theorem $\ell_\alpha(\hat{\zeta}(t)) = 0$, and hence $\zeta([T, T'])$ lies in the closure of $\mathcal{S}(\alpha)$. On the other hand, given that $\hat{\tau} = \emptyset$, part (2) of Theorem 2.2 asserts that $\zeta(t) \in \text{Teich}(S)$ for all $t \in [T, T']$ except possibly the points $\{t_l\}_{l=0}^{k+1} \cap [T, T']$. This contradiction finishes the proof of the fact that $\ell_\alpha(r(T))$ is bounded below by a positive number, depending on T, but independent of the ray r. $\square$

Lemma 3.15. *There exists a function $f_2 : [0, \infty) \to \mathbb{R}^+$ such that for any geodesic ray r as above $d_\alpha(r(0), r(t)) \leq f_2(t)$ for all $t \in [0, \infty)$.*

Proof. Suppose that such a function does not exist. Then there is a sequence of geodesic rays r_n constructed as above and a $T > 0$, so that $\lim_{n\to\infty} d_\alpha(r_n(0), r_n(T)) = \infty$. Then, since $r_n(0) = Z$ for all $n \geq 1$ we have that $\sup_t \ell_\alpha(r_n(t)) \geq \text{inj}(Z) > 0$. Then, by Theorem 2.3 we have that $\inf_{t\in[0,T]} \ell_\alpha(r_n(t)) \to 0$ as $n \to \infty$. But this contradicts the fact that $\inf_{t\in[0,T]} \ell_\alpha(r_n(t)) \geq \inf_{t\in[0,T]} f_1(t) > 0$ for all $n \geq 1$ by Lemma 3.15. The assertion of the lemma now follows from this contradiction. $\square$

With these two lemmas in place, we now impose our final growth conditions on $\{e_k\}_k$. Let f_1, f_2 be the functions from Lemmas 3.14 and 3.15, and for $k \in \mathbb{N}$ let

$$F_{1,k} = \min\{f_1(s) \mid s \in [t_k, t_{k+2}]\} \tag{3.11}$$

$$F_{2,k} = \max\{f_2(s) \mid s \in [t_k, t_{k+2}]\}. \tag{3.12}$$

As our last growth requirement for $\{e_k\}_k$, we assume that,

$$\lim_{k\to\infty} \frac{F_{2,k} - 2\log F_{1,k}}{e_k} = 0. \tag{3.13}$$

3.6. **Limit sets.** For the remainder of the paper, we let $\{e_i^h\}_{i=0}^\infty$ with $e_i^h \geq K_i$, for $h = 0, 1$ and all $i \in \mathbb{N}$, where K_i is from Lemma 3.1. Let $\{e_k\}_k$, $\{\gamma_k\}_k$, $\{t_k\}_k$, $\{t'_k\}_k$, $\{X_k\}_k$ be as in Notation 3.2, and assume that $\{e_k\}_k$ satisfies (3.9) and (3.13).

Theorem 3.16. *The limit set of r in the Thurston compactification of* Teich(S) *is the* 1*–simplex* $[[\bar{\lambda}_0], [\bar{\lambda}_1]]$ *of projective classes of measures supported on $\lambda_0 \cup \lambda_1$.*

For curves $\delta, \gamma \in \mathcal{C}_0(S)$ and any time $s \in [0, \infty)$ let

$$\ell_\delta(\gamma, s) = \text{i}(\delta, \gamma)\Big(w_\gamma(r(s)) + \ell_\gamma(r(s))\, \text{tw}_\gamma(\delta, r(s))\Big). \tag{3.14}$$

Now suppose that $\{s_k\}_k$ is a sequence such that $s_k \in [t'_k, t'_{k+1}]$. Pass to a subsequence $\{s_k\}_{k\in\mathcal{K}}$ so that $r(s_k) \to [\bar{\nu}]$ in the Thurston compactification (to avoid cluttering the notation with additional subscripts, we have chosen to index a subsequence using a subset $\mathcal{K} \subset \mathbb{N}$). Let $\{u_k\}_{k\in\mathcal{K}}$ be a scaling sequence, so that

$$\lim_{k\to\infty} u_k \ell_\delta(r(s_k)) = i(\delta, \bar{\nu}),$$

for all curves δ.

