CONVEXITY OF BALLS IN OUTER SPACE

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Abstract. In this paper we study the convexity properties of geodesics and balls in Outer space equipped with the Lipschitz metric. We introduce a class of geodesics called balanced folding paths and show that, for every loop $\alpha$, the length of $\alpha$ along a balanced folding path is not larger than the maximum of its lengths at the end points. This implies that out-going balls are weakly convex. We then show that these results are sharp by providing several counterexamples.

1. Introduction

Let $\text{Out}(F_n)$ be the group of outer automorphisms of a free group of rank $n$, and let Outer space $\text{CV}_n$ be the space of marked metric graphs of rank $n$. The Outer space, which is a simplicial complex with an $\text{Out}(F_n)$ action, was introduced by Culler-Vogtmann [CV86] to study $\text{Out}(F_n)$ as an analogue of the action of mapping class group on Teichmüller space or the action of a lattice on a symmetric space. The Outer space can be equipped with a natural asymmetric metric, namely the Lipschitz metric. For points $x, y \in \text{CV}_n$, $d(x, y) = \inf_{\phi} \log(L_{\phi})$ where $\phi: x \rightarrow y$ is a difference of markings map from $x$ to $y$ and $L_{\phi}$ is the Lipschitz constant of the map $\phi$. The geometry of $\text{CV}_n$ equipped with the Lipschitz metric is closely related to the large scale geometry of $\text{Out}(F_n)$ and has recently been the subject of extensive study (see for example, [Vog15, AK11, BF12]).

In this paper, we examine the convexity properties of geodesics and balls in $\text{CV}_n$. However, we need to be careful with our definitions since the metric $d$ is asymmetric (the ratio of $d(x, y)$ and $d(y, x)$ can be arbitrarily large [AKB12]) and there maybe many geodesics connecting two points in $\text{CV}_n$. We introduce the notion of a balanced folding path along which we have more control over the lengths of loops. Recall that, a geodesic in $\text{CV}_n$ (not necessarily parametrized with unit speed) is a map $\gamma: [a, b] \rightarrow \text{CV}_n$. For $a \leq t \leq b$, let $x = \gamma(a)$ and $y = \gamma(b)$, we have $d(x, \gamma(t)) + d(\gamma(t), y) = d(x, y)$.

We often denote the image of $\gamma$ by $[x, y]$. The length of a loop $\alpha$ in a graph $x$ is denoted by $|\alpha|_x$ and we use $|\alpha|_t$ to denote the length of $\alpha$ at $\gamma(t)$. The balanced folding paths from $x$ to $y$ is denoted by $[x, y]_{bf}$. We show that, lengths of loops along a balanced folding path satisfy a weak notion of convexity.

Theorem 1.1. Given points $x, y \in \text{CV}_n$, there exists a geodesic $[x, y]_{bf}$ from $x$ to $y$ so that, for every loop $\alpha$, and every time $t$, $|\alpha|_t \leq \max(|\alpha|_x, |\alpha|_y)$.
The proof of this theorem is by construction. We then apply Theorem 1.1 to convexity of balls. There are two different notions of a round ball in $\text{CV}_n$. For $x \in \text{CV}_n$ and $R > 0$, we define the out-going ball of radius $R$ centered at $x$ to be

$$B_{\text{out}}(x, R) = \{ y \in \text{CV}_n \mid d(x, y) \leq R \}.$$ 

As an immediate corollary of Theorem 1.1 we have

**Theorem 1.2.** Given a point $x \in \text{CV}_n$, a radius $R > 0$ and points $y, z \in B_{\text{out}}(x, R)$,

$$[y, z]_{\text{bf}} \subset B_{\text{out}}(x, R).$$

That is, the ball $B_{\text{out}}(x, R)$ is weakly convex.

**Corollary 1.3.** The intersection of any number of out-going balls is connected.

**Proof.** The same balanced folding path is contained in all balls. □

We will also show that these theorems are sharp in various ways by providing examples of how possible stronger statements fail. There are other ways to choose a geodesic connecting $y$ to $z$, for example, the *standard geodesic* which is a concatenation of a rescaling of the edges and a *greedy folding path* (see Section 2 for definitions). In fact, when there is a greedy folding path connecting $y$ to $z$, Dowdall-Taylor [DT14, Corollary 3.3] following Bestvina-Feighn [BF14, Lemma 4.4] have shown that the lengths of loops are quasi-convex. However, we will show that these paths do not satisfy the conclusion of Theorem 1.1 and a standard path or a greedy folding path with end points in $B_{\text{out}}(x, R)$ may leave the ball.

Here is the summary of our counter-examples.

**Theorem 1.4.** Theorems 1.1 and 1.2 are sharp.

1. (Lengths cannot be made convex.) There are points $x, y \in \text{CV}_n$ and a loop $\alpha$ so that along any geodesic connecting $x$ to $y$, the length of $\alpha$ is not a convex function of distance in $\text{CV}_n$.

2. (The ball $B_{\text{out}}$ is not quasi-convex) This is true even when one restricts attention to (non-greedy)-folding paths. Namely, for any $R > 0$, there are points $x, y, z \in \text{CV}_n$ and there is a folding path $[y, z]_{\text{ng}}$ connecting $y$ to $z$ so that $y, z \in B_{\text{out}}(x, 2)$ and $[y, z]_{\text{ng}} \not\subset B_{\text{out}}(x, R)$.

That is, a folding path with end points in $B_{\text{out}}(x, 2)$ can travel arbitrarily far away from $x$.

3. (Standard geodesics could behave very badly) There exists a constant $c > 0$ such that, for every $R > 0$, there are points $x, y, z \in \text{CV}_n$ and a standard geodesic $[y, z]_{\text{std}}$ connecting $y$ to $z$ such that $y, z \in B_{\text{out}}(x, R)$ and $[y, z]_{\text{std}} \not\subset B_{\text{out}}(x, 2R - c)$.

That is, the standard geodesic path can travel nearly twice as far from $x$ as $y$ and $z$ are from $x$.

4. (Greedy folding paths may not stay in the ball) For every $R > 0$, there are points $x, y, z \in \text{CV}_n$, $n \geq 6$, where $y$ and $z$ are connected by a greedy folding path $[y, z]_{\text{gf}}$ such that $y, z \in B_{\text{out}}(x, R)$ but $[y, z]_{\text{gf}} \not\subset B_{\text{out}}(x, R)$.
Construction of a balanced folding path. Given an optimal difference of markings map \( \phi: x \to y \) where the tension graph is the whole \( x \), there are many folding paths connecting \( x \) to \( y \). We need a controlled and flexible way to construct a folding path between \( x \) to \( y \). To this end, we introduce a notion of a speed assignment (see Section 2.4) which describes how fast every illegal turn in \( x \) folds. Given a speed assignment, one can write a concrete formula for the rate of change of the length of a loop \( \alpha \) (Lemma 3.2). To prove Theorem 1.1, we need to find a speed assignment so that, whenever \( |\alpha_y| < |\alpha_x| \), the derivative of length of \( \alpha \) is negative and if \( |\alpha_y| > |\alpha_x| \) the length \( \alpha \) does not grow too fast.

A difference of markings map \( \phi: x \to y \) (again assuming the tension graph is the whole \( x \)) can be decomposed to a quotient map \( \bar{\phi}: x \to \bar{y} \) which is a local isometry and a scaling map \( \bar{y} \to y \). Our approach is to determine the contribution \( \ell_\tau \) of every sub-gate \( \tau \) in \( x \) to the length loss from \( x \) to \( y \). For an easy example, consider \( F_3 = \langle a, b, c \rangle \), let \( x \) be a rose with 3 pedals where the edges are labeled \( ac^2 \), \( bc \) and \( c \) and the edge lengths are \( \frac{1}{2}, \frac{1}{3} \) and \( \frac{1}{6} \) and let \( y \) be the rose with labels \( a \), \( b \) and \( c \) and edges lengths all \( \frac{1}{6} \). Then \( y \) is obtained from \( x \) by wrapping \( ac^2 \) around \( c \) twice and \( bc \) around \( c \) once. The length loss going from \( x \) to \( y \) is

\[
|x| - |y| = 1 - \frac{1}{2} = \frac{1}{2}.
\]

Here, the sub-gate \( \langle bc, c \rangle \) is contributing \( \frac{1}{2} \) to the length loss and the sub-gate \( \langle ac^2, c \rangle \) is contributing \( 2 \times \frac{1}{6} \) to the length loss. Of course, in general, the definition of length loss contribution needs to be much more subtle.

These length loss contributions are then used to determine the appropriate speed assignment. That is, we fold each sub-gates proportional to the length loss they eventually induce (see Section 3). For instance, in the above example, the sub-gate \( \langle ac^2, c \rangle \) should be folded with twice the speed of the sub-gate \( \langle bc, c \rangle \).

Decorated difference of markings map. If the tension graph of \( \phi: x \to y \) is a proper subset of \( x \), there is no folding path between \( x \) to \( y \). As we see in part (3) of Theorem 1.4, standard paths are not suitable for our purposes. Instead, we emulate a folding path even in this case. Namely, we introduce a notion of decorated difference of markings map. That is, by adding decoration to \( x \) and \( y \) (marked points are added to \( x \) and some hair is added to \( y \)), we can ensure that the difference of markings map is tight. Then, we show, a folding path can be defined as before and the discussion above carries through (see Section 4).

A criterion for uniqueness of geodesics. To prove part one of Theorem 1.4 and Theorem 1.6 below, we need to know what all the geodesics connecting \( x \) to \( y \) are accounted for. In general this is hard to characterize. Instead, we focus on a case where there is a unique geodesic connecting \( x \) to \( y \). It is not hard to prove the uniqueness of geodesic in special cases, however, we prove a general statement giving a criterion for uniqueness. A "yo-yo" is an illegal turn formed by a one-edge loop and a second edge, with no other edges incident to the vertex of this illegal turn. We say a folding path from \( x \) to \( y \) is rigid if at every point along the path there is exactly one illegal turn and it is not a yo-yo.

**Theorem 1.5.** For points \( x, y \in \mathrm{CV}_n \), the geodesic from \( x \) and \( y \) is unique if and only if there exists a rigid folding path connecting \( x \) to \( y \).
The in-coming balls. In the last section we examine the convexity of in-coming balls. For $x \in \text{CV}_n$ and $R > 0$, we define the in-coming ball of radius $R$ centered at $x$ to be

$$B_{\text{in}}(x, R) = \{y \in \text{CV}_n \mid d(y, x) \leq R\}.$$ 

We show that a ball $B_{\text{in}}(x, R)$, in general, is not even weakly quasi-convex. That is,

**Theorem 1.6.** For any constant $R > 0$, there are points $x, y, z \in \text{CV}_n$ such that, $y, z \in B_{\text{in}}(x, 2)$ but, for any geodesic $[y, z]$ connecting $y$ to $z$,

$$[y, z] \not\subset B_{\text{in}}(x, R).$$

Once again we use Theorem 1.5; we construct an example where there is a unique geodesic between $y$ and $z$ and show that it can go arbitrarily far out.

