GEODESICS IN THE MAPPING CLASS GROUP

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Abstract. We construct explicit examples of geodesics in the mapping class group and show that the shadow of a geodesic in mapping class group to the curve graph does not have to be a quasi-geodesic. We also show that the quasi-axis of a pseudo-Anosov element of the mapping class group may not have the strong contractibility property. Specifically, we show that, after choosing a generating set carefully, one can find a pseudo-Anosov homeomorphism $\phi$, a sequence of points $w_k$ and a sequence of radii $r_k$ so that the ball $B(w_k, r_k)$ is disjoint from a quasi-axis $a_\phi$ of $\phi$, but for any projection map from mapping class group to $a_\phi$, the diameter of the image of $B(w_k, r_k)$ grows like $\log(r_k)$.

1. Introduction

Let $S$ be a surface of finite type and let $\text{Map}(S)$ denote the (pure) mapping class group of $S$, that is, the group of orientation preserving self homeomorphisms of $S$ fixing the punctures of $S$, up to isotopy. This is a finitely generated group [7] and, after choosing a generating set, the word length turns $\text{Map}(S)$ into a metric space. The geometry of $\text{Map}(S)$ has been a subject of extensive study. Most importantly, in [15], Masur and Minsky gave an estimate for the word length of a mapping class using the sub-surface projection distances and constructed efficient quasi-geodesics in the mapping class group, called hierarchy paths, connecting the identity to any given mapping class. The starting point of the construction of a hierarchy path is a geodesic in the curve graph of $S$ which is known to be a Gromov hyperbolic space [14]. Hence, by construction, the shadow of a hierarchy path to the curve graph is nearly a geodesic.

It may seem intuitive that any geodesic in the mapping class group should also have this property, considering that similar statements have been shown to be true in other settings. For example, it is known that the shadow of a geodesic in Teichmüller space with respect to the Teichmüller metric is a re-parametrized quasi-geodesic in the curve graph [14]. The same is true for any geodesic in Teichmüller space with respect to the Thurston metric [12], for any line of minima in Teichmüller space [9], for a grafting ray [5], or for the set of short curves in a hyperbolic 3–manifold homeomorphic to $S \times \mathbb{R}$ [17]. However, it is difficult to construct explicit examples of geodesics in $\text{Map}(S)$ and so far, all estimates for the word length of an element have been up to a multiplicative error.

In this paper, we argue that one should not expect geodesics in $\text{Map}(S)$ to be well-behaved in general. Changing the generating set changes the metric on $\text{Map}(S)$ significantly and a geodesic with respect to one generating set is only a quasi-geodesic with respect to another generating set. Since $\text{Map}(S)$ is not Gromov hyperbolic (it contains flats), its quasi-geodesics are not well behaved in general. Similarly, one should not expect that the geodesics with respect to an arbitrary generating set to behave well either.

We make this explicit in the case where $S = S_{0,5}$ is the five-times punctured sphere. Consider the curves $\alpha_1, \ldots, \alpha_5$ depicted in Figure 1. Fix an integer $n \gg 1$ (to be determined in the proof of Theorem 1.3), and consider the following generating set for $\text{Map}(S)$

$$S_n = \left\{ D_{\alpha_i, s_{i,j}} : i, j \in \mathbb{Z}_5, |i - j| = 1 \pmod 5 \right\}$$
where \( s_{i,j} = D_{\alpha_i}^{-1} D_{\alpha_j} \), and \( D_\alpha \) is a Dehn twist around a curve \( \alpha \). Since we are considering the pure mapping class group, the set \( \{ D_{\alpha_i} \}_{i=1}^5 \) already generates \( \text{Map}(S) \). We denote the distance on \( \text{Map}(S) \) induced by the generating set \( \delta_n \) by \( d_{\delta_n} \). By an \( \delta_n \)-geodesic, we mean a geodesic with respect to this metric.

**Theorem 1.1.** There is an \( n \gg 1 \) so that, for every \( K, C > 0 \), there exists an \( \delta_n \)-geodesic \( \gamma : [0,m] \to \text{Map}(S) \) so that the shadow of \( \gamma \) to the curve graph \( \mathcal{C}(S) \) is not a re-parametrized \((K,C)\)-quasi-geodesic.

Even though the mapping class group is not Gromov hyperbolic, it does have hyperbolic directions. There are different ways to make this precise. For example, Behrstock proved [2] that in the direction of every pseudo-Anosov, the divergence function in \( \text{Map}(S) \) is super-linear. Another way to make this notion precise is to examine whether geodesics in \( \text{Map}(S) \) have the contracting property.

This notion is defined analogously with Gromov hyperbolic spaces where, for every geodesic \( \mathcal{G} \) and any ball disjoint from \( \mathcal{G} \), the closest point projection of the ball to \( \mathcal{G} \) has a uniformly bounded diameter. However, often it is useful to work with different projection map. We call a map

\[
\text{Proj}: X \to \mathcal{G}
\]

from a metric space \( X \) to any subset \( \mathcal{G} \subset X \) a \((d_1,d_2)\)-projection map, \( d_1, d_2 > 0 \), if for every \( x \in X \) and \( g \in \mathcal{G} \), we have

\[
d_X(\text{Proj}(x), g) \leq d_1 \cdot d_X(x,g) + d_2.
\]

This is a weak notion of projection since \( \text{Proj} \) is not even assumed to be coarsely Lipschitz. By the triangle inequality, the closest point projection is always a \((2,0)\)-projection.

**Definition 1.2.** A subset \( \mathcal{G} \) of a metric space \( X \) is said to have the contracting property if there is a constant \( \rho < 1 \), a constant \( B > 0 \) and a projection map \( \text{Proj}: X \to \mathcal{G} \) such that, for any ball \( B(x,R) \) of radius \( R \) disjoint from \( \mathcal{G} \), the projection of a ball of radius \( \rho R \), \( B(x,\rho R) \), has a diameter at most \( B \).

\[
diam_{\delta_n}(\text{Proj}(B(x,\rho R))) \leq B.
\]

We say \( \mathcal{G} \) has the strong contracting property if \( \rho \) can be taken to be 1.

The axis of a pseudo-Anosov element has the contracting property in many settings. This has been shown to be true in the setting of Teichmüller space by Minsky [16], in the setting of the pants complex by Brock, Masur, and Minsky [4] and in the setting of the mapping class group by Duchin and Rafi [9].

In work by Arzhantseva, Cashen and Tao, they asked if the axis of a pseudo-Anosov element in the mapping class group has the strong contracting property and showed that a positive answer would imply that the mapping class group is growth tight [1]. Additionally, using the work of Yang [19], one can show that if one pseudo-Anosov element has a strongly contracting axis with respect to some generating set, then a generic element in mapping class group has a strongly
contracting axis with respect to this generating set. Similar arguments would also show that, if
one pseudo-Anosov element has a strongly contracting axis with respect to some generating set,
then the mapping class group with respect to this generating set has purely exponential growth.