By Corollary 3.11 and Proposition 3.12, the curves $\gamma_k, \gamma_{k+1}, \alpha$ form a uniformly bounded length pants decomposition on $r(s_k)$. Consequently, we may appeal to [CRS08, Lemmas 7.2, 7.3] to obtain the following expansion for the length of the curve $\delta \in \mathcal{C}_0(S)$ at $r(s_k)$,

$$\begin{aligned} \ell_\delta(r(s_k)) &= \ell_\delta(\gamma_k, s_k) + \ell_\delta(\gamma_{k+1}, s_k) + \ell_\delta(\alpha, s_k) \\ &\quad + O\left(\mathrm{i}(\delta, \gamma_k)\right) + O\left(\mathrm{i}(\delta, \gamma_{k+1})\right) + O\left(\mathrm{i}(\delta, \alpha)\right) \end{aligned} \tag{3.15}$$

where the constant of the O notation depends only on the uniform upper bounds for the lengths of γ_k, γ_{k+1}, and α.

The next proposition shows that only two of the terms in (3.15) are actually relevant.

Proposition 3.17. *With notation as above, and $\delta \in \mathcal{C}_0(S)$ with $i(\delta, \alpha) \neq 0$, we have*

$$i(\delta, \bar{\nu}) = \lim_{k\to\infty} u_k \ell_\delta(r(s_k)) = \lim_{k\to\infty} u_k(\ell_\delta(\gamma_k, s_k) + \ell_\delta(\gamma_{k+1}, s_k)).$$

For this, we will need the following lemma.

Lemma 3.18. *With notation as above,*

$$w_{\gamma_k}(r(s_k)) + \ell_{\gamma_k}(r(s_k))\, \mathrm{tw}_{\gamma_k}(\delta, r(s_k)) \overset{+}{\asymp} \ell_{\gamma_k}(r(s_k)) e_k,$$

where the constant in the coarse equation depends on δ, but not on k.

Proof. By Corollary 3.11,

$$\ell_{\gamma_k}(r(s_k)), \ell_{\gamma_{k+2}}(r(s_k)) \leq C'.$$

Since $\mathrm{i}(\gamma_k, \gamma_{k+2}) = 1$, the Collar Lemma [Bus10, §4] implies that

$$w_{\gamma_k}(r(s_k)) \overset{*}{\asymp} 1 \quad \text{and} \quad \ell_{\gamma_k}(r(s_k)) \overset{*}{\asymp} 1.$$

By Lemma 3.13 we have $\mathrm{tw}_{\gamma_k}(\delta, r(s_k)) \overset{+}{\asymp} e_k$, and so the lemma follows. □

Proof of Proposition 3.17. First, observe that by Corollary 3.7 (and since α is a fixed curve and $e_k \to \infty$), we have

$$\lim_{k\to\infty} \frac{\mathrm{i}(\delta, \gamma_k)}{e_k} = 0, \lim_{k\to\infty} \frac{\mathrm{i}(\delta, \gamma_{k+1})}{e_k} = 0, \text{ and } \lim_{k\to\infty} \frac{\mathrm{i}(\delta, \alpha)}{e_k} = 0. \tag{3.16}$$

As in the proof of Lemma 3.18, $\ell_{\gamma_k}(r(s_k)) \overset{*}{\asymp} 1$, and so $\ell_{\gamma_k}(r(s_k)) e_k \to \infty$ and $\mathrm{i}(\delta, \gamma_k) \to \infty$. From Lemma 3.18 and the definition, we have $\ell_\delta(\gamma_k, s_k) \overset{*}{\succeq} e_k$.

Moreover, $u_k\ell_\delta(r(s_k)) \to \mathrm{i}(\delta,\bar\nu) > 0$, then by (3.15), $u_k \overset{*}{\preceq} \frac{1}{e_k}$. Combining this with (3.16) and appealing to (3.15) again, we see that

$$i(\delta,\bar\nu) = \lim_{k\to\infty} u_k\ell_\delta(r(s_k)) = \lim_{k\to\infty} u_k(\ell_\delta(\gamma_k,s_k) + \ell_\delta(\gamma_{k+1},s_k) + \ell_\delta(\alpha,s_k)).$$