Analogies with Teichmüller space. The problem addressed in this paper has a long history in the setting of Teichmüller space. Let $(T(\Sigma), d_T)$ be the Teichmüller space of a surface $\Sigma$ equipped with the Teichmüller metric. It was claim by Kravetz [Kra59] that round balls in Teichmüller space are convex and he used it to give a positive answer to the Nielsen-realization problem. However, his proof turned out to be incorrect. Even-though the Nielsen-realization problem was solved by Kerckhoff [Ker83], it was open for many years whether or not the round balls are convex and was only resolved recently; It was shown in [LR11] that round balls in $(T(\Sigma), d_T)$ are quasi-convex and it was shown in [FBR16] that there are non-convex balls in the Teichmüller space. The problem is still open for the Teichmüller space equipped with the Thurston metric [Thu86], which is an asymmetric metric and is more directly analogous to the Lipschitz metric in $\text{CV}_n$. Hence, the weak convexity in the case of Outer space is somewhat surprising.

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2. Preliminaries

2.1. Outer space. Let $F_n$ be a free group of rank $n$ and let $\text{Out}(F_n)$ be the outer automorphism group of $F_n$. Let $\text{cv}_n$ be the space of free, minimal actions of $F_n$ by isometries on metric simplicial trees [CV86]. Two such actions are considered isomorphic if there is an equivariant isometry between the corresponding trees. Equivalently we think of a point in $\text{cv}_n$ as the quotient metric graph of the tree by the corresponding action. The quotient graph is marked, that is, its fundamental group is identified (up to conjugation) with $F_n$.

The Culler-Vogtmann Outer space, $\text{CV}_n$, (or simply the Outer space) is the subspace of $\text{cv}_n$ consisting of all marked metric graphs of total length $1$. Let $x$ be a metric graph of total length $1$, in which every vertex has degree at least $3$. Let $R_n$ be the graph of $n$ edges that are all incident to one vertex. A marking is a homotopy equivalence $f: R_n \to x$. Two marked graphs $f: R_n \to x$ and $f': R_n \to x'$ are equivalent if there is an isometry $\phi: x \to x' \Sigma$ such that $\phi \circ f \simeq f'(\text{homotopic})$. When the context is clear, we often drop the marking out of the notation and simply write $x \in \text{CV}_n$. In this paper, we refer to metric graphs as $x, y$, etc. and the corresponding trees as $T_x, T_y$, etc. We also use $\tilde{\phi}: T_x \to T_y$ for the lift of $\phi$.

The set of marked metric graphs that are isomorphic as marked graphs to a given point $x \in \text{CV}_n$ makes up an open simplex in $\text{CV}_n$ which we denote $\Delta_x$. Outer space $\text{CV}_n$ consists of simplices with missing faces. The group $\text{Out}(F_n)$ acts on $\text{CV}_n$ by precomposing the marking: for an element $g \in \text{Out}(F_n)$, $(x, f)g = (x, f \circ g)$. This is a simplicial action.
2.2. Lipschitz metric. A map $\phi: x \to y$ is a difference of markings map if $\phi \circ f_x \simeq f_y$. We will only consider Lipschitz maps and we denote by $L_\phi$ the Lipschitz constant of $\phi$. The Lipschitz metric on $\text{CV}_n$ is defined to be:

$$d(x, y) := \inf_{\phi} \log L_\phi$$

where the infimum is taken over all differences of markings maps. There exists a non-unique difference of markings map that realizes the infimum [FM11]. Since such a difference of markings map is homotopic rel vertices to a map that is linear on edges, we also use $\phi$ to denote the representative that realizes the infimum and is linear on edges and refer to such a map as an optimal map from $x$ to $y$. Given an optimal map $\phi: x \to y$, the slope of $e \in x$ associated to $\phi$ is the metric length of $\phi(e) \in y$. For the remainder of the paper, we always assume difference of markings maps are optimal.

By a loop or an immersed loop in $x$, we mean a free homotopy class of a map from the circle into $x$, or equivalently, a conjugacy class in $\mathbb{F}_n$. Meanwhile we use a simple loop to mean a union of edges in a graph that forms a circle with no repeated vertices. Both kinds of loops can be identified with a conjugacy class in the free group, call it $\alpha$. We use $|\alpha|_x$ to denote the metric length of the shortest representative of $\alpha$ in $x$, where $x$ can be a point in $\text{CV}_n$ or $\text{cv}_n$, depending on the context. It is shown [FM11] that if $x, y \in \text{CV}_n$ then the distance $d(x, y)$ can be computed as:

$$d(x, y) = \sup_{\alpha} \log \frac{|\alpha|_y}{|\alpha|_x},$$

where the sup is over all conjugacy classes that are realized as loops in $x$. In fact, it is shown in [FM11] that:

**Theorem 2.1.** Given two points in Outer space $x$ and $y$, the immersed loop that represents $\alpha$ which realizes the supremum can be taken from a finite set of sub-graphs of $x$ of the following forms

- simple loops
- figure-eight: an immersed loop where there is exactly one vertex with two pre-images in the circle
- dumbbell: an immersed loop that crosses edges in two disjoint loops once and edges in the connecting arc twice.

This result implies we can compute distances between two points by calculating the ratio $|\alpha|_y / |\alpha|_x$ for a finite set of immersed loops.

2.3. Train track structure. It is often convenient to use a difference of markings map that has some additional structure. We define

$$\lambda(\alpha) = \frac{|\alpha|_y}{|\alpha|_x}$$

to be the the stretch factor of a shortest immersed loop that represents $\alpha$. For an optimal map $\phi: x \to y$, since it is linear on edges, one can define

$$\lambda(e) = \frac{||\phi(e)||}{|e|_x}$$

to be the stretch factor of an edge $e$ and define the tension sub-graph, $x_{\phi}$, to be the sub-graph of $x$ consisting of maximally stretched edges.
Let $\phi$ be an optimal map. A direction at a vertex $v \in x_\phi$ is a germ of geodesic path $[0, \epsilon] \to x_\phi$ sending 0 to $v$. Let $D(v)$ be the set of all directions at $v$. Now $\phi$ induces a map:

$$\phi_* : D(v) \to D(\phi(v))$$

since it sends a geodesic $\gamma : [0, \epsilon] \to x_\phi$ to a geodesic $\phi \circ \gamma : [0, \epsilon] \to x_\phi$. Thus we have an equivalence relation on $D(v)$:

$$d \sim d' \iff \phi_*(d) = \phi_*(d').$$

A gate at $v$ is an equivalence class. The size of a gate $\tau$ is denoted $|\tau|$ and is defined to be the number of directions in the equivalence class. An unordered pair $\{d, d'\}$ of distinct directions at a vertex $v$ of $x_\phi$ is called a turn. The turn $\{d, d'\}$ is $\phi$-illegal if $d$ and $d'$ belong to a same gate and is legal otherwise. The set of gates at $x_\phi$ is also called the illegal turn structure on $x_\phi$ induced by $\phi$.

**Definition 2.2.** A sub-gate is a subset of directions in a gate (including the gate itself). The set of all sub-gates of $x \in \text{CV}_n$ under the difference of markings map $\phi$ is denoted $T_\phi$, or simply $T$ if the associated map is clear from the context. With this terminology we can view an illegal turn as a sub-gate of size 2. A speed assignment is an assignment of non-negative real numbers $s_\tau$ to all elements of $T_\phi$ of size two and denote the assignment

$$\mathcal{S} = \left\{ s_\tau \mid \tau \in T, |\tau| = 2 \right\}.$$

For a gate $\tau = d_1, d_2, d_3$, we might have $s_{\{d_1, d_2\}}$ and $s_{\{d_2, d_3\}}$ are larger than $s_{\{d_1, d_3\}}$. But if we identify the edges associated to $d_1$ and $d_2$ as well as $d_2$ and $d_3$ alone some segment of size $\epsilon$, then the edges associated to $d_1$ and $d_3$ are also identified. To address this issue, we add some extra assignment of the $\mathcal{S}$.

For $s > 0$, we say directions $d$ and $d'$ as a point $v$ are $s$–equivalent if there is a sequence of direction $d = d_1, \ldots, d_k$ and gate $\tau_1, \ldots, \tau_{k-1}$ so that $\{d_i, d_{i+1}\} \subset \tau_i$ and $s_{\tau_i} \geq s$. We say a speed assignment is coherent if for every gate $\tau = d_1, \ldots, d_\ell$ and every $s > 0$

$$\forall \quad 0 \leq i, j \leq k \quad d_i, \text{ and } d_j \text{ are } s\text{–equivalent} \implies s_{\tau} \geq s.$$

For every speed assignment $\mathcal{S}$ there is a coherent speed assignment $\mathcal{S}_c$ where the $s$ equivalent classes are identical. In fact, we can define

$$s^c_\tau = \sup \left\{ s > 0 \mid \text{every } d, d' \in \tau \text{ are } s\text{–equivalent} \right\}.$$

Note that replacing $s_\tau$ with $s^c_\tau$ does not change the $s$–equivalent classes since we are increasing the speed only for directions that are already $s$–equivalent. We call $\mathcal{S}_c$ the associated coherent speed assignment to $\mathcal{S}$.

For an immersed loop $\alpha_x \in x$, the set of illegal turns of $\alpha$ is denoted $T_\alpha(\phi)$, or $T_\alpha$ when the associated map is clear from the context. Since $\alpha$ is immersed, $T_\alpha$ is a priori a multi-set because an illegal turn $\tau = \{d, d'\}$ can appear in $T_\alpha$ more than once.

An illegal turn structure is moreover a train track structure if there are at least two gates at each vertex. For any two points $x, y \in \text{CV}_n$, there exists an optimal map $\phi : x \to y$ such that $x_\phi$ has a train track structure [FM11], which then allow us to use immersed loops such as the candidate loops in Theorem 2.1 to compute Lipschitz distances between points.
2.4. Folding paths. In this section we construct a family of paths called folding paths. The general definition can be found in [BF14], but we introduce it here in the language that is adapted to this paper. Given an optimal difference of markings map $\phi: x \to y$ with $x_\phi = x$, we obtain a train track structure. An immersed path in $x_\phi = x$ is legal if whenever it passes through a vertex, the entering and the exiting gates are distinct. A legal loop in a graph is a legal immersed path whose last vertex coincides with the first vertex on the path. A legal segment is a legal immersed path whose last vertex does not coincide with its first vertex.