However, using our specific generating set, we show that this does not always hold:

**Theorem 1.3.** For every $d_1, d_2 > 0$, there exists an $n \gg 1$, a pseudo-Anosov map $\phi$, a constant
$c_n > 0$, a sequence of elements $w_k \in \text{Map}(S)$ and a sequence of radii $r_k > 0$ where $r_k \to \infty$ as $k \to \infty$ such that the following holds. Let $\alpha_\phi$ be a quasi-axis for $\phi$ in $\text{Map}(S)$ and let $\text{Proj}_{\alpha_\phi}: \text{Map}(S) \to \alpha_\phi$ be any $(d_1, d_2)$–projection map. Then the ball of radius $r_k$ centered at $w_k$, $B(w_k, r_k)$, is disjoint
from $\alpha_\phi$ and $\text{diam}_{S_n} \left( \text{Proj}_{\alpha_\phi} \left( B(w_k, r_k) \right) \right) \geq c_n \log(r_k)$.

We remark that, since $\alpha_\phi$ has the contracting property [9], the diameter of the projection can
grow at most logarithmically with respect to the radius $r_k$ (see Corollary 5.5), hence the lower-
bound achieved by the above theorem is sharp.

**Outline of proof.** To find an exact value for the word-length of an element $f \in \text{Map}(S)$, we
construct a homomorphism

$$h: \text{Map}(S) \to \mathbb{Z},$$

where a large value for $h(f)$ guarantees a large value for the word length of $f$. At times, this
lower bound is realized and an explicit geodesic in $\text{Map}(S)$ is constructed (see Section 2). The
pseudo-Anosov element $\phi$ is defined as

$$\phi = D_{\alpha_5}D_{\alpha_4}D_{\alpha_3}D_{\alpha_2}D_{\alpha_1}.$$

In Section 2, we find an explicit invariant train-track for $\phi$ to show that $\phi$ is a pseudo-Anosov. In
Section 3, we use the geodesics constructed in Section 2 to show that the shadows of geodesics
in $\text{Map}(S)$ are not necessarily quasi-geodesics in the curve complex. In Section 5, we begin by
showing that $\phi$ acts loxodromically on $\text{Map}(S)$, that is, it has a quasi-axis $\alpha_\phi$ which fellow travels
the path $\{\phi^i\}$. We finish Section 5 by showing that the bound in our main theorem is sharp. In
Section 6, we set up and complete the proof of 1.3.

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## 2. Finding Explicit Geodesics

In this section, we develop the tools needed to show that certain paths in $\text{Map}(S)$ are geodesics.
We emphasize again that, in our paper, $S$ is the five-times punctured sphere and $\text{Map}(S)$ is the pure
mapping class group. That is, all homeomorphisms are required to fix the punctures point-wise.

By a *curve* on $S$ we mean a free homotopy class of a non-trivial, non-peripheral simple closed
curve. Fix a labelling of the 5 punctures of $S$ with elements of $\mathbb{Z}_5$, the cyclic group of order 5. Any
curve $\gamma$ on $S$ cuts the surface into two surfaces; one copy of $S_0,3$ containing two of the punctures
from $S$ and one copy of $S_0,4$ which contains three of the punctures from $S$.

**Definition 2.1.** We say that a curve $\gamma$ on $S$ is an $(i, j)$–curve, $i, j \in \mathbb{Z}_5$, if the component of $(S - \gamma)$
that is a three-times punctured sphere contains the punctures labeled $i$ and $j$. Furthermore, if
$|i - j| \equiv 1 \mod 5$ we say that $\gamma$ *separates two consecutive punctures*, and if $|i - j| \equiv 2 \mod 5$ we
say that $\gamma$ *separates two non-consecutive punctures*. 


In [13], Luo gave a simple presentation of the mapping class group where the generators are the set of all Dehn twists

\[ S = \{ D_\gamma : \gamma \text{ is a curve} \} \]

and the relations are of a few simple types. In our setting, we only have the following two relations:

- (Conjugating relation) For any two curves \( \beta \) and \( \gamma \),
  \[ D_{D_\beta(\beta)} = D_\gamma D_\beta D_\gamma^{-1}. \]

- (The lantern relation) Let \( i, j, k, l, m \) be distinct elements in \( \mathbb{Z}_5 \) and \( \gamma_{i,j}, \gamma_{j,k}, \gamma_{k,i} \) and \( \gamma_{l,m} \) be curves of the type indicated by the indices. Further assume that each pair of curves among \( \gamma_{i,j}, \gamma_{j,k}, \gamma_{k,i} \) intersect twice and that they are all disjoint from \( \gamma_{l,m} \). Then
  \[ D_{\gamma_{i,j}} D_{\gamma_{j,k}} D_{\gamma_{k,i}} = D_{\gamma_{l,m}}. \]

Using this presentation, we construct a homomorphism from \( \text{Map}(S) \) into \( \mathbb{Z} \).

**Theorem 2.2.** There exists a homomorphism \( h : \text{Map}(S) \to \mathbb{Z} \) whose restriction to the generating set \( S \) is as follows:

\[
\begin{align*}
D_\gamma &\mapsto 1 & \text{if } \gamma \text{ separates two consecutive punctures} \\
D_\gamma &\mapsto -1 & \text{if } \gamma \text{ separates two non-consecutive punctures}
\end{align*}
\]

**Proof.** To show that \( h \) extends to a homomorphism, it suffices to show that \( h \) preserves the relations stated above.

First, we check the conjugating relation. Let \( \beta \) and \( \gamma \) be a pair of curves. Since, \( D_\gamma \) is a homeomorphism fixing the punctures, if \( \beta \) is an \((i, j)\)-curve, so is \( D_\gamma(\beta) \). In particular, \( h(D_{D_\beta(\beta)}) = h(D_\gamma) \). Hence,

\[
h(D_{D_\beta(\beta)}) = h(D_\beta) = h(D_\gamma) + h(D_\beta) - h(D_\gamma) = h(D_\gamma) + h(D_\beta) + h(D_\gamma^{-1}) = h(D_\gamma D_\beta D_\gamma^{-1}).
\]

We now show that \( h \) preserves the lantern relation. For any three punctures of \( S \) labeled \( i, j, k \in \mathbb{Z}_5 \), two of these punctures are consecutive. Without loss of generality, suppose \( |i - j| = 1 \mod 5 \). There are two cases:

1. Assume \( k \) is consecutive to one of \( i \) or \( j \). That is, without loss of generality, suppose \( |j - k| = 1 \mod 5 \). Then \( |i - k| = 2 \mod 5 \) and the remaining two punctures, \( l \) and \( m \), are consecutive: \( |l - m| = 1 \mod 5 \). Thus

\[
h(D_{\gamma_{i,j}} D_{\gamma_{j,k}} D_{\gamma_{k,i}}) = h(D_{\gamma_{i,j}}) + h(D_{\gamma_{j,k}}) + h(D_{\gamma_{k,i}}) = 1 + 1 + (-1) = 1 = h(D_{\gamma_{l,m}}).
\]