By similar reasoning, to eliminate the last term (and thus prove the proposition), it suffices to prove

$$\lim_{k\to\infty} \frac{\ell_\delta(\alpha,s_k)}{e_k} = 0. \tag{3.17}$$

To do this, first note that by Lemma 3.14, $\ell_\alpha(r(s_k)) \geq f_1(s_k) \geq F_{1,k}$, and so by the Collar Lemma we have

$$w_\alpha(r(s_k)) \overset{+}{\asymp} -2\log(\ell_\alpha(r(s_k))) \leq -2\log(F_{1,k}).$$

By Lemma 3.15, we also have

$$\mathrm{tw}_\alpha(\delta, r(s_k)) \overset{+}{\asymp} d_\alpha(r(0), r(s_k)) \leq f_2(s_k) \leq F_{2,k},$$

where the additive constant depends on δ. Therefore, since $F_{1,k} \overset{+}{\preceq} 1$, and since $\ell_\alpha(r(s_k))$ is uniformly bounded by Proposition 3.12, we have

$$\ell_\delta(\alpha,s_k) \overset{+}{\preceq} \mathrm{i}(\delta,\alpha)\Big(-2\log(F_{1,k}) + F_{2,k}\Big),$$

with additive error that again depends on δ. By our growth condition (3.13), since $\mathrm{i}(\delta,\alpha)$ does not depend on k, we have

$$\lim_{k\to\infty} \frac{\ell_\delta(\alpha,s_k)}{e_k} \leq \lim_{k\to\infty} \frac{\mathrm{i}(\delta,\alpha)\Big(-2\log(F_{1,k}) + F_{2,k}\Big)}{e_k} = 0.$$

This proves (3.17), and hence the proposition. □

Continue to let $\{s_k\}_{k\in\mathcal{K}}$ be a sequence with $s_k \in [t'_k, t'_{k+1}]$ as above, and suppose that $r(s_k) \to [\bar\nu]$ as $k\to\infty$ in the Thurston compactification and that $\{u_k\}_k$ is a scaling sequence. We set

$$x(s_k) = w_{\gamma_k}(r(s_k)) + \ell_{\gamma_k}(r(s_k))\,\mathrm{tw}_{\gamma_k}(\gamma_{\bar k}, r(s_k)),$$

and

$$y(s_k) = w_{\gamma_{k+1}}(r(s_k)) + \ell_{\gamma_{k+1}}(r(s_k))\,\mathrm{tw}_{\gamma_{k+1}}(\gamma_{\overline{k+1}}, r(s_k)).$$

Lemma 3.19. *For any $\delta \in \mathcal{C}_0(S)$ with $i(\delta,\alpha) \neq 0$, we have*

$$\lim_{k\to\infty} \frac{x(s_k)i(\delta,\gamma_k) + y(s_k)i(\delta,\gamma_{k+1})}{\ell_{\gamma_k}(\delta,s_k) + \ell_{\gamma_{k+1}}(\delta,s_k)} = 1.$$

For $s_k = t'_k$, we have

$$\lim_{k\to\infty} \frac{x(t'_k)i(\delta,\gamma_k)}{\ell_{\gamma_k}(\delta,t'_k) + \ell_{\gamma_{k+1}}(\delta,t'_k)} = 1.$$

Proof. As in the proof of Lemma 3.13

$$\mathrm{tw}_{\gamma_k}(\delta, r(s_k)) \overset{+}{\asymp} \mathrm{tw}_{\gamma_k}(\gamma_{\bar{k}}, r(s_k))$$

and

$$\mathrm{tw}_{\gamma_{k+1}}(\delta, r(s_k)) \overset{+}{\asymp} \mathrm{tw}_{\gamma_k}(\gamma_{\overline{k+1}}, r(s_k))$$

where the implicit constant in these coarse equations depends on δ.