Given a train track structure and an associated speed assignment $S = \{s_\tau\}_{|\tau|=2}$, we define a folding path $\{x_t\}$, for small $t \geq 0$. The optimal difference of markings map $\phi_t: x \to x_t$ is a composition of a quotient map $\bar{\phi}_t: x \to \bar{x}_t$ and a scaling map $\tau_t \to x_t$. For $t$ small enough, the quotient graph $\bar{\tau}_t$ of $x$ is obtained from $x$ as follows. For every sub-gate $\tau$ at vertex $v$ with $|\tau| = 2$ and two points $u, w$ on the two edges $e_u, e_w$ of the sub-gate $\tau$, we identify $u$ and $w$ if $|v, u|_x = |v, w|_x \leq ts_\tau$ (here $|\cdot, \cdot|$ measures the length of the segment in the graph $x$). The graph $\bar{\tau}_t$ inherits a natural metric so that this quotient map is a local isometry on each edge of $x$. Since $\bar{\tau}_t \in CV_n$, let $x_t$ be the projective class of $\bar{\tau}_t$ in $CV_n$, and let $\phi_t: x \to x_t$ be the composition $\bar{\phi}_t$ and the appropriate scaling.

Notice that in $x_t$ it is possible for edges in a sub-gate $\tau$ to be identified along a segment that is longer than $ts_\tau$, depending on the identification of other edges in the gate containing $\tau$. However we still say $x_t$ constructed this way is the folding path associated with $S = \{s_\tau\}_{|\tau|=2}$. That is, different speed assignments may result in the same folding path.

We assume $t_1$ is small enough so that the combinatorial type of $x_t$ does not change on the interval $(0, t_1)$. Also, for $t$ small enough, any $u$ and $w$ as above are also identified under $\phi$ because $e_u$ and $e_w$ are in the same gate. Hence, $\phi \circ \phi_t^{-1}$ is a well defined map. We always assume $t$ small enough that is true and define the left-over map $\psi_t$ at time $t$ to be defined by

$$\psi_t: x_t \to y, \quad \psi_t = \phi \circ \phi_t^{-1}$$

Note that the norm of the derivatives of the map $\phi: x \to y$ is constant along $x$ because it is an optimal map and $x_\phi = x$ hence $\phi$ stretches $x$ by the same amount everywhere. The same is true for $\phi_t$. Therefore, the norm of the derivative of $\psi_t: x_t \to y$ is also constant along $x_t$ and in fact it is the ratio of the norm of the derivative of $\phi$ over that of $\phi_t$. That is,

$$d(x_t, y) = d(x, y) - d(x, x_t).$$

We call such a path $\{x_t\}$ a geodesic starting from $x$ towards $y$ since it does not necessarily reach $y$. To summarize, we have shown:

**Proposition 2.3.** Assume a difference of markings map $\phi: x \to y$ gives a train track structure on $x$. And assume any speed assignment $S = \{s_\tau\}_{|\tau|=2}$, there is $t_1 > 0$ and a geodesic $\gamma_S: [0, t_1] \to CV_n$ starting from $x$ towards $y$ where the graph $x_t = \gamma(t)$ is obtained by folding every gate $\tau$ at speed $s_\tau$.

Note that, when we say the combinatorial type of $x_t$ does not change, it does not mean it is the same as $x$ or even $x_{t_1}$. Typically, the geodesic segment $\gamma_S$ starts from $x$ which lies on the boundary of simplex in $CV_n$ (not necessarily of maximal dimension) travels in the interior of this simplex and stops when it hits the boundary of the simplex. At this point, if there is a new speed assignment, the folding could continue.
In general, a folding path from $x$ to $y$ is denoted $[x, y]_t$. If all elements of the speed assignment are equal then the path is called a greedy folding path and denoted $[x, y]_{gf}$. In Section 3 we construct a specific type of folding path whose speed assignment reflects the contribution of each sub-gate to the total length loss along the path. However, one has to be careful to extend the local construction described here to a geodesic connecting $x$ to $y$. In Section 5 we extend the local construction to a global construction.

2.5. Standard geodesic. Another important class of geodesic paths to consider is the class standard geodesic paths. For two points $x, y \in CV_n$, there may not exist a folding path connecting them. There is, however, a non-unique standard geodesic, denoted $[x, y]_{std}$, from $x$ to $y$ [BF14]. In [BF14, Proposition 2.5], Bestvina and Feighn give a detailed construction of such a standard geodesic, which we summarize briefly here. First, take an optimal map $\phi: x \to y$ and consider the tension sub-graph $x_\phi$. Let $\Delta_x \subset CV_n$ denote the smallest simplex containing $x$. By shortening some of the edges outside of $x_\phi$ (and rescaling to maintain total length 1), one may then find a point $x' \in \Delta_x$ in the closed simplex $\Delta_x$ together with an optimal difference of markings $\phi': x' \to y$ whose tension graph $x'_\phi'$ is all of $x'$ and such that

$$d(x, y) = d(x, x') + d(x', y)$$

If $\gamma_1$ denotes the linear path in $\Delta_x$ from $x$ to $x'$ (which when parameterized by arc length is a directed geodesic) and $\gamma_2 = \gamma'\phi'$ denotes the folding path from $x'$ to $y$ induced by $\phi'$, it follows from the equation above that the concatenation $\gamma_1\gamma_2$ is a directed geodesic from $x$ to $y$ which is called a standard geodesic from $x$ to $y$, which we denote $[x, y]_{std}$.

2.6. Unique geodesics. We would like to show that, in certain situations, all geodesics connecting a pair of points have some property. Here, we give a criterion for when the geodesic between two points is unique. For a difference of markings map $\phi: x \to y$, we define a yo-yo to be an illegal turn $\langle e, \overline{e} \rangle$, $e \neq \overline{e}$, at a vertex $v$ satisfying the following (see Fig. 1):

- The edge $\overline{e}$ forms a loop at $v$.
- There are no other edges attached to $v$.

![Figure 1. A yo-yo illegal turn](image)

We say a folding path $\gamma_t: [a, b] \to CV_n$ is rigid if, for every $t \in [a, b]$, there is a difference of markings map $\phi_t: \gamma_t(t) \to y$ that induces a train-track structure which has exactly one illegal turn, and that illegal turn is not a yo-yo. We show that unique geodesics are exactly rigid folding paths.

**Theorem 2.4.** For points $x, y \in CV_n$, where $n \geq 3$, the geodesic from $x$ to $y$ is unique if and only if there exists a rigid folding path $\gamma_t$ connecting $x$ to $y$.

**Proof.** Let $\gamma_t: [0, a] \to CV_n$ be a rigid folding path connecting $x$ to $y$. This implies, in particular, that there is a difference of markings map $\phi_t: x \to y$ where tension sub-graph of $\phi_t$ is all of $x$ and where the train-track structure associated to $\phi_t$ has one, non-yo-yo illegal turn. We need the following combinatorial statement.
Claim. Every edge, and every legal segment $P = \{e_1, e_2\}$ in $x$ is a subpath of an immersed $\phi_r$-legal loop $\alpha$ in $x$.

Proof of Claim. Let $\tau = \{e, \overline{e}\}$ denote the only illegal turn in $x$ at vertex $v$. The graph $x \setminus e$ either has a vertex of degree one, or every vertex of $x \setminus e$ has degree at least two. In the first case, $e, \overline{e}$ forms a yo-yo, which contradicts the assumption. In the second case, since every turn in $x \setminus e$ is $\phi_r$-legal and every vertex has two or more gates, every edge and every length-2 legal segment is part of an immersed legal loop. Similarly, in $x \setminus \tau$, every edge and every length-2 legal segment is part of an immersed legal loop. Since every edge is contained either in $x \setminus e$ or in $x \setminus \tau$, it is part of an immersed $\phi_r$-legal loop in $x$.

Given a legal segment $P = \{e_1, e_2\}$ of lengths 2, one of the following holds:

- $P \subset x \setminus e$
- $P \subset x \setminus \tau$
- $\{e_1, e_2\} = \{e, \overline{e}\}$, and the segment starts and ends at vertex $v$
- $\{e_1, e_2\} = \{e, \tau\}$, and the segment starts at $v$, and ends at a different vertex $w$

For the first two cases, we have already established that $P$ is a subpath of an immersed $\phi_r$-legal loop $\alpha$ in $x$. In the third case, $e_1$ and $e_2$ both start and end at $v$. Let $d_1$ and $d_2'$ be the directions associated to $e_1$ and $d_2$ and $d_2'$ be the ends associated to $e_2$. Assume $\{d_1, d_2\}$ is the only illegal turn. There is loop $\alpha$ consisting of $e_1$ and $e_2$ where $d_1$ is not followed by $d_2$. We take $\alpha$ to be our immersed loop. For the fourth case, we can assume $e_1$ starts and ends at $v$ and $e_2$ starts at $v$ and ends in $w$. Then there has to be another edge $e_3$ connecting $v$ to a vertex $u$ otherwise, $\{e, \tau\}$ forms a yo-yo. If $u$ can be connected to $w$, then this path and $P$ form a legal loop. Otherwise, we find an immersed loop passing through $w$ and an immersed loop passing through $u$. Then a concatenation of these two loops, two copies of $e_2$ and $e_3$ each and the loop $e_1$ forms an immersed legal loop containing $P$. This finishes the proof.

Let $z$ be a point that lies on a possibly different geodesic connecting $x$ to $y$, that is

$$(2) \quad d(x, z) + d(z, y) = d(x, y).$$

Let $\phi: x \to z$ be a difference of markings map that gives rise to a standard path $\gamma_{\text{std}}$. We decomposed $\phi = \phi_1 \circ \phi_2$, where $\phi_1: x \to w$ represents the scaling segment of the standard path. Assume that the tension graph of $\phi$ is not all of $x$ and consider an edge $e \notin x_\phi$.

By the claim, there exists a $\phi_r$-legal immersed loop $\alpha$ containing $e$. Then

$$|\alpha|_x e^{d(x,y)} = \left(|\alpha|_x e^{d(x,z)}\right) e^{d(z,y)}$$

$$(3) \quad < |\alpha|_x e^{d(z,y)} \leq |\alpha|_y.$$

But $\alpha$ is $\phi_r$-legal, hence

$$|\alpha|_x e^{d(x,y)} = |\alpha|_y.$$
This is a contradiction. Thus $x_\phi = x$, which implies $\phi_1$ is degenerate, $w = x$ and $\phi = \phi_2$. That is, there is a folding path $\gamma$ connecting $x$ to $z$.