2. Otherwise, \( |j - k| = 2 \mod 5 \) and \( |i - k| = 2 \mod 5 \), so that the remaining two punctures, \( l \) and \( m \), are nonconsecutive: \( |l - m| = 2 \mod 5 \). Thus

\[
h(D_{\gamma_{i,j}} D_{\gamma_{j,k}} D_{\gamma_{k,i}}) = h(D_{\gamma_{i,j}}) + h(D_{\gamma_{j,k}}) + h(D_{\gamma_{k,i}}) = 1 + (-1) + (-1) = (-1) = h(D_{\gamma_{l,m}}).
\]

Thus, \( h \) preserves the lantern relation. \( \square \)

Now, we switch back to the generating set \( S_n \) given in the introduction. The homomorphism of Theorem 2.2 gives a lower bound on the word length of elements in \( \text{Map}(S) \). Note that

\[
h(s_{i,j}) = (n - 1) \quad \text{and} \quad h(D_{\alpha_i}) = 1.
\]
Lemma 2.3. For any \( f \in \text{Map}(S) \), let
\[
h(f) = q(n - 1) + r
\]
for integer numbers \( q \) and \( r \) where \( 0 \leq |r| < \frac{2n-1}{2} \). Then \( \|f\|_{\delta_n} \geq |q| + |r| \).

Proof. First we show that, if \( h(f) = a(n-1) + b \) for integers \( a \) and \( b \), then \( |a| + |b| \geq |q| + |r| \). To see this, consider such a pair \( a \) and \( b \) where \( |a| + |b| \) is minimized. If \( a < q \), then \( |b| \geq |(n-1)+r| > \frac{2n-1}{2} \). Therefore, we can increase \( a \) by 1 and decrease \( b \) by \( n - 1 \) to decrease the quantity \( |a| + |b| \), which is a contradiction. Similarly, if \( a > q \), then \( |b| > (n-1)/2 \) and we can decrease \( a \) by 1 and increase \( b \) by \( n - 1 \) to decrease the quantity \( |a| + |b| \), which again is a contradiction. Hence, \( a = q \) and subsequently \( b = r \).

Now, write \( f = g_1g_2 \ldots g_k \), where \( g_i \in \delta_n \) or \( g_i^{-1} \in \delta_n \) and \( k = \|f\|_{\delta_n} \). For each \( g_i \), \( h(g_i) \) takes either value 1, \((-1)\), \((n-1)\) or \((1-n)\). Hence, there are integers \( a' \) and \( b' \) so that
\[
h(f) = h(g_1) + h(g_2) + \ldots + h(g_k) = a'(n-1) + b',
\]
where \( k \geq |a'| + |b'| \). But, as we saw before, we also have \( |a'| + |b'| \geq |q| + |r| \). Hence \( k \geq |q| + |r| \). \( \square \)

This lemma allows us to find explicit geodesics in \( \text{Map}(S) \). We demonstrate this with an example.

Example 2.4. Let \( f = D_{\alpha_1}^{n-1} \in \text{Map}(S) \). We have,
\[
h(f) = n^k - 1 = (n-1)(n^{k-1} + n^{k-2} + \ldots + n^2 + n + 1).
\]
Therefore, by Lemma 2.3 \( \|f\|_{\delta_n} \geq n^{k-1} + n^{k-2} + \ldots + n^2 + n + 1 \). On the other hand, (assuming \( k \) is even to simplify notation), we have
\[
D_{\alpha_1}^{n-1} = \left(D_{\alpha_1}^{n} D_{\alpha_2}^{n-1} \right) \left(D_{\alpha_2}^{n-1} D_{\alpha_3}^{n-1} \right) \ldots \left(D_{\alpha_i}^{n} D_{\alpha_{i+1}}^{n-1} \right)
\]
\[
= s_{1,2}^{(k-1)} s_{2,1}^{(k-2)} \ldots s_{1,2}^{n} s_{2,1}.
\]
Since we used exactly \((n^{k-1} + n^{k-2} + \ldots + n + 1)\) elements in \( \delta_n \), we have found a geodesic path. However, notice there is a second geodesic path from the identity to \( f \) (which works for every \( k \)), namely:
\[
D_{\alpha_1}^{n-1} = \left(D_{\alpha_1}^{n} D_{\alpha_2}^{n} \right) \left(D_{\alpha_2}^{n} D_{\alpha_3}^{n} \right) \ldots \left(D_{\alpha_{k-1}}^{n} D_{\alpha_{k+1}}^{n} \right)
\]
\[
= s_{1,2}^{(k-1)} s_{2,3}^{(k-2)} \ldots s_{k-1,1}^{n} s_{k,k+1}.
\]
This shows that geodesics are not unique in \( \text{Map}(S) \). Either way, we have established that
\[
\|D_{\alpha_1}^{n-1}\|_{\delta_n} = n^{k-1} + n^{k-2} + \ldots + n + 1.
\]

We now use a similar method to compute certain word lengths that will be useful later in the paper. Define
\[
\phi = D_{\alpha_1} D_{\alpha_2} D_{\alpha_3} D_{\alpha_4} D_{\alpha_5}.
\]
We will show in the next section that \( \phi \) is a pseudo-Anosov element of \( \text{Map}(S) \). We also use the notation
\[
\phi^{k/5} = D_{\alpha_1}^{m_k} \ldots D_{\alpha_5}^{m_k}
\]
where indices are considered to be in \( \mathbb{Z}_5 \). This is accurate when \( k \) is divisible by 5 but we use it for any integer \( k \). For a positive integer \( k \), define
\[
m_k = n^k + n^{k-1} + \ldots + n + 1 \quad \text{and} \quad \ell_k = n^k - n^{k-1} - n^{k-2} - \ldots - n - 1,
\]
and let \( w_k = D_{\alpha_1}^{\ell_k} \) and \( u_k = D_{\alpha_1}^{\ell_k} \). Additionally, we will define
\[
v_k = D_{\alpha_1}^{-\frac{k+1}{2}} D_{\alpha_2}^{-\frac{k-1}{2}}.
\]
We will show that $u_k$ and $w_k$ are closer to a large power of $\phi$ than the identity even though they are both just a power of a Dehn twist.