According to Corollary 3.11, $\ell_{\gamma_k}(r(s_k)) \leq C'$. From the preceding coarse equations and Lemma 3.18, we have

$$x(s_k) \overset{+}{\asymp} w_{\gamma_k}(r(s_k)) + \ell_{\gamma_k}(r(s_k))\, \mathrm{tw}_{\gamma_k}(\delta, r(s_k)) \overset{+}{\asymp} \ell_{\gamma_k}(r(s_k)) e_k. \tag{3.18}$$

Since $e_k \to \infty$ as $k \to \infty$, the following is immediate:

$$\lim_{k\to\infty} \frac{x(s_k)\mathrm{i}(\delta,\gamma_k)}{\ell_\delta(\gamma_k, s_k)} = \lim_{k\to\infty} \frac{x(s_k)}{w_{\gamma_k}(r(s_k)) + \ell_{\gamma_k}(r(s_k))\, \mathrm{tw}_{\gamma_k}(\delta, r(s_k))} = 1. \tag{3.19}$$

Similar to (3.18) we have

$$y(s_k) \overset{+}{\asymp} w_{\gamma_{k+1}}(r(s_k)) + \ell_{\gamma_{k+1}}(r(s_k))\, \mathrm{tw}_{\gamma_{k+1}}(\delta, r(s_k)) = \frac{\ell_{\gamma_{k+1}}(\delta, s_k)}{\mathrm{i}(\delta, \gamma_{k+1})}. \tag{3.20}$$

By (3.8) and the growth condition (3.9) we have

$$\lim_{k\to\infty} \frac{\mathrm{i}(\delta, \gamma_{k+1})}{e_k} = 0.$$

After passing to a subsequence, there are two cases to consider:

Case 1. There exists $R > 0$ so that $y(s_k) \leq R$ for all k.

In this case, appealing to (3.16) and (3.20) we have

$$0 = \lim_{k\to\infty} \frac{y(s_k)\mathrm{i}(\delta, \gamma_{k+1})}{e_k} = \lim_{k\to\infty} \frac{\ell_{\gamma_{k+1}}(\delta, s_k)}{e_k},$$

and thus

$$\begin{aligned}\lim_{k\to\infty} \frac{x(s_k)\mathrm{i}(\delta,\gamma_k) + y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{\ell_{\gamma_k}(\delta, s_k) + \ell_{\gamma_{k+1}}(\delta, s_k)} &= \lim_{k\to\infty} \frac{\frac{x(s_k)\mathrm{i}(\delta,\gamma_k)}{e_k} + \frac{y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{e_k}}{\frac{\ell_{\gamma_k}(\delta,s_k)}{e_k} + \frac{\ell_{\gamma_{k+1}}(\delta,s_k)}{e_k}} \\ &= \lim_{k\to\infty} \frac{x(s_k)\mathrm{i}(\delta,\gamma_k)}{\ell_{\gamma_k}(\delta, s_k)} = 1.\end{aligned}$$

Case 2. $\lim_{k\to\infty} y(s_k) = \infty$.

Here, we can argue as for $x(s_k)$, appealing to (3.20) to deduce that

$$\lim_{k\to\infty} \frac{y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{\ell_{\gamma_{k+1}}(\delta, s_k)} = 1.$$

Combined with (3.19) we have

$$\lim_{k\to\infty} \frac{x(s_k)\mathrm{i}(\delta,\gamma_k) + y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{\ell_{\gamma_k}(\delta, s_k) + \ell_{\gamma_{k+1}}(\delta, s_k)} = 1.$$

These two cases prove the first claim of the lemma. For the second claim, when $s_k = t'_k$, we note that by Corollary 3.10 we have

$$\ell_{\gamma_{k-1}}(r(t'_k)), \ell_{\gamma_{k+1}}(r(t'_k)) \le C',$$

and so by the Collar Lemma (as in the proof of Lemma 3.18) we have

$$w_{\gamma_{k+1}}(r(t'_k)) \overset{*}{\asymp} 1 \quad \text{and} \quad \ell_{\gamma_{k+1}}(r(t'_k)) \overset{*}{\asymp} 1$$

so since $\mathrm{tw}_{\gamma_{k+1}}(\delta, r(t'_k)) \overset{+}{\asymp} 1$ by Lemma 3.13, it follows that $y(t'_k)$ is uniformly bounded, and thus as in Case 1, we deduce

$$\lim_{k\to\infty} \frac{x(t'_k)\mathrm{i}(\delta,\gamma_k)}{\ell_{\gamma_k}(\delta, t'_k) + \ell_{\gamma_{k+1}}(\delta, t'_k)} = 1,$$