Let $u = \gamma(s)$ be the last point along $\gamma$ where $\gamma_r$ and $\gamma$ agree, let $\psi_{r,s}: u \to y$ be the left over difference of markings map associated to $\gamma_r$ and $\psi_s: u \to z$ be the left over difference of markings map associated to $\gamma$. That is, the path $\gamma|_{[0,s]}$ is a (possibly degenerate) sub-path of $\gamma_r$, but at $u$, there is a $\psi_s$-illegal turn $\tau$ that is $\psi_{r,s}$-legal. Consider the segment $P$ consists of the pair of edges that form $\tau$. This segment is legal in $\psi_{r,s}$, and hence by the claim, there exists a $\psi_{r,s}$-legal immersed loop $\alpha$ containing $P$.

That is, $\alpha$ is not stretching maximally from $\psi_s(s)$ to $\psi_s(s + \epsilon)$ and an identical to the argument to above gives a contradiction. Thus, the path $\psi_s$ is subpath of $\gamma_r$ and $z$ lies on $\gamma_r$.

![Figure 2](image.png)

**Figure 2.** A yo-yo illegal turn gives rise to two geodesic paths.

We now show the other direction, that is, we establish that uniqueness of a geodesic implies that it is a rigid folding path. Consider a standard geodesic $[x,y]_{\text{std}}$ from $x$ to $y$. Again, the path $[x,y]_{\text{std}}$ is by definition a concatenation of a rescaling path $\gamma_1$ and a folding path $\gamma_2$.

Suppose that, on $\gamma_2$ there are two illegal turns at some point. Then [FM11] shows that folding the two illegal turns at different speeds renders different geodesic paths, hence obstructing uniqueness. Otherwise, suppose $\gamma_2$ contains a yo-yo at some time $s_0$. Then a segment $\gamma_2|[s_0,s_1]$ can be replaced with a different geodesic $\tilde{\gamma}_2$. Referring to Fig. 2, the geodesic $\gamma_2|[s_0,s_1]$ is obtained from folding the yo-yo, labeled $\tau$ from $\gamma_2(s_0)$ to $\gamma_2(s_1)$. However, one can also fold the edge $e$ first at $\sigma_1$ to the point $\tilde{\gamma}_2(s)$, $s \in (s_0, s_1)$, and then fold $\sigma_2$ to reach $\gamma_2(s_2) = \tilde{\gamma}_2(s_2)$. Thus, the geodesic $\gamma_2|[s_0,s_1]$ is not a unique geodesic connecting its end points.

Consider now the segment $\gamma_1$. Suppose there is more than one edge that is not in $x_\phi$. Then we can choose how fast to rescale the lengths of these edges rendering multiple geodesics with same end points as $\gamma_1$. Otherwise, suppose $e$ is the only edge that is not in $x_\phi$. Similar to the paths illustrated in Fig. 2, $e$ can be folded onto one of its neighboring edges in a zigzag manner such that the end graph is isomorphic to the end point of $\gamma_1$. Thus $\gamma_1$ is never a unique geodesic connecting its end points, unless it is degenerate, that is $\gamma = \gamma_2$.  


To sum up, for a standard geodesic to be unique, $\gamma_1$ is necessarily degenerate and $\gamma_2$ contains only one non-yo-yo illegal turn at any point. That is to say, it is a rigid folding path.

**Remark 2.5.** Note that, the first part of the proof still works for rank $n = 2$. That is, if points $x$ and $y$ are connected via a rigid folding path, that path is the unique geodesic from $x$ to $y$. However, in CV$_2$, even when there is a yo-yo, the geodesic is unique. In fact, the paths $\gamma_2$ and $\hat{\gamma}_2$ described in Fig. 2 are identical in CV$_2$.

3. We strong convexity

The purpose of this section is to prove the following:

**Theorem 3.1.** Given a difference of markings map $\phi: x \to y$, where $x_\phi = x$, there exists a speed assignment $S$ defining a folding path $\gamma: [0,t_1] \to CV_n$ starting at $x$ towards $y$ so that, for every loop $\alpha$ and every time $t \in [0,t_1]$, 

$$|\alpha|_t \leq \max(|\alpha|_x, |\alpha|_y).$$

Recall that a speed assignment is a set $S = \{s_\tau\}_{|\tau| = 2}$ of speeds assigned to all sub-gates of size 2. For $t$ small enough, there is a quotient map $\bar{\phi}_t: x \to \bar{x}_t$ (that is, $\bar{\phi}_t$ is an isometry along the edges of $x$) where the edges in gate $\tau$ are identified along a subsegment of length $t s_\tau$. Let $|S|$ be the speed at which $\bar{x}_t$ is losing length, that is, 

$$|S| = \frac{1 - \text{total length of } \bar{x}_t}{t}.$$

Note that, this is a constant for small values of $t$. Also, if $S'$ is the associated coherent speed assignment, then $|S| = |S'|$ because the define the same quotient graph $\bar{x}_t$.

**Lemma 3.2.** Let $[x,y]_t$ be a folding path associated to a difference of markings map $\phi: x \to y$ and a coherent speed assignment $S = \{s_\tau\}_{\tau \in T}$. Then, for every loop $\alpha$, the derivative of the length of $\alpha$ along this path equals

$$|\alpha|'_t = |\alpha|_t - 2 \sum_{\tau \in T_\alpha(\alpha)} \frac{s_\tau}{|S|},$$

where the derivative is taken with respect to distance in $CV_n$.

**Proof.** For every $\tau \in T_\alpha$, there are two sub-edges of $\alpha$ of length $t s_\tau$ that are identified under the quotient map $\bar{\phi}_t: x \to \bar{x}_t$. And $x_t$ is obtained from $\bar{x}_t$ by a scaling of factor $\frac{1}{1-t|S|}$. Since $S$ is coherent, the length loss of the loop $\alpha$ associated for to every $\tau \in T_\alpha$ is exactly $2t s_\tau$. Hence

$$|\alpha|_t = \frac{|\alpha|_x - 2t \sum_{\tau \in T_\alpha} s_\tau}{1 - t|S|},$$

and, for $s > t$,

$$|\alpha|_s - |\alpha|_t = \frac{(s - t)|S||\alpha|_x - 2(s - t) \sum_{\tau \in T_\alpha} s_\tau}{(1 - s|S|)(1 - t|S|)}.$$  

Also $d(x_t, x_s) = \log \left( \frac{(1-s|S|)}{(1-t|S|)} \right)$. That is, when $(s-t)$ is small,

$$d(x_t, x_s) = \log \left( \frac{1 - t|S|}{1 - s|S|} \right) = \log \left( 1 + \frac{(s - t)|S|}{1 - s|S|} \right) \sim \frac{(s - t)|S|}{1 - s|S|}.$$
Therefore,

\[ |\alpha|_t = \lim_{s \to t} \frac{|\alpha_s| - |\alpha_t|}{d(x_t, x_s)} = \frac{|\alpha|_x - 2\sum_{\tau \in T_\alpha} s_\tau}{1 - t|S|}. \tag{6} \]

On the other hand, replacing, \(|\alpha|_t\) in the right-hand side of (4) with the expression in Equation (5), we get

\[ |\alpha|_t - 2\sum_{\tau \in T_\alpha(a)} s_\tau |S| = \frac{|\alpha|_x - 2t\sum_{\tau \in T_\alpha} s_\tau}{1 - t|S|} - 2\sum_{\tau \in T_\alpha(a)} \frac{s_\tau}{|S|} = \frac{|\alpha|_x - 2\sum_{\tau \in T_\alpha} s_\tau}{1 - t|S|}. \tag{7} \]

The right hand sides of Equation (6) and Equation (7) are the same, hence the left hand sides are equal. But this is what was claimed. \(\Box\)

For given \(x\) and \(y\), our goal is to find an appropriate speed assignment so that, for every loop \(\alpha\), if \(|\alpha|_y \leq |\alpha|_x\) then \(|\alpha|_{t=0} \leq 0\). To this end, we will define values \(\ell_\tau\) that quantifies the contribution of sub-gate \(\tau\) to the total length loss from \(x\) to \(y\) and then use these to define a speed assignment. For the remainder of this section, let \(\tilde{y} \in cv_\alpha\) be the representative in the projective class of \(y\) so that the associated change of marking map \(\tilde{\phi}: x \to \tilde{y}\) restricted to every edge is a length preserving immersion. Also, let \(\Phi: T_x \to T_{\tilde{y}}\) be a lift of \(\tilde{\phi}\).

Consider a point \(p \in T_{\tilde{y}}\) and let \(\text{Pre}(p) \subset T_x\) denote the set of pre-images of \(p\) under the map \(\Phi\) and let \(\text{CH}(p)\) denote the convex hull of \(\text{Pre}(p)\) in \(T_x\). Assume \(p\) is generic, and thus, \(\text{Pre}(p)\) does not contain any vertex in \(T_x\). We give \(\text{CH}(p)\) a tree structure where there are no degree 2 vertices; some edges of \(\text{CH}(p)\) may consist of several edges in \(T_x\). The tree \(\text{CH}(p)\) also inherits its illegal turn structure from \(T_x\), however, an edge of \(\text{CH}(p)\) may contain one or more illegal turns. Also, note that, since all end points of \(\text{CH}(p)\) map to \(p\), \(\text{CH}(p)\) does not contain any legal path connecting its end vertices.

We denote the set of sub-gates of \(T_x\) by \(\Theta\). For each sub-gate \(\sigma \in \Theta\), we assign a weight \(c(\sigma, p)\) to \(\sigma\) which measures how much of the branching of \(\text{CH}(p)\) is due to \(\sigma\). We do this inductively.

**Step 0.** Note that all vertices of degree 1 in \(\text{CH}(p)\) are in \(\text{Pre}(p)\). But some points in \(\text{Pre}(p)\) may lie on the interior of an edge of \(\text{CH}(p)\). As Step 0, we cut \(\text{CH}(p)\) along these point to decompose \(\text{CH}(p) = \sqcup_{i=1}^m T_i\). The vertices of degree 1 in \(T_i\) are exactly \(\text{Pre}(p) \cap T_i\) and \(T_i\) is the convex hull of \(\text{Pre}(p) \cap T_i\). This decomposes \(\Theta = \sqcup_{i=1}^m \Theta_i\).

For a tree \(T_i\), \(i = 1, \ldots, m\), a vertex is an outer vertex if it has degree 1 and an edge is an outer edge if one of its vertices has degree one. All other edges are called inner edges. We now assign a weight to each sub-gate in \(\Theta_i\). Let \(T = T_i\).

**Step 1.** Consider all the outer edges of \(T\). If there is an edge \(e\) that contains illegal turns \(\sigma_1, \ldots, \sigma_k\), then we define \(c(\sigma, p) = \frac{1}{k}\) and we remove this edge from \(T\). The total weight assignment is 1 and the number of end vertices of \(T\) is reduced by one. After we remove this edge we remove any vertices that have degree two, such that in the resulting graph, each edge is again a topological edge, consistent with our initial condition. Note that, as in \(T\), the interior of an edge in the resulting tree may contain illegal turns. We apply Step 1 as many times as possible.