**Proposition 2.5.** For $u_k$ and $w_k$ as above, we have

$$\|w_k \phi^{-(k+1)/5}\|_{\delta_n} = \|w_k v_k\|_{\delta_n} = n^{k-1} + 2n^{k-2} + \ldots + (k-1)n + k,$$

and

$$\|\phi^{k/5} u_k\|_{\delta_n} = n^{k-1} - n^{k-3} - 2n^{k-4} - \ldots - (k-3)n - (k-2) + 1.$$

**Proof.** Note that

$$h\left(w_k \phi^{-(k+1)/5}\right) = (n^k + n^{k-1} + \ldots + n + 1) - (k+1)$$

$$= (n-1)(n^{k-1} + 2n^{k-2} + \ldots + (k-1)n + k).$$

Lemma 2.3 implies that

$$\|w_k \phi^{-(k+1)/5}\|_{\delta_n} \geq n^{k-1} + 2n^{k-2} + \ldots + (k-1)n + k.$$

On the other hand, since $m_k - 1 = n m_{k-1}$, we have

$$w_k \phi^{-(k+1)/5} = D_{\alpha_1}^{m_k} \left(D_{\alpha_1}^{-1} D_{\alpha_2}^{-1} \ldots D_{\alpha_k}^{-1}\right)$$

$$= D(m_{k-1}) \left(D_{\alpha_1}^{-1} D_{\alpha_2}^{-1} \ldots D_{\alpha_k}^{-1}\right)$$

$$= s_{1,2}^{m_{k-1}} D_{\alpha_1}^{m_{k-1}} \left(D_{\alpha_2}^{-1} D_{\alpha_3}^{-1} \ldots D_{\alpha_k}^{-1}\right)$$

$$= s_{1,2}^{m_{k-1}} s_{2,3}^{m_{k-2}} D_{\alpha_2}^{m_{k-2}} \left(D_{\alpha_3}^{-1} D_{\alpha_4}^{-1} \ldots D_{\alpha_k}^{-1}\right)$$

$$\ldots$$

$$= s_{1,2}^{m_{k-1}} s_{2,3}^{m_{k-2}} \ldots s_{k-1,k}^{m_1}s_{k,k+1}.$$

Therefore,

$$\|w_k \phi^{-(k+1)/5}\|_{\delta_n} = m_{k-1} + \ldots + m_1 + 1 = n^{k-1} + 2n^{k-2} + \ldots (k-1)n + k.$$

To show that

$$\|w_k v_k\|_{\delta_n} = n^{k-1} + 2n^{k-2} + \ldots (k-1)n + k$$

is as above, but in place of applying $s_{i,i+1}$ for $1 \leq i \leq k$, we alternate between applying $s_{1,2}$ and $s_{2,1}$ to find

$$w_k v_k = s_{1,2}^{m_{k-1}} s_{2,1}^{m_{k-2}} \ldots s_{1,2}^{m_1}s_{2,1},$$

which proves our claim. Similarly, we have

$$h(\phi^{k/5} u_k) = k + (n^k - n^{k-1} + \ldots - n - 1)$$

$$= (n-1)(n^{k-1} - n^{k-3} - 2n^{k-4} - \ldots - (k-3)n - (k-2)) + 1,$$

and Lemma 2.3 implies

$$\|\phi^{k/5} u_k\|_{\delta_n} \geq n^{k-1} - n^{k-3} - 2n^{k-4} - \ldots - (k-3)n - (k-2) + 1.$$
On the other hand, since $\ell_k + 1 = n\ell_{k-1}$, we have

$$
\phi^{k/5} u_k = (D_{\alpha_k} \ldots D_{\alpha_2} D_{\alpha_1}) D_{\alpha_1}^{\ell_k} \\
= (D_{\alpha_k} \ldots D_{\alpha_3} D_{\alpha_2}) D_{\alpha_1}^{\ell_{k-1}} \\
= (D_{\alpha_k} \ldots D_{\alpha_4} D_{\alpha_3}) D_{\alpha_1}^{\ell_{k-2}} s_{1,2}^{\ell_{k-2}} \\
= (D_{\alpha_k} \ldots D_{\alpha_5} D_{\alpha_4}) D_{\alpha_1}^{\ell_{k-3}} s_{1,2}^{\ell_{k-3}} \\
= \ldots \\
= D_{\alpha_k}^{\ell_{k-1}} s_{k-1,k}^{\ell_{k-1}} \ldots s_{2,3}^{\ell_{k-2}} s_{1,2}^{\ell_{k-1}} \\
= t_{k+1} s_{k,k+1} s_{k-1,k} \ldots s_{2,3} s_{1,2}^{\ell_{k-1}}.
$$

Therefore,

$$
\|u_k \phi^{k/5}\|_{\mathcal{D}_a} = \ell_{k-1} + \ldots + \ell_1 + 2 \\
= n^{k-1} - n^{k-3} - 2n^{k-4} - \ldots - (k-3)n - (k-2) + 1.
$$

This is because the coefficient of $n^i$ is 1 is $\ell_i$ and is $(-1)$ in $\ell_{k}, \ldots, \ell_{i+1}$. Summing up, we get $-(k-i-1)$ as the coefficient of $n^i$. \hfill \square

### 3. The Pseudo-Anosov Map $\phi$

In this section, we introduce the pseudo-Anosov map $\phi$ which will be used in the proof of Theorem \ref{thm:pseudo-Anosov}.

Define

$$
\phi = D_{\alpha_k} D_{\alpha_4} D_{\alpha_2} D_{\alpha_1}.
$$

We check that $\phi$ is, in fact, a pseudo-Anosov.

**Theorem 3.1.** The map $\phi$ is pseudo-Anosov.

**Proof.** In order to prove that $\phi$ is a pseudo-Anosov map, we find a train track $\tau$ on $S$ so that $\phi(\tau)$ is carried by $\tau$ and show that the matrix representation of $\phi$ in the coordinates given by $\tau$ is a Perron-Frobenius matrix (see \textit{[13]} for basic information about train-tracks).

The series of images in Fig.\ref{fig:train-track} depict the train track $\tau$ and its images under successive applications of Dehn twists associated to $\phi$. We see that $\phi(\tau)$ is indeed carried by $\tau$ and, keeping track of weights on $\tau$, we calculate that the induced action on the space of weights on $\tau$ is given by the following matrix.

$$
A = \begin{pmatrix}
3 & 2 & 0 & 0 & 2 \\
6 & 3 & 6 & 4 & 0 \\
4 & 2 & 3 & 2 & 0 \\
12 & 8 & 6 & 3 & 6 \\
6 & 4 & 4 & 2 & 3
\end{pmatrix}
$$

Note that the space of admissible weights on $\tau$ is the subset of $\mathbb{R}^5$ given by positive real numbers $a, b, c, d$ and $e$ such that $a + b + c = c + d$. The linear map described above preserves this subset. The square of the matrix $A$ is strictly positive, which implies that the matrix is a Perron-Frobenius matrix. In fact, the top eigenvalue is

$$
\lambda = \sqrt{13} + 2\sqrt{2\sqrt{13} + 7} + 4
$$

that is associated to a unique irrational measured lamination $F$ carried by $\tau$ that is fixed by $\phi$. We now argue that $F$ is filling. Note that, curves on $S$ are in one-to-one association with simple
arcs connecting one puncture to another. We say an arc is carried by \( \tau \) if the associated curve is carried by \( \tau \). If \( F \) is not filling, it is disjoint from some arc \( \omega \) connecting two of the punctures. Modifying \( \omega \) outside of a small neighborhood of \( \tau \), we can produce an arc that is carried by \( \tau \).
In fact, for any two cusps of the train-track \( \tau \), either an arc going clock-wise or counter-clockwise connecting these two cusps can be pushed into \( \tau \). Hence, we can replace the portion of \( \omega \) that is outside of a small neighborhood of \( \tau \) with such an arc to obtain an arc \( \omega' \) that is still disjoint from \( F \) but is also carried by \( \tau \). Hence, if \( F \) is not filling, it is disjoint from some arc (and thus some curve) carried by \( \tau \). But \( F \) is the unique lamination carried by \( \tau \) that is fixed under \( \phi \) which is a contradiction. This implies that \( \phi \) is pseudo-Anosov. \( \square \)