completing the proof. □

We are now ready for the

Proof of Theorem 3.16. First, we show that $[\bar\lambda_0]$ and $[\bar\lambda_1]$ are in the limit set Λ of r. Consider the sequence of times $\{t'_{2k}\}$ and pass to a subsequence so that $r(t'_{2k}) \to [\bar\nu]$ in the Thurston compactification and let $\{u_k\}$ be a scaling sequence for $r(t'_{2k})$. Let δ be any curve with $\mathrm{i}(\delta,\bar\nu) \ne 0$ and $\mathrm{i}(\delta,\alpha) \ne 0$. By the second part of Lemma 3.19, together with Proposition 3.17 we have

$$\begin{aligned} 1 &= \lim_{k\to\infty} \frac{x(t'_{2k})\mathrm{i}(\delta,\gamma_{2k})}{\ell_{\gamma_{2k}}(\delta,t'_{2k}) + \ell_{\gamma_{2k+1}}(\delta,t'_{2k})} = \lim_{k\to\infty} \frac{u_k x(t'_{2k})\mathrm{i}(\delta,\gamma_{2k})}{u_k(\ell_{\gamma_{2k}}(\delta,t'_{2k}) + \ell_{\gamma_{2k+1}}(\delta,t'_{2k}))} \\ &= \frac{\lim_{k\to\infty} \mathrm{i}(\delta, u_k x(t'_{2k})\gamma_{2k})}{\mathrm{i}(\delta,\bar\nu)}. \end{aligned}$$

Therefore $\lim_{k\to\infty} \mathrm{i}(\delta, u_k x(t'_{2k})\gamma_{2k}) = \mathrm{i}(\delta,\bar\nu)$. We apply this to a set of curves $\delta_1,\dots,\delta_N$ sufficient for determining a measured lamination, and so deduce that $\lim_{k\to\infty} u_k x(t'_{2k})\gamma_{2k} = \bar\nu$.

On the other hand, $[\gamma_{2k}] \to [\bar\lambda_0]$, hence $[\bar\nu] = [\bar\lambda_0]$, and so $[\bar\lambda_0]$ is in Λ. A similar argument using the sequence $\{t'_{2k+1}\}$ shows that $[\bar\lambda_1] \in \Lambda$.

Now suppose that $\{s_k\}_k$ is an arbitrary sequence so that $r(s_k) \to [\bar\nu]$ and let $\{u_k\}_k$ be a scaling sequence. Adjusting indices and passing to a subsequence we can assume that $s_k \in [t'_k, t'_{k+1}]$ for all $k \in \mathcal{K}$ (some subset $\mathcal{K} \subset \mathbb{N}$). Arguing as above, appealing to the first part of Lemma 3.19 and Proposition 3.17 we have

$$\begin{aligned} 1 &= \lim_{k\to\infty} \frac{x(s_k)\mathrm{i}(\delta,\gamma_k) + y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{\ell_{\gamma_k}(\delta,s_k) + \ell_{\gamma_{k+1}}(\delta,s_k)} = \lim_{k\to\infty} \frac{u_k x(s_k)\mathrm{i}(\delta,\gamma_k) + u_k y(s_k)\mathrm{i}(\delta,\gamma_{k+1})}{u_k(\ell_{\gamma_k}(\delta,s_k) + \ell_{\gamma_{k+1}}(\delta,s_k))} \\ &= \frac{\lim_{k\to\infty} \mathrm{i}\big(\delta, u_k(x(s_k)\gamma_k + y(s_k)\gamma_{k+1})\big)}{\mathrm{i}(\delta,\bar\nu)}. \end{aligned}$$

So, $\lim_{k\to\infty} u_k(x(s_k)\gamma_k + y(s_k)\gamma_{k+1}) = \mathrm{i}(\delta, \bar{\nu})$, and as above

$$\bar{\nu} = \lim_{k\to\infty} u_k(x(s_k)\gamma_k + y(s_k)\gamma_{k+1}) = \lim_{k\to\infty} u_k x(s_k)\gamma_k + \lim_{k\to\infty} u_k y(s_k)\gamma_{k+1}.$$

Therefore $[\bar{\nu}] \in [[\bar{\lambda}_0], [\bar{\lambda}_1]]$. That is, Λ is contained in $[[\bar{\lambda}_0], [\bar{\lambda}_1]]$, and contains the endpoints. Moreover, Λ is connected, so it is the entire 1–simplex, as was desired. □

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