We observe that the remaining subtree still has the property that it does not contain any legal paths connecting its end vertices. To see this, noticing that since we removed exactly one topological edge, the degree of the vertex at which we moved this edge is still
two or higher. This implies we did not create a new leaf by removing one edge, which means a legal path the would have appeared after this step already exists before the step. But that contradicts our assumption.

**Step 2.** We claim that there is a vertex \( v \) of \( T \) and a sub-gate \( \sigma \) at \( v \) so that \( \sigma \) contain all but one edge incident to \( v \) and all edges of \( \sigma \) are outer edges (the remaining edge can be either inner or outer). To see this, consider the longest embedded path \( v_0, v_1, \ldots, v_m \) in \( T \). Then, all but one of the edges incident to \( v_1 \) are outer edges otherwise the path can be made longer. In fact, all these outer-edges have to be in some sub-gate \( \sigma \) otherwise there is a legal path of length 2. This proves the claim.

There are two cases. If all edges incident to \( v \) are in the same gate \( \tau \) (which contains \( \sigma \) and the remaining edge) then we define \( c(\tau, p) = |\tau| - 1 \) and we remove \( v \) and all edges incident to \( v \). The number of ends of \( T \) goes down by \( |\sigma| = |\tau| - 1 \).

Otherwise, there are two gates at \( v \), namely \( \sigma \) and a gate with one edge. In this case, we define \( c(\sigma, p) = |\sigma| - 1 \) and delete all the edges associated to \( \sigma \) from \( T \). The vertex \( v \) survives this process and the number of ends of the tree is reduced by \( |\sigma| - 1 \). The remaining tree still has the property the it does not contain any legal path connecting its end vertices because any such path could be extended to a legal path in the original tree.

We proceed in this way, by first applying Step 1 as many times as possible, and when that is not possible Step 2. This is a finite process since each time the number of edges in \( T \) decreases. When we cannot apply either Step 1 or Step 2, what is left has no inner edges or outer edges with illegal turns. Therefore it only has outer edges without illegal turns. If there outer edges belong to more than one gate, we have a legal path from end vertices to end vertices, which is contradictory to our assumption. Therefore we necessarily have a one-gate vertex with a finite set of edges. In this case we define \( c(\sigma, p) = |\sigma| - 1 \) for that gate \( \sigma \). Again, the number of ends is one more than weight assigned. To summarize, the weight assigned at every step is equal to the number of end vertices removed. We have shown

**Lemma 3.3.** For every \( T_i \) as above, we have

\[
\sum_{\sigma \in \Theta_i} c(\sigma, p) = |\text{Pre}(p) \cap T_i| - 1.
\]

This implies

**Lemma 3.4.** For any generic point \( p \in T_y \), we have

\[
\sum_{\sigma \in \Theta} c(\sigma, p) = |\text{Pre}(p)| - 1
\]

where the sum is over all sub-gates in \( T_x \).

*Proof.* Firs, we see that

\[
(8) \quad \sum_{i=1}^{m} |\text{Pre}(p) \cap T_i| = |\text{Pre}(p)| + (m - 1)
\]
Figure 3. \( c(\sigma, p) \) is computed iteratively by applying Step 1 as many times as possible and then apply Step 2, and then repeat.

This is because the number of points on the interior of \( \text{CH}(p) \) is \((m - 1)\) and these points are counted twice in the right hand side of the above equation. We now have

\[
\sum_{\sigma \in \Theta} c(\sigma, p) = \sum_{i=1}^{m} \sum_{\sigma \in \Theta_i} c(\sigma, p)
\]

(Lemma 3.3)

\[
= \sum_{i=1}^{m} \left( |\text{Pre}(p) \cap T_i| - 1 \right)
\]

\[
= \sum_{i=1}^{m} \left( |\text{Pre}(p) \cap T_i| \right) - m
\]

(Equation (8))

\[
= |\text{Pre}(p)| + (m - 1) - m = |\text{Pre}(p)| - 1.
\]

which is as we claimed. \( \square \)

Remark 3.5. The result of the algorithm is not unique. Different gates may be assigned different values depending on the order in which we remove outer edges containing illegal turns. We proceed without claiming uniqueness and prove that all the resulting paths are weakly convex.

Example 3.6. The example in Fig. 3 illustrate the definition of \( c(\sigma, p) \). Suppose \( \text{CH}(p) \) is as shown in the leftmost graph, with seven outer edges and seven illegal turns, marked \{\( \sigma_1, \sigma_2, \ldots, \sigma_7 \}\). First apply Step 1. There are two possible candidates for Step 1. The illegal turns that occur as part of a topological edge are \( \sigma_6 \) and \( \sigma_7 \), which belong to different topological edges. It does not matter which one of them is assigned first. After applying Step 1 twice, we get

\[
c(\sigma_6, p) = 1 \text{ and } c(\sigma_7, p) = 1.
\]

After deleting these two topological edges, \( \sigma_5 \) no longer exists, therefore

\[
c(\sigma_5, p) = 0.
\]

Next, Step 2 picks out either \( \sigma_1 \) or \( \sigma_4 \). Suppose we start with \( \sigma_4 \), which contains 3 outer edges, thus

\[
c(\sigma_4, p) = 3 - 1 = 2.
\]
After deleting these three outer edges of $\sigma_4$, we apply Step 1 again and observe that there is a new topological edge with two illegal turns $\sigma_2$ and $\sigma_3$. Thus

$$c(\sigma_2, p) = c(\sigma_3, p) = \frac{1}{2}.$$  

And the topological edge is deleted at the end of the step. Lastly we have only one gate with two outer edge, thus

$$c(\sigma_1, p) = 1.$$  

It can be verified that

$$\sum_{\sigma \in \Theta} c(\sigma, p) = 1 + 1 + 0 + 2 + \frac{1}{2} + \frac{1}{2} + 1 = 6 = |\text{Pre}(p)| - 1.$$  

We now use the $c(\sigma, p)$ functions to define the length loss functions:

$$\ell_\sigma = \int_{T_{\bar{y}}} c(\sigma, p) \, dp$$

where the integral is taken with respect to the length in $T_{\bar{y}}$. Note that $c(\sigma, p)$ is defined only for a generic $p$ but the integral is still defined. Even though this is integral over a non-compact set, for every $\sigma$, the set of points $p$ where $c(\sigma, p)$ is non-zero is compact and hence the integral is finite. Also, since our construction is equivariant, for any sub-gate $\tau \in T$ (Recall $T$ is the set of all sub-gates in $x$ under the map $\phi$.) we can define $\ell_\tau = \ell_\sigma$ where $\sigma$ is any lift of $\tau$ to $T_x$.

The number $\ell_\tau$ represent how much of the length loss from $x$ to $\bar{y}$ we are attributing to the sub-gate $\tau$. In particular, we have

**Lemma 3.7.** For $\overline{\phi} : x \to \bar{y}$ and $\ell_\tau$ defined as above, we have

$$\sum_{\tau \in T} \ell_\tau = |x| - |ar{y}|$$  

**Proof.** We denote points in $\overline{y}$ by $q$ and $\text{Pre}(q)$ represents the pre-image of $q$ under $\overline{\phi}$. Since the map $\overline{\phi}$ is locally a length preserving immersion, we have

$$1 = |x| = \int_{\bar{y}} |\text{Pre}(q)| \, dq.$$
Let $T_0 \subset T_y$ be a tree that is a fundamental domain of action $F_n$ on $T_y$ and let $\Theta_0$ be a finite subset of $\Theta$ that contains exactly one lift for every $\tau \in T$. Now,

$$ \sum_{\tau \in T} \ell_{\tau} = \sum_{\sigma \in \Theta_0} \ell_{\sigma} = \sum_{\sigma \in \Theta_0} \int_{T_y} c(\sigma, p) \, dp $$

$$(T_y = \cup_g g(T_0))$$

(c(\cdot, \cdot) is equivariant)

$$\sum_{\sigma \in \Theta_0} \int_{T_0} c(g^{-1}(\sigma), p) \, dp$$

$$(\Theta = \cup_g g(\Theta_0))$$

(Lemma 3.4)

$$\int_{T_0} |Pre(p)| - 1 \, dp$$

which is what was claimed in the lemma.

Next, we use length loss contributions $\ell_{\tau}$ to define a speed assignment $S$. For a sub-gate $\tau$ with $|\tau| = 2$, define

$$ s_{\tau} = \sum_{\tilde{\tau} \supseteq \tau} \frac{\ell_{\tilde{\tau}}}{|\tilde{\tau}| - 1} $$

where the sum is over all sub-gates $\tilde{\tau}$ containing $\tau$. We are dividing $\ell_{\tilde{\tau}}$ by $(|\tilde{\tau}| - 1)$ because if you fold edges of $\tilde{\tau}$ along a segment of length $t$, the length loss is larger by factor $(|\tilde{\tau}| - 1)$. Define other the speed for other gates so that $S = \{s_{\tau}\}_{|\tau|=2}$ is a coherent. Then $S$ is our desired speed assignment.

**Example 3.8.** We illustrate the computation of $s_{\tau}$ with the example mentioned in the introduction. Consider $F_3 = \langle a, b, c \rangle$. Let $x$ be a rose with three petals. The three edges we refer to as $e_1, e_2, e_3$. The edge $e_1$ is labeled $ac^2$, the edge $e_2$ is labeled $bc$, the edge $e_3$ is labeled $c$. The edge lengths are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, respectively. The graph $y$ is a rose of three petals with labels $\{a, b, c\}$ and lengths $\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$. The construction is such that $\phi: x \to y$ satisfies $x_\phi = x$. $y$ is obtained from $x$ by wrapping $ac^2$ around $c$ twice and $bc$ around $c$ once.

$T_y$ contains three types of edges. If the point $p$ is on an edge labeled $a$ or $b$, then the pre-image contains only one copy of $p$, and $c(\tau, p) = 0$ for all $\tau$. If $p$ is on the $c$-edge, then $CH(p)$ is as shown in Fig. 4, where the four pre-images of $p$ are marked with a circle.