4. Shadow to Curve Complex not a Quasi-Geodesic

The curve graph \( \mathcal{C}(S) \) is a graph whose vertices are curves on \( S \) and whose edges are pairs of disjoint curves. We assume every edge has length one turning \( C \). This implies that, for a pair of curves \( \alpha \) and \( \tau \), \( \phi(\alpha) \) is the unique lamination carried by \( \tau \) that is fixed under \( \phi \) which is a contradiction. This implies that \( \phi \) is pseudo-Anosov.

We also talk about the distance between subsets of \( \mathcal{C}(S) \) using the same notation. That is, for two sets of curves \( \mu_0, \mu_1 \subset \mathcal{C}(S) \) we define

\[
d_{\mathcal{C}(S)}(\mu_0, \mu_1) = \max_{\gamma_0 \in \mu_0, \gamma_1 \in \mu_1} d_{\mathcal{C}(S)}(\gamma_0, \gamma_1).
\]

**Definition 4.1.** The **shadow map** from the mapping class group to the curve complex is the map defined as:

\[
\Upsilon: \text{Map}(S) \to \mathcal{C}(S) \quad f \to f(\alpha_1).
\]

The shadow map from \( \text{Map}(S) \) equipped with \( d_{\mathcal{C}_n} \) to the curve complex is 4-Lipschitz:

**Lemma 4.2.** For any \( f \in \text{Map}(S) \), we have

\[
d_{\mathcal{C}(S)}(\alpha_1, f\alpha_1) \leq 4\|f\|_{\mathcal{C}_n}.
\]

In particular, the Lipschitz constant of the shadow map is independent of \( n \).

**Proof.** It is sufficient to prove the lemma for elements of \( \mathcal{D}_n \). Consider \( D_{\alpha_i} \in \mathcal{D}_n \). If \( i(\alpha_i, \alpha_1) = 0 \) then

\[
d_{\mathcal{C}(S)}(\alpha_1, D_{\alpha_i}(\alpha_1)) = d_{\mathcal{C}(S)}(\alpha_1, \alpha_1) = 0.
\]

If \( i(\alpha_i, \alpha_1) = 2 \), then there is a curve \( \alpha_j \) that disjoint from both \( \alpha_1 \) and \( \alpha_i \) and hence \( \alpha_j \) is also disjoint \( D_{\alpha_i}(\alpha_1) \). Therefore, \( d_{\mathcal{C}(S)}(\alpha_1, D_{\alpha_i}(\alpha_1)) = 2 \).

Now consider the element \( s_{i,i+1} \in \mathcal{D}_n \). Note that \( s_{i,i+1}^{-1}\alpha_i = \alpha_i \). Hence,

\[
d_{\mathcal{C}(S)}(\alpha_1, s_{i,i+1}\alpha_1) \leq d_{\mathcal{C}(S)}(\alpha_1, \alpha_i) + d_{\mathcal{C}(S)}(\alpha_i, s_{i,i+1}\alpha_1)
\]

\[
\leq 2 + d_{\mathcal{C}(S)}(s_{i,i+1}\alpha_i, \alpha_1)
\]

\[
\leq 2 + d_{\mathcal{C}(S)}(\alpha_i, \alpha_1) \leq 2 + 2 = 4
\]

Thus, we have proven our claim. \( \square \)

Using this lemma and the theorems from Section 3, we show that the shadow of geodesics from the mapping class group to the curve complex are not always quasi-geodesics.

**Theorem 4.3.** For all \( K \geq 1, C \geq 0 \), there exists a geodesic in the mapping class group, whose shadow to the curve complex is not a \((K,C)\)-quasi-geodesic.
Proof. Recall that, for a positive integer $k$, we have
\[ m_k = n^{k-1} + n^{k-2} + \ldots + n + 1, \quad \ell_k = n^k - n^{k-1} - n^{k-2} - \ldots - n - 1, \]
and $w_k = D_{\alpha_1}^{n_k}$ and $u_k = D_{\alpha_1}^{k}$. Note that $m_k - \ell_k = n^k$. Hence, we can write
\[ D_{\alpha_1}^{n_k} = (w_{k-1} \phi^{-k/5})(\phi^{k/5} u_k). \]
Also,
\[ h(D_{\alpha_1}^{n_k}) = n^k = (n - 1)(n^{k-1} + n^{k-2} + \ldots + n + 1) + 1. \]
Therefore by Lemma 2.3
\[ \|D_{\alpha_1}^{n_k}\|_{\delta_n} \geq n^{k-1} + n^{k-2} + \ldots + n + 2. \]
But, from Theorem 2.5 we have
\[ \|w_{k-1} \phi^{-k/5}\|_{\delta_n} = n^{k-2} + 2n^{k-3} + \ldots + (k - 2)n + (k - 1). \]
and
\[ \|u_k \phi^{k/5}\|_{\delta_n} = n^{k-1} - n^{k-3} - 2n^{k-4} - \ldots - (k - 3)n - (k - 2) + 1. \]
The sum of the word lengths of the two elements is
\[ n^{k-1} + n^{k-2} + \ldots + n + 2 \]
which is equal to the lower bound found in Equation 3. Thus
\[ \|D_{\alpha_1}^{n_k}\|_{\delta_n} = \|w_{k-1} \phi^{-k/5}\|_{\delta_n} + \|\phi^{k/5} u_k\|_{\delta_n} \]
which means there is a geodesic connecting $D_{\alpha_1}^{n_k}$ to the identity that passes through $\phi^{k/5} u_k$.

Since $\phi$ is a pseudo-Anosov map, there is a lower-bound on its translation distance along the curve graph (see Theorem 4.6 from [14]). Namely, there is a constant $\sigma > 0$ so that, for every $m$,
\[ d_{\mathcal{C}(S)}(\alpha_1, \phi^m \alpha_1) \geq \sigma m. \]
Also, $u_k \alpha_1 = \alpha_1$ which implies
\[ d_{\mathcal{C}(S)}(\alpha_1, \phi^{k/5} u_k \alpha_1) = d_{\mathcal{C}(S)}(\alpha_1, \phi^{k/5} \alpha_1) \geq \sigma \frac{k}{5}. \]
That is,
\[ \Upsilon(\text{id}) = \Upsilon(D_{\alpha_1}^{n_k}) = \alpha_1. \]
However, $\Upsilon(\phi^{k/5} u_k)$ is at least distance $\frac{3k}{5}$ away from $\alpha_1$. Therefore, choosing $k$ large compared with $\sigma$, $K$ and $C$, we see that the shadow of this geodesic (the one connecting id to $D_{\alpha_1}^{n_k}$ which passes through $\phi^{k/5} u_k$) to $\mathcal{C}(S)$ is not a $(K,C)$-quasi-geodesic.