$CH(p)$ has two gates. One contains a black and green edge, which we denote $\sigma_{e_1e_3}$. The other gate contains three edges, black, green and blue, and we denote the gate $\sigma_{e_1e_2e_3}$. At Step 1,

$$ c(\sigma_{e_1e_3}, p) = 1; $$

at Step 2,

$$ \sigma_{e_1e_2e_3} = 2. $$

Next we compute the length loss function by integrating $c(\cdot, \cdot)$ over $T_y$. In this case, the only non-zero component of the integral is when integrating over edge labeled $c$, which
Figure 4. The $CH(p)$ of Example 3.8 where $p$ is a point on the edge labeled $c$ in $y$

has length $\frac{1}{6}$, therefore:

\[
\ell_{e_1e_3} = 1 \times \frac{1}{6} = \frac{1}{6}, \\
\ell_{e_1e_2e_3} = 2 \times \frac{1}{6} = \frac{1}{3}.
\]

Indeed, it is the case that

\[
\frac{1}{3} + \frac{1}{6} = \sum_{\sigma} \ell_{\sigma} = x - \gamma = 1 - \frac{1}{2} = \frac{1}{2}
\]

Based on the length loss functions we compute the folding speed of all sub-gates in $x$. Again we can denote a sub-gate in $x$ by the edges in the gate, so we have

\[
s_{e_1e_3} = \ell_{e_1e_3} + \frac{1}{2} \ell_{e_1e_2e_3} = \frac{1}{6} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{3} \\
\]

\[
s_{e_1e_2} = s_{e_2e_3} = \frac{1}{2} \ell_{e_1e_2e_3} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}
\]

That is to say, since $ac^2$ wraps over $c$ twice while $bc$ wraps over $c$ once, infinitesimally, the folding associated to the former is twice as fast.

**Lemma 3.9.** For the speed assignment $S$ above, we have

\[|S| \leq \sum_{\tau \in \mathcal{T}} \ell_{\tau}.\]

**Proof.** We organize the argument by considering one gate $\tau$ and its sub-gates only. The length losses at different gates add up, hence, it is sufficient to prove the lemma one gate at the time. For the rest of the argument, let $\tau$ be a fixed gate in $\mathcal{T}$.

By an $\epsilon$–neighbourhood of a gate $\tau$ we mean the intersection of an $\epsilon$–ball around the vertex associated to $\tau$ with the edges associated to $\tau$. For $\epsilon$ small enough, the $\epsilon$–neighbourhood of $\tau$ is a tree with one vertex $v$ and $|\tau|$ edges $e_1, \ldots, e_{|\tau|}$ of size $\epsilon$. Choose $t > 0$ small enough so that, for every $1 \leq i, j \leq |\tau|$, $ts_{i,j} < \epsilon$.

The image of this $\epsilon$–neighbourhood in $\mathcal{T}_\epsilon$ is a quotient of the $\epsilon$–neighbourhood of $\tau$ after identifying $e_i$ and $e_j$ along a segment of length $t s_{i,j}$ starting from the vertex $v$. In fact, it is enough to do $|\tau| - 1$ identifications. Namely, we choose the pair of edges $e_1, e_1'$ that are identified along the longest segments (that is, for $\tau_1 = \{e_1, e_1'\}$, $s_{\tau_1}$ is maximal among
all sub-gates of $\tau$ of size 2). We put $e_1$ and $e'_1$ in the same group and all other edges in separate individual groups. Continue in this way for $i = 2, \ldots, (|\tau| - 1)$, we choose a pair of edges $e_i, e'_i$ from different groups so that for $\tau_i = \{e_i, e'_i\}$, $s_{\tau_i}$ is maximal among all such pairs and combine the groups associated to $e_i$ and $e'_i$ into one group. After $|\tau| - 1$ steps, we have only one group (see Figure 5).

![Diagram](image)

**Figure 5.** We identify the edges of $\tau_i$ along a segment of length $t s_{\tau_i}$ to obtain the image of the $\epsilon$–neighborhood of $\tau$ in $x$ to $\bar{x}$.

The image of the $\epsilon$–neighborhood of $\tau$ in $x$ to $\bar{x}$ can be obtained from the $\epsilon$–neighborhood of $\tau$ in $x$ by identifying $e_i$ and $e'_i$ along a segment of length $t s_{\tau_i}$ because any other identification between two groups is along a smaller segment. That is,

$$|S\tau| = \sum_{i=1}^{|\tau|-1} s_{\tau_i}. $$

The way $\tau_i$ are chosen, every sub-gate $\tau' \subset \tau$ contains at most $(|\tau'| - 1)$ of these $\tau_i$. This is because, after $(|\tau'| - 1)$ sub-gates of $\tau'$ are chosen in the above process, every edge in $\tau'$ is already in the same group. Therefore, letting $T\tau$ to be the set of sub-gates of $\tau$, we have

$$\sum_{i=1}^{|\tau|-1} s_{\tau_i} = \sum_{i=1}^{|\tau|-1} \sum_{\tau' \supseteq \tau_i} |\tau'| - 1 = \sum_{\tau' \in T\tau} \left\lfloor \frac{|\tau'|}{|\tau'| - 1} \right\rfloor \cdot \sum_{\tau' \subseteq \tau' \subseteq \tau} \ell_{\tau'}. $$

Combining the last two equations, we have

$$|S\tau| \leq \sum_{i=1}^{|\tau|-1} \sum_{\tau' \subseteq \tau} \ell_{\tau'} = \sum_{\tau' \subseteq \tau} \ell_{\tau'}. $$

This finishes the proof.

**Lemma 3.10.** Let $v \subset T_x$ be a vanishing path and let $\Theta_v$ be the set of sub-gates in $T_x$ that appear along $v$.

$$|v|_x \leq 2 \sum_{\sigma \in \Theta_v} s_{\sigma}. $$

**Proof.** Since $v$ is a vanishing path, $\Phi$ sends its end-points to some point $p \in T\bar{\gamma}$. We claim

$$1 \leq \sum_{\sigma \in \Theta_v} \sum_{\delta \supseteq \sigma} \frac{c(\delta, p)}{|\delta| - 1}. $$
This is because \( v \) is contained in \( \text{CH}(p) \). If \( v \) passes through an edge \( e \) of \( \text{CH}(p) \) with an illegal turn, then the sum
\[
1 = \sum_{\sigma \in \Theta_v} c(\sigma, p) \leq \sum_{\sigma \in \Theta_v} c(\sigma, p)
\]
and the claim follows. Otherwise, a sub-gates \( \sigma \in \Theta_v \) is contained in a sub-gate \( \tilde{\sigma} \) that appears in \( \text{CH}(p) \), where all the associated edges are legal, and hence, following the algorithm, \( c(\tilde{\sigma}, p) = |\tilde{\sigma}| - 1 \). Again the claim follows. Note that the map \( \Phi \) from \( v \) to \( \Phi(v) \) is 2-to-1. Hence
\[
|v|_x = \int_{\Phi(v)} 2 \, dp
\]
(Using the claim)
\[
\leq 2 \int_{\Phi(v)} \sum_{\sigma \in \Theta_v, \tilde{\sigma} \supseteq \sigma} c(\tilde{\sigma}, p) \frac{1}{|\tilde{\sigma}| - 1} \, dp
\]
(Enlarging the domain of integration)
\[
= 2 \sum_{\sigma \in \Theta_v, \tilde{\sigma} \supseteq \sigma} \ell_\sigma \frac{1}{|\tilde{\sigma}| - 1} = 2 \sum_{\sigma \in \Theta_v} s_\sigma.
\]
And we are done. \( \square \)

**Proof of Theorem 3.1.** Let \( S \) be the speed assignment defined in Equation (9), let \( t_1 > 0 \) be a time for which the folding with the speed \( S \) is defined (see Proposition 2.3) and let \( \alpha \) be any loop. Denote the geodesic representative of \( \alpha \) in \( x \) with \( \alpha_x \) and in \( y \) with \( \alpha_y \).

- The segments \( u_i \) are vanishing paths in \( x \).
- The segments \( \Phi(w_i) \) are immersed and \( \alpha_y = \bigcup_i \Phi(w_i) \).

In particular,
\[
|\alpha|_x = |\alpha|_y + \sum_i |v_i|_x.
\]
Let \( u_i \) be a lift of \( u_i \) to \( T_x \). We can assume \( v_i \) are completely disjoint from each other. Since \( w_i \) are all legal, there is a one-to-one correspondence between sub-gates in \( T_\alpha \) and in \( \bigcup_i \Theta_{w_i} \). Thus, Lemma 3.10 implies
\[
\sum_i |u_i|_x = \sum_i |v_i|_x \leq 2 \sum_i \sum_{\sigma \in \Theta_{w_i}} s_\sigma = 2 \sum_{\tau \in T_\alpha} s_\tau.
\]
(10)

We have
\[
\frac{|\alpha|_x - \sum_i |u_i|_x}{|y|} = |\alpha|_y = |\alpha|_x + (|\alpha|_y - |\alpha|_x)
\]
\[
|\alpha|_x - \sum_i |u_i|_x = |\alpha|_x|\bar{y}| + (|\alpha|_y - |\alpha|_x)|\bar{y}|.
\]
By Equation (10), we replacing \( \sum_i |u_i|_x \) with \( 2 \sum_{\tau \in T_\alpha} s_\tau \), we get
\[
|\alpha|_x(1 - |\bar{y}|) - 2 \sum_{\tau \in T_\alpha} s_\tau \leq (|\alpha|_y - |\alpha|_x)|\bar{y}|.
\]
But $(1 - |\gamma|) = \sum_{\tau \in T} \ell_{\tau} \geq |S|$, therefore
\[
|\alpha|_{x} - 2 \sum_{\tau \in T_{\alpha}} \frac{s_{\tau}}{|S|} \leq (|\alpha|_{y} - |\alpha|_{x}) \frac{|\gamma|}{1 - |\gamma|}.
\]

Note that
\[
d(x, y) = \log \frac{1}{|\gamma|} \implies |\gamma| = e^{-d(x, y)} \implies \frac{|\gamma|}{1 - |\gamma|} = \frac{1}{e^{d(x, y)} - 1}.
\]

Hence, letting $V_{\alpha} = 2 \sum_{\tau \in T_{\alpha}(\alpha)} \frac{s_{\tau}}{|S|}$, we have
\[
(11) \quad |\alpha|_{x} - V_{\alpha} \leq \frac{|\alpha|_{y} - |\alpha|_{x}}{e^{d(x, y)} - 1}.
\]

Now re-parametrize the folding path with arc-length and denote the new parameter with $s$. Solving the differential equation given in Lemma 3.2, we have
\[
(12) \quad |\alpha|_{s} = |\alpha|_{s} - V_{\alpha} \implies |\alpha|_{s} = (|\alpha|_{x} - V_{\alpha}) e^{s} + V_{\alpha}.
\]

Note that, if $|\alpha|_{y} \leq |\alpha|_{x}$ then by Equation (11) the rate of change of the length is negative and $|\alpha|_{t} \leq |\alpha|_{x}$. If $|\alpha|_{y} \geq |\alpha|_{x}$ then,
\[
|\alpha|_{s} = (|\alpha|_{x} - V_{\alpha}) e^{s} + V_{\alpha}
\]
\[
= (|\alpha|_{x} - V_{\alpha})(e^{s} - 1) + |\alpha|_{x}
\]

(Equation (11))
\[
\leq (|\alpha|_{y} - |\alpha|_{x}) \frac{e^{s} - 1}{e^{d(x, y)} - 1} + |\alpha|_{x}
\]

$s \leq d(x, y))$
\[
\leq |\alpha|_{y}.
\]

That is, in either case, $|\alpha|_{t} \leq \max (|\alpha|_{x}, |\alpha|_{y})$. This finishes the proof.

\[
\square
\]

4. DECORATED DIFFERENCE OF MARKINGS MAP

In Section 3, we constructed a balanced folding path starting from $x$ towards $y$ assuming that there is a difference of markings map $\phi: x \to y$ such that $x_{\phi} = x$. In general, such a difference of marking map does not exit. In this section, we modify the difference of markings map such that, for any two points $x, y \in CV_{n}$, one can still define a notion of a folding path from $x$ to $y$. Such a modified difference of markings map will be called a decorated difference of markings map. Recall that, given $\phi: x \to y$, one can construct a standard geodesics where first every edge $e$ outside of $x_{\phi}$ is shortened. For a decorated difference of markings map, we instead create an illegal turn in the interior of the edge $e$. The folding of that illegal turn effectively shortens the length of $e$.