5. Axis of a Pseudo-Anosov in the Mapping Class Group

Consider the path
\[ \mathcal{A}_\phi : \mathbb{Z} \to \text{Map}(S), \quad i \to \phi^i. \]
Since $\|\phi\|_{\delta_n} \leq 5$, then $\|\phi^i\|_{\delta_n} \leq 5i$. Also, using Lemma 4.2 and Equation 4 we get
\[ \|\phi^i\|_{\delta_n} \geq \frac{1}{4} d_{\mathcal{C}(S)}(\alpha_1, \phi^i \alpha_1) \geq \frac{i \sigma}{4}. \]
Therefore,
\[ \frac{i \sigma}{4} \leq \|\phi^i\|_{\delta_n} \leq 5i. \]
This proves the following lemma.

Lemma 5.1. The path $\mathcal{A}_\phi$ is a quasi-geodesic in $(\text{Map}(S), d_{\delta_n})$ for every $n$ with uniform constants.
We abuse notation and allow $A_\phi$ to denote both the map, and the image of the map in $\text{Map}(S)$. For $i, j \in \mathbb{Z}$, let $g = g_{i,j}$ be a geodesic in $(\text{Map}(S), d_{S_n})$ connecting $\phi^i$ to $\phi^j$. Let $\mathcal{E} = \Upsilon \circ g$ be the shadow of $g$ to the curve complex and let

$$
\text{Proj}_\mathcal{E} : \text{Map}(S) \to \mathcal{E}
$$

be the composition of $\Upsilon$ and the closest point projection from $\mathcal{C}(S)$ to $\mathcal{E}$. The following theorem, proven in more generality by Duchin and Rafi [9, Theorem 4.2], is stated for geodesics $g_{i,j}$ and the path $\mathcal{E}$.

**Theorem 5.2.** The path $\mathcal{E}$ is a quasi-geodesic in $\mathcal{C}(S)$. Furthermore, there exists a constant $B_n$ which depends on $n$ and $\phi$, and a constant $B$ depending only on $\phi$ such that the following holds. For $x \in \text{Map}(S)$ with $d_{S_n}(x, g) > B_n$, let $r = d_{S_n}(x, g)/B_n$ and let $B(x, r)$ be the ball of radius $r$ centered at $x$ in $(\text{Map}(S), d_{S_n})$. Then

$$
diam_{\mathcal{C}(S)} \left( \text{Proj}_\mathcal{E} \left( B(x, r) \right) \right) \leq B.
$$

In the proof of [9] Theorem 4.2, it can be seen that $B_n$ ($B_1$ in their notation) is dependent on the generating set since $B_n$ is taken to be large with respect to the constants from the Masur and Minsky distance formula which depend on the generating set $[13]$. Let $\delta$ be a fixed generating set for $\text{Map}(S)$. Then the word lengths of elements in $\mathcal{S}_n$ in terms of $\delta$ grow linearly in $n$ with respect to $\delta$. Hence, the constants involved in the Masur-Minsky distance formula also change linearly in $n$. That is, $B_n \approx n$. Also, one can see that the constant $B$ ($B_2$ in their proof) depends only on $\phi$ and the hyperbolicity constant of the curve graph, but not the generating set.

Since, $A_\phi$ is a quasi-geodesic, Theorem 5.2 and the usual Morse argument implies the following.

**Proposition 5.3.** The paths $A_\phi[i,j]$ and $g_{i,j}$ fellow travel each other and the constant depends only on $n$. That is, there is a bounded constant $\delta_n$ depending on $n$ such that

$$
\delta_n \geq \max \left( \max_{p \in A_\phi[i,j]} \min_{q \in A_\phi[i,j]} d_{S_n}(p, q), \max_{p \in A_\phi[i,j]} \min_{q \in A_\phi[i,j]} d_{S_n}(p, q) \right).
$$

We now show that $\phi$ acts loxodromically in $(\text{Map}(S), d_{S_n})$. That is, there exists a geodesic $A_\phi$ in $(\text{Map}(S), d_{S_n})$ that is preserved by a power of $\phi$. This is folklore theorem, but we were unable to find a reference for it in the literature. The proof given here follows the arguments in [3] Theorem 1.4] where Bowditch showed that $\phi$ acts loxodromically on the curve graph, which is more difficult since the curve graph is not locally finite. Bowditch’s proof in turn follows the arguments of Delzant [8] for a hyperbolic group.

**Proposition 5.4.** There is a geodesic

$$
a_\phi : \mathbb{Z} \to \text{Map}(S)
$$

that is preserved by some power of $\phi$. We call the geodesic $a_\phi$ the quasi-axis for $\phi$.

**Proof.** The statement is true for the action of any pseudo-Anosov homeomorphism in any mapping class group equipped with any word metric coming from a finite generating set. We only sketch the proof since it is a simpler version of the argument given in [3].

Let $\mathcal{L}(i,j)$ be the set of all geodesics connecting $\phi^i$ to $\phi^j$. Note that every point on every path in $\mathcal{L}(i,j)$ lies in the $\delta_n$–neighborhood of $A_\phi$. Letting $i \to \infty$, $j \to -\infty$ and using a diagonal limit argument (Map(S) is locally finite) we can find bi-infinite geodesics that are the limits of geodesic segments in sets $A_\phi[i,j]$. Let $\mathcal{L}$ be the set of all such bi-infinite geodesics. Then $\phi(\mathcal{L}) = \mathcal{L}$ and every geodesic in $\mathcal{L}$ is also contained in the $\delta_n$–neighborhood of $A_\phi$. Let $\mathcal{L}/\phi$ represent the set of all edges which appear in a geodesic in $\mathcal{L}$ up to the action of $\phi$. Then $\mathcal{L}/\phi$ is a finite set.

Choose an order for $\mathcal{L}/\phi$. We say a geodesic $g \in \mathcal{L}$ is lexicographically least if for all vertices $x, y \in g$, the sequence of $\phi$-classes of directed edges in the segment $g_0 \subset g$ between $x$ and $y$ is
lexicographically least among all geodesic segments from $x$ to $y$ that are part of a geodesic in $L$. Let $L_L$ be the set lexicographically least elements of $L$. We will show that every element of $L_L$ is preserved by a power of $\phi$.

Let $P$ be the cardinality of a ball of radius $\delta_n$ in $(\text{Map}(S), d_{\delta_n})$. We claim that $|L_L| \leq P^2 + 1$. Otherwise, we can find $P^2 + 1$ elements of $L_L$ which all differ in some sufficiently large compact subset $N_{\delta_n}(S_{\phi})$, the $\delta_n$–neighborhood of $S_{\phi}$. In particular, we can find $x, y \in N_{\delta_n}(S_{\phi})$ so that each of these $P^2 + 1$ geodesics has a subsegment connecting a point in $N_{\delta_n}(x)$ to a point in $N_{\delta_n}(y)$, and these subsegments are all distinct. But then, at least two such segments must share the same endpoints, which means they cannot both be lexicographically least.

Since $\phi$ permutes elements of $L_L$, each geodesic in $L_L$ is preserved by $\phi^{P^2+1}$. □

As before, we use the notation $a_{\phi}$ to denote both the map and the image of the map in $\text{Map}(S)$. We now show that the projection of a ball that is disjoint from $a_{\phi}$ to $a_{\phi}$ grows at most logarithmically with the radius of the ball proving that Theorem 1.3 is sharp.

**Corollary 5.5.** There are uniform constants $c_1, c_2 > 0$ so that, for $x \in \text{Map}(S)$ and $R = d_{\delta_n}(x, a_{\phi})$, we have

$$\text{diam}_{G(S)} \left( \text{Proj}_{a_{\phi}}(\text{Ball}(x, R)) \right) \leq c_1 \log(n) \log(R) + c_2 n.$$  

**Proof.** Consider $y \in \text{Ball}(x, R - B_n)$. Let $N$ be the smallest number so that there is a sequence of points along the geodesic connecting $x$ to $y$

$$x = x_0, x_1, \ldots, x_N = y$$

so that

$$d_{\delta_n}(x_i, x_{i+1}) \leq \frac{d_{\delta_n}(x_i, a_{\phi})}{B_n}.$$  

Then,

$$d_{\delta_n}(x_{i+1}, a_{\phi}) \geq d_{\delta_n}(x_i, a_{\phi}) - d_{\delta_n}(x_i, x_{i+1})$$

$$\geq d_{\delta_n}(x_i, a_{\phi}) - \frac{d_{\delta_n}(x_i, a_{\phi})}{B_n} \geq d_{\delta_n}(x_i, a_{\phi}) \left(1 - \frac{1}{B_n}\right).$$

Hence,

$$d_{\delta_n}(x_i, a_{\phi}) \geq R \left(1 - \frac{1}{B_n}\right)^i.$$  

Since $N$ is minimum

$$d_{\delta_n}(x_i, x_{i+1}) + 1 \geq \frac{d_{\delta_n}(x_i, a_{\phi})}{B_n},$$

which implies

$$d_{\delta_n}(x_i, x_{i+1}) \geq \frac{R}{B_n} \left(1 - \frac{1}{B_n}\right)^i - 1.$$  

Since $d_{\delta_n}(x, y) \leq R - B_n$,

$$d_{\delta_n}(x_i, a_{\phi}) \geq R - d_{\delta_n}(x, x_i) \geq R - d_{\delta_n}(x, y) \geq B_n.$$  

Applying Theorem 5.2 to $r_i = d_{\delta_n}(x_i, a_{\phi})/B_n$ and $x_{i+1} \in \text{Ball}(x_i, r_i)$ we get

$$d_{G(S)}(\text{Proj}_{a_{\phi}}(x_i), \text{Proj}_{a_{\phi}}(x_{i+1})) \leq B,$$

and hence,

$$d_{G(S)}(\text{Proj}_{a_{\phi}}(x), \text{Proj}_{a_{\phi}}(y)) \leq Bc_n' \log R.$$  

(5)
Now, for any $y' \in \text{Ball}(x,R)$ there is a $y \in \text{Ball}(x,R - B_n)$ with $d_{\delta_n}(y,y') \leq B_n$. But $\Upsilon$ is $4$-Lipschitz and the closest point projection from $\mathcal{C}(S)$ to $\mathcal{Z}_\phi$ is also Lipschitz with a Lipschitz constant depending on the hyperbolicity constant of $\mathcal{C}(S)$. Therefore,

\begin{equation}
  d_{\mathcal{C}(S)}(\text{Proj}_{\mathcal{C}(S)}(y), \text{Proj}_{\mathcal{C}(S)}(y')) \leq c''B_n ,
\end{equation}

where $c''$, the Lipschitz constant for $\text{Proj}_{\mathcal{C}(S)}$ is a uniform constant. By letting

\[ c_n = \max(Bc'_n, Buc''n) \geq \log(n), \]

the Corollary follows from Equation (5) and Equation (6) and the triangle inequality. \qed

6. \textsc{The logarithmic lower-bound}

In this section, we will show that the quasi-axis $a_\phi$ of the pseudo-Anosov map $\phi$ does not have the strongly contracting property proving Theorem 1.3 from the introduction.

\textbf{Definition 6.1.} Given a metric space $(X,d_X)$, a subset $\mathcal{G}$ of $X$ and constants $d_1, d_2 > 0$, we say a map $\text{Proj} : X \to \mathcal{G}$, a $(d_1,d_2)$-projection map if for every $x \in X$ and $g \in \mathcal{G}$,

\[ d_X(\text{Proj}(x), g) \leq d_1 \cdot d_X(x,g) + d_2. \]

To prove this theorem, notice first that the geodesic found in Section 2 may not determine the nearest point of $a_\phi$ to $w_k = D_{m_1}^m$, where $m_k = n^k + n^{k-1} + \ldots + n + 1$.

\textbf{Lemma 6.2.} If $\phi^{p_k}$ is the nearest point of $a_\phi$ to $w_k$, then $p_k \geq k/5$.

\textbf{Proof.} Consider a point $\phi^m$ on $a_\phi$ where $m < k/5$. Applying the homomorphism $h$ we have

\[ h(w_k \phi^{-m}) = (m_k - 5m) > (m_k - k) = h(w_k \phi^{-k/5}). \]

But $(m_k - k)$ is divisible by $(n-1)$. Hence, if we write $(m_k - m) = q (n-1) + r$, where $|r| \leq \frac{(n-1)}{2}$, we have

\[ |q| \geq \frac{m_k - k}{n-1}, \quad \text{and} \quad |r| \geq 0. \]

Lemma 2.3 implies that $\|w_k \phi^{-m}\|_{\delta_n} > \|w_k \phi^{-k/5}\|_{\delta_n}$, which means the closest point in $a_\phi$ to $w_k$ is some point $\phi^{p_k}$ where $p_k \geq k/5$. \qed

Let $R_k = d_{\delta_n}(w_k, \phi^{p_k}) = d_{\delta_n}(w_k, a_\phi)$ and $\Delta_k = d_{\delta_n}(w_k, \phi^{(k+1)/5})$.