4.1. Decorating the graphs. Consider a pair of points $x, y \in CV_{n}$ and an optimal map $\phi: x \to y$. Recall that $x_{\phi}$ is the tension sub-graph of the map $\phi$, i.e. the sub-graph of $x$ consisting of edges that are stretched by a factor $\lambda$, where $\log \lambda = d(x, y)$. We start by decorate the graph $x$(See Fig. 6 for the decoration described here).

Let $e$ be an edge outside of $x_{\phi}$, say connecting $v_{0}$ to $v_{3}$. Add two subdividing vertices $v_{1}$ and $v_{2}$ to $e$ so that the following holds: Let $e_{i,j}$ denote the edge with end vertices $v_{i}$ and $v_{j}$. Then, we require that
\[
|e_{0,1}|_{x} = |e_{1,2}|_{x}, \quad \text{and} \quad \lambda |e_{2,3}|_{x} = |\phi(e)|_{y}.
\]
This is always possible since $\lambda|e_x < |\phi(e)|_y$. We call $v_1$ and $v_2$ pseudo-vertices and refer to the graph $x$ with all the pseudo-vertices added as $x^d$.

![Figure 6. Decoration of edges in $x$ and $y$.](image)

Next we decorate $y$. For every edge and vertex $v_0 \in x$ as above, we attach a new edge of length $\lambda|e_{0,1}|_x$ to $y$ at the point $\phi(v_0) \in y$. The other end of this new edge is incident to an added degree-one vertex. Thus we have added a leaf at $\phi(v_0)$. We label this edge, oriented away from $\phi(v_0) \in y$, by $\epsilon_e$, and the same edge with opposite orientation by $\epsilon_{e}^{-1}$. We do this for every edge in $x$ that is not in $x_\phi$ and denote the resulting decorated graph by $y^d$.

We now modify the optimal map $\phi: x \to y$ to $\phi^d: x^d \to y^d$. Define $\phi^d$ to be a map that is linear on every edge with slope $\lambda$. For edges in $x_\phi$ the two maps agree. For every edge $e$ outside of $x_\phi$, we have

$$
\phi^d(e_{0,1}) = \epsilon_e \\
\phi^d(e_{1,2}) = \epsilon_{e}^{-1} \\
\phi^d(e_{2,3}) = \phi(e)
$$

Note that, the vertex $v_1$ is a one-gate vertex.

4.2. Folding paths under decorated difference of markings maps. We can use the decorated graphs to construct a folding path from $x$ to $y$. The graphs $x^d, y^d$ are both constructed from $x, y$ by adding vertices and hairs, respectively. The graph $y$ is the core graph of $y^d$. The graphs and maps $\phi^d: x^d \to y^d$ define a train track structure on $x^d$ that is slightly different from its definition in Section 2: in $x^d$, each point either has two gates or is mapped to the end of a leaf. When the context is clear we also write the decorated difference of marking map as $\phi^d: x \to y$.

Let $\mathcal{T}$ be the set of sub-gates of $\phi^d$. As done previously, we can fold $x^d$ according to a given speed assignments $S = \{s_\tau\}_{|\tau|=2}$ by identifying edges associated to $\tau$ along a sub-segment of length $t s_\tau$ for $t$ small enough, to obtain a quotient map $\overline{\phi}^d_t: x^d \to \overline{x}^d_t$. The graph is a union of a core graph $\overline{x}_t \in cv$ and a hair of size $t s_\tau$ associated to each $\tau$. Let $x_t$ be the point in $CV_\pi$ that is a normalization $\overline{x}_t$ to a graph of total length 1 and let $x^d_t$ be the scaling of $\overline{x}^d_t$ by the same factor. Let $\phi^d_t: x^d \to x^d_t$ be the composition of $\overline{\phi}^d_t$ and scaling and let $L(\phi^d_t)$ denote the Lipschitz constant of $\phi^d_t$. By definition of distance, $\log L(\phi^d_t) \leq d(x, x_t)$. But also, $L(\phi^d_t)$ is the scaling factor form $\overline{x}_t$ to $x_t$ and the length of any loop that is legal in $x^d$ increases by this factor from $x$ to $x_t$. Thus,

$$
\log L(\phi^d_t) = d(x, x_t).
$$

As before, for $t$ small enough, the left over map

$$
\psi^d_t: x^d_t \to y^d, \quad \psi^d_t = \phi^d \circ (\phi^d_t)^{-1}
$$
is defined because, for small $t$, points that are identified under $\phi^d_t$ are also identified under $\phi^d$. Also, since the Lipschitz constant of these maps are constant, we have

$$L\left(\psi^d_t\right) = \frac{L\left(\phi^d\right)}{L\left(\phi^d_t\right)} \implies d(x_t, y) \leq \log L\left(\psi^d_t\right) \leq d(x, y) - d(x, x_t).$$

But $d(x, x_t) + d(x_t, y) \leq d(x, y)$ by the triangle inequality. Therefore,

$$d(x, x_t) + d(x_t, y) = d(x, y)$$

and hence the path $\{x_t\}$ is a geodesic starting from $x$ towards $y$. To summarize, similar to Proposition 2.3 we have

**Proposition 4.1.** Given any two points $x, y$ in $\text{CV}_n$ there exists a decorated difference of markings map $\phi^d: x \to y$ such that $x^d_{\phi^d} = x^d$. Furthermore, any speed assignment $S$ defines a geodesic $\gamma: [0, t_1] \to \text{CV}$ starting from $x$ towards $y$, for some $t_1 > 0$.

**Remark 4.2.** The decorated difference of markings map is non-unique in several ways. First, it depends on the difference of markings map $\phi$ which is not unique. Also, the pseudo-vertices can be added at either end or in the middle of an edge. That is, there is a family of decorated folding paths connecting any pair of points.

We now prove an analogue of Theorem 3.1 for decorated folding paths following closely the constructions and arguments of Section 3. We define the length loss function $\ell_\tau$ at each illegal turns based on the decorated difference of markings map from $x^d$ to $y^d$ as we do in Section 3. The associated function $s_\tau$ is also identically defined as in Equation (9) and it defines a geodesic from $x$ towards $y$ as in Proposition 4.1. We keep track of which illegal turns are from the original graph $x$ and which are at the pseudo-vertices. Let $T^C$ denotes the set of all sub-gates in the undecorated graph $x$ (as a subset of $x^d$), let $T^H$ denotes the set of sub-gates at the pseudo-vertices of $x^d$ and let $T$ be the union of $T^C$ and $T^H$. We always assume $t \in [0, t_1]$ (Proposition 4.1), in particular, $x_t$ is in the same simplex at $x$. From $x^d$ to $\pi_t^d$, after time $t$, we define analogously:

$$|S|^d = \frac{1 - |\pi_t^d|}{t}.$$  

However, our actual target graph is the core graph $\pi_t$ of $\pi_t^d$. Observe that the length of $\pi_t$ is the length of $\pi_t^d$ minus the length of the set of hairs. The length of each hair associated to a sub-gate $\tau$ is $ts_\tau$ by construction, thus

$$|\pi_t| = |\pi_t^d| - t \sum_{\tau' \in T^H} s_{\tau'} = |\pi_t^d| - t \sum_{\tau' \in T^H} \ell_{\tau'}.$$  

In the second equality, the assertion $s_{\tau'} = \ell_{\tau'}$ comes from the fact that at each pseudo-vertex there is only one sub-gate that contains the only illegal turn $\tau$. We also define the length loss speed $|S|$ with respect to $\pi_t$:

$$|S| = \frac{1 - |\pi_t|}{t}.$$  

Substitute Equation (14) into Equation (15), we have

$$|S| = |S|^d + \sum_{\tau' \in T^H} \ell_{\tau'}.$$
By Lemma 3.7 and Lemma 3.9 we have
\[ 1 - |\mathbf{y}| = \sum_{\tau \in T} \ell_\tau > |S|^d. \]

By Equation (16) that becomes
\[ (17) \quad 1 - |\mathbf{y}| = \sum_{\tau \in T} \ell_\tau + \sum_{\tau' \in T^H} \ell_{\tau'} > |S|. \]

Now we are ready to prove the main theorem of this section:

**Theorem 4.3.** Given a difference of markings map \( \phi^d : x^d \to y^d, x, y \in \text{CV}_n \), where \( x^d = x^d \), there exists a speed assignment \( S \) defining a folding path \( \gamma : [0, t_1] \to \text{CV}_n \) starting at \( x \) towards \( y \) so that, for every loop \( \alpha \) and every time \( t \in [0, t_1] \),
\[ |\alpha|_t \leq \max (|\alpha|_x, |\alpha|_y). \]

**Proof.** We follow the proof of Theorem 3.1. Let \( S \) be the speed assignment describe above and \( \gamma \) be the associated geodesic starting from \( x \) towards \( y \) coming from Proposition 4.1. Recall that a decorated difference of markings map fold \( x \) into a graph with hairs \( \mathbf{x}^d \) and where the core graph of \( \mathbf{x}^d \) is \( x_t \). By Lemma 3.2, for a given loop \( \alpha \),
\[ \dot{\alpha}|_t = 0 = |\alpha|_x - 2 \sum_{\tau \in T_\phi(\alpha)} s_\tau |S|. \]

By Lemma 3.7,
\[ \sum_{\tau \in T} \ell_\tau = |x| - |\mathbf{x}|. \]