\textbf{Proof of Theorem 1.3.} For fixed $d_1, d_2 > 0$, let $\text{Proj}_{a_\phi} : \text{Map}(S) \to a_\phi$ be any $(d_1,d_2)$-projection map. Fix $n$ large enough so that

\begin{equation}
  \sigma > \frac{5 d_1}{n - 1}.
\end{equation}

Choose the sequence $\{k_i\} = \{2n^i - 3\}$ and recall that

\[ v_{k_i} = D_{\alpha_i}^{k_i+1} D_{\alpha_2}^{k_i+1}. \]

By Example 2.4 (notice that $\frac{k_i + 1}{n} = n^i - 1$)

\begin{equation}
  d_{\delta_n}(v_{k_i}, \text{id}) = \|v_{k_i}\|_{\delta_n} = \frac{k_i + 1}{n - 1}.
\end{equation}

and by Proposition 2.5 we have

\[ d_{\delta_n}(w_{k_i}, v_{k_i}) = \Delta_k. \]

Consider a ball $B(w_{k_i}, r_i)$ of radius $r_i = R_{k_i} - (\delta_n + 1)$ around $w_{k_i}$. This ball is disjoint from $a_\phi$ since $a_\phi$ and $a_\phi$ are $\delta_n$-fellow-travellers by 5.3 and $R_{k_i} = d_{\delta_n}(w_{k_i}, a_\phi)$. For the rest of the proof, we refer to Figure 3.
Since $h$ is a homomorphism, we have

$$h(w_k, \phi^{k_i/5}) = h(w_k, \phi^{-p_k}) + h(\phi^{p_k}, \phi^{-k_i/5})$$

Theorem 2.5 showed

$$h(w_k, \phi^{k_i/5}) = (n-1)\Delta_{k_i},$$

from Lemma 2.3 we have

$$h(w_k, \phi^{-p_k}) \leq (n-1)R_{k_i},$$

and since $\|\phi\|_{\delta_n} = 5$, we have

$$h(\phi^{p_k}, \phi^{-k_i/5}) \leq 5p_{k_i} - k_i.$$ 

The above equations imply:

$$\Delta_{k_i} - R_{k_i} \leq \frac{5p_{k_i} - k_i}{n-1}.$$ 

Consider a point $p$ on the geodesic from $w_{k_i}$ to $v_{k_i}$ such that $d_{\delta_n}(w_{k_i}, p) = r_i$, i.e. such that

$$d_{\delta_n}(p, v_{k_i}) = \Delta_{k_i} - r_i = \Delta_{k_i} - (R_{k_i} - \delta_{n} - 1) \leq \frac{5p_{k_i} - k_i}{n-1} + \delta_{n} + 1.$$ 

This and Equation (8) imply

$$d_{\delta_n}(id, p) \leq \frac{k_i + 1}{n-1} + \frac{5p_{k_i} - k_i}{n-1} + \delta_{n} + 1 = \frac{5p_{k_i} + 1}{n-1} + \delta_{n} + 1.$$ 

Since $A_{\phi}$ and $A_{\Phi}$ are $\delta_n$-fellow-travellers by 5.3, there exists a point $x_0 \in A_{\phi}$ in the $\delta_n$ neighborhood of the identity. Thus $d_{\delta_n}(p, x_0) \leq \frac{5p_{k_i} + 1}{n-1} + 2\delta_n + 1$ and

$$d_{\delta_n}(id, proj_{A_{\phi}}(p)) \leq d_{\delta_n}(id, x_0) + d_{\delta_n}(x_0, proj_{A_{\phi}}(p)) \leq d_{\delta_n}(p, x_0) + d_1 \cdot d_{\delta_n}(x_0, p) + d_2 \leq 5 \cdot \frac{p_{k_i}}{n-1} + A_p,$$

where $A_p$ is a constant depending on $\delta_n$, $d_1$ and $d_2$ but is independent of $k_i$. Similarly, we consider a point $q$ on the geodesic from $w_{k_i}$ to $\phi^{p_k}$ such that $d_{\delta_n}(w_{k_i}, q) = r_i$. Again, since $A_{\phi}$ and $A_{\Phi}$
are $\delta_n$-fellow-travellers by 5.3 there exists an $x_1 \in \mathcal{A}_\phi$ such that $d_{\delta_n}(\phi^{p_{k_i}}, x_1) \leq \delta_n$, and thus $d_{\delta_n}(q, x_1) \leq 2\delta_n + 1$. Therefore

$$d_{\delta_n}(\phi^{p_{k_i}}, \text{Proj}_{\mathcal{A}_\phi}(q)) \leq d_{\delta_n}(\phi^{p_{k_i}}, x_1) + d_{\delta_n}(x_1, \text{Proj}_{\mathcal{A}_\phi}(q))$$

$$\leq \delta_n + 1 \cdot (2\delta_n + 1) + d_2 \leq A_q$$

where, again, $A_q$ depends on $\delta_n$, $d_1$ and $d_2$ but is independent of $k_i$. Since $p, q \in B(w_{k_i}, r_{i})$, we have

$$\text{diam}_{\delta_n} \left( \text{Proj}_{\mathcal{A}_\phi}(B(w_{k_i}, r_{i})) \right) \geq d_{\delta_n} \left( \text{Proj}_{\mathcal{A}_\phi}(p), \text{Proj}_{\mathcal{A}_\phi}(q) \right)$$

$$\geq d_{\delta_n} \left( \text{id}, \phi^{p_{k_i}} \right) - d_{\delta_n} \left( \text{id}, \text{Proj}_{\mathcal{A}_\phi}(p) \right) - d_{\delta_n} \left( \text{Proj}_{\mathcal{A}_\phi}(q), \phi^{p_{k_i}} \right)$$

But $d_{\delta_n}(\text{id}, \phi^{p_{k_i}}) \geq \sigma p_{k_i}$. By combining this fact and equations 9 and 10 we find

$$\text{diam}_{\delta_n} \left( \text{Proj}_{\mathcal{A}_\phi}(B(w_{k_i}, r_{i})) \right) \geq \sigma p_{k_i} - \frac{5d_1 p_{k_i}}{n-1} - A_p - A_q$$

$$= p_{k_i} \left( \sigma - \frac{5d_1}{n-1} \right) - A_p - A_q.$$ 

By our assumption on $n$ (Equation (7)) this expression is positive and goes to infinity at $p_{k_i} \to \infty$. But, for $n$ large enough, $r_i \leq R_{k_i} \leq \Delta_{k_i} \leq n^{k_i}$. Also, $p_{k_i} \geq \frac{k_i}{5}$. Hence,

$$\frac{5p_{k_i}}{\log n} \geq \log(r_{i}).$$

Hence, there is a constant $c_n$ so that

$$\text{diam}_{\delta_n} \left( \text{Proj}_{\mathcal{A}_\phi}(B(w_{k_i}, r_{i})) \right) \geq c_n \log r_{i}.$$ 

This finishes the proof. \qed

References


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