However, since our final target is \( \mathbf{y} \), we separate the sub-gates \( \tau \) in the core graph and the sub-gates \( \tau' \) at the pseudo-vertices. Let \( T^C \) denotes the set of all sub-gates in the core graph and \( T^H \) denotes the set of sub-gates at the pseudo-vertices; \( T \) is a union of \( T^C \) and \( T^H \). We have \( |\mathbf{x}| - |\mathbf{y}| = \sum_{\tau' \in T^H} \ell_{\tau'} \). Therefore we can rewrite Lemma 3.7 as
\[ \sum_{\tau \in T} \ell_\tau + \sum_{\tau' \in T^H} \ell_{\tau'} = \sum_{\tau \in T^C} \ell_\tau + 2 \sum_{\tau' \in T^H} \ell_{\tau'} = |x| - |\mathbf{y}|. \]

By Equation (17), instead of Lemma 3.9, we have
\[ \sum_{\tau \in T} \ell_\tau + \sum_{\tau' \in T^H} \ell_{\tau'} \geq \sum_{\tau \in T^C} \ell_\tau + 2 \sum_{\tau' \in T^H} \ell_{\tau'} \geq |S|. \]

Therefore, for every loop \( \alpha \) such that \( |\alpha|_y \leq |\alpha|_x \),
\[ |\alpha|_y = \frac{|\alpha|_x - \sum_i |u_i|_x}{|\mathbf{y}|} \leq |\alpha|_x \]
\[ = |\alpha|_x - \sum_i |u_i|_x \leq |\alpha|_x |\mathbf{y}| = |\alpha|_x \left( 1 - \sum_{\tau \in T^C} \ell_\tau - 2 \sum_{\tau' \in T^H} \ell_{\tau'} \right) \]
\[ = |\alpha|_x \left( \sum_{\tau \in T^C} \ell_\tau + 2 \sum_{\tau' \in T^H} \ell_{\tau'} \right) \leq \sum_i |u_i|_x \leq 2 \sum_{\tau \in T_\alpha} s_\tau. \]
But $|S| \leq \sum_{\tau \in T \cap \mathcal{L}} \ell_{\tau} + 2 \sum_{\tau' \in \mathcal{H}} \ell_{\tau'}$, therefore

$$|\alpha|_x \leq 2 \sum_{\tau \in \tau_x} \frac{s_{\tau}}{|S|}.$$ 

It follows that $|\alpha|_x < 2 \sum_{\tau \in \tau_x} \frac{s_{\tau}}{|S|}$ and $\dot{|\alpha|}_{t=0} \leq 0$. 

\[\square\]

5. Construction of the balanced folding path

In this section we prove Theorem 1.1 restated below:

**Theorem 5.1.** Given points $x, y \in \text{CV}_n$, there exists a geodesic $[x, y]_{bf}$ from $x$ to $y$ so that, for every loop $\alpha$, and every time $t$,

$$|\alpha|_t \leq \max(|\alpha|_x, |\alpha|_y).$$

**Proof.** Given $x$ and $y$, we consider the decorated graphs $x^d$ and $y^d$ and the decorated difference of markings map $\phi^d: x^d \to y^d$. Applying Theorem 4.3, we obtain a geodesic $\gamma: [0, t_1] \to \text{CV}_n$ starting from $x$ towards $y$. Now we consider the pair of points $x_{t_1}$ and $y$ and apply Theorem 4.3 again to continue the geodesic to an interval $[t_1, t_2]$. Continuing in this way, we either reach $y$ after finitely many steps or limit to a point $x' \in \text{CV}_n$. Note that every point $x_t$ along this path has the property that

$$d(x, x_t) + d(x_t, y) = d(x, y)$$

and the set of such points is a compact subset of $\text{CV}_n$. Hence, the same holds for $x'$. In particular $x'$ is a point in $\text{CV}_n$ and the geodesic does not exit $\text{CV}_n$.

Now, we can apply Theorem 4.3 to the pair $x'$ and $y$ and continue the geodesic even further, getting closer to $y$. This results in a geodesic connecting $x$ to $y$ because if the process stops at some points $x''$ before $y$, then $x''$ is still in the compact set defined by Equation (18) and we could apply Theorem 4.3 again to go further.

We re-parametrize this geodesic by arc-length to obtain $\gamma: [0, d] \to \text{CV}_n$, $d = d(x, y)$ (and use the parameter $s$ to emphasize this fact). Let $\Sigma \subset [0, d]$ be the closure of set of times, each of which is an endpoint of an interval coming from an application of Theorem 4.3. Note that, if $s \in \Sigma$, then an interval to the right of $s$ is not in $\Sigma$. Hence, $\Sigma$ is a well ordered set. That is, $[0, d]$ is a union of the interiors of countably many intervals coming from Theorem 4.3 and the countable, well ordered set $\Sigma$ which includes 0 and $d$.

For any loop $\alpha$, we prove the theorem using transfinite induction on $\Sigma$. That is, for every time $s \in \Sigma$, we show

$$|\alpha|_s \leq \max(|\alpha|_x, |\alpha|_y).$$

The theorem for other times then follows from Theorem 4.3.

Equation (19) is clearly true for $s = 0$. For any $s \in \Sigma$, assume Equation (19) holds for every $s' \in \Sigma$ with $s' < s$. We need to show that it also holds for $s$. There are two cases. If $s$ is an end point of an interval $[s', s]$ coming from Theorem 4.3, then, by Theorem 4.3

$$|\alpha|_s \leq \max(|\alpha|_{s'}, |\alpha|_y)$$

and by the assumption of induction

$$|\alpha|_{s'} \leq \max(|\alpha|_x, |\alpha|_y)$$

and the conclusion follows.
Otherwise, there is a sequence $s_i \in \Sigma$, with $s_i < s$, so that $s_i \to s$. By the assumption of induction, we have

$$|\alpha|_{s_i} \leq \max(|\alpha|_x, |\alpha|_y).$$

But the length of $\alpha$ is a continuous function over $CV_n$. Taking a limit, we obtain the theorem. \qed

6. NON-CONVEXITY

In this section we present examples that combine to prove Theorem 1.4. Some of the examples are done in low rank free groups, however, they can clearly be generalized to higher rank. First we show that there are points in Outer space such that no geodesic between them gives rise to convex length functions for all curves.

**Proposition 6.1.** There are points $x, y \in CV_n$ and a loop $\alpha$ so that along any geodesic connecting $x$ to $y$, the length of $\alpha$ is not a convex function of distance in $CV_n$.

**Proof.** We construct a simple example in $CV_2$. Let $a$ and $b$ be generators for $F_2$. Let $x \in CV_2$ be a graph that consists of two simple loops labeled $a$ and $b$, wedged at a vertex $v$ where each loop has length $\frac{1}{2}$ (a rose with two pedals). Let $\overline{y}$ be a quotient of $x$ obtained by identifying two subsegments of length $\frac{1}{8}$ in the loop labeled $a$. Then $\overline{y}$ is a rank 2 graph in the shape of a dumbbell with total length $\frac{7}{8}$, where the $b$-loop has length $\frac{1}{2}$ and the $a$-loop has a length $\frac{1}{4}$. Let $y$ be $\overline{y}$ rescaled to have length 1 (by a factor $\frac{8}{7}$). There is a rigid folding path from $x$ to $y$ because $\overline{y}$ was obtained from $x$ by identifying two sub-edges. Hence, this path $[x, y]_t$ is the unique geodesic connecting $x$ to $y$ (see Remark 2.5). Let $\alpha$ be the loop representing element $a \in F_2$. Then

$$|\alpha|_x = \frac{1}{2} \quad \text{and} \quad |\alpha|_y = \frac{8}{7} \cdot \frac{1}{4} = \frac{2}{7}.$$ 

Consider the length of $|\alpha|_t$ of $\alpha$ along this folding path. By Lemma 3.2, the derivative of the length of $\alpha$ at $x$ is:

$$|\alpha|_t \bigg|_{t=0} = |\alpha|_x - 2 = \frac{1}{2} - 2 < 0.$$ 

And since the length of $\alpha$ is decreasing the derivative stays negative. In fact, for $s > 0$

$$|\alpha|_t \bigg|_{t=s} = |\alpha|_s - 2$$

is a decreasing function as well. Therefore $|\alpha|_t$ is concave along this folding path. That is, there is no geodesic between $x$ and $y$ on which the length of $\alpha$ is a convex function of distance. \qed

We now examine if Theorem 1.1 holds for other geodesics connecting two points in $CV_n$. We start by looking at a general folding path and we show that a folding path with end points in a small ball can still go arbitrarily far away from the center of the ball.

**Proposition 6.2.** For any $R > 0$, there are points $x, y, z \in CV_3$ and there is a folding path $[y, z]_{ng}$ connecting $y$ to $z$ so that

$$y, z \in B_{out}(x, 2) \quad \text{and} \quad [y, z]_{ng} \not\subset B_{out}(x, R).$$
Proof. The example presented below is in CV$_3$. For higher rank Outer spaces, one can modify the example to roses with more loops such that the optimal map outside of the simple loops labeled $a,b,c$ is identity.

Consider constants $\epsilon > 0$, $\delta > 0$ and an integer $m > 0$ so that
\[ \epsilon \leq \delta \ll 1 \quad \text{and} \quad m\delta = 1 - 3\delta. \]
Assume $\mathbb{F}_3$ is generated by elements $a, b$ and $c$ and let $x, y, z, w$ be points in CV$_3$ which are wedges of simple loops with lengths and labels summed up in the table below.

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$y$</th>
<th>$w$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>label</td>
<td>$a$</td>
<td>$ab$</td>
<td>$ab$</td>
<td>$a$</td>
</tr>
<tr>
<td>length</td>
<td>$\frac{\epsilon}{2}$</td>
<td>$\delta + \delta^2$</td>
<td>$\frac{1+\delta}{3}$</td>
<td>$\frac{\delta}{2}$</td>
</tr>
<tr>
<td>label</td>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
</tr>
<tr>
<td>length</td>
<td>$\frac{1}{2}$</td>
<td>$\delta$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>label</td>
<td>$c$</td>
<td>$cb^m$</td>
<td>$c$</td>
<td>$c$</td>
</tr>
<tr>
<td>length</td>
<td>$\frac{1-\epsilon}{2}$</td>
<td>$1 - 2\delta - \delta^2$</td>
<td>$\frac{1-\delta}{3}$</td>
<td>$\frac{1-\delta}{2}$</td>
</tr>
</tbody>
</table>

Note that if, in $y$, we fold the edge labeled $cb^m$ $m$–times around $b$ (without rescaling), we obtain a graph $\overline{w}$ with labels $ab, b$ and $c$ and lengths $(\delta + \delta^2)$, $\delta$ and
\[(1 - 2\delta - \delta^2) - m\delta = (1 - 2\delta - \delta^2) - (1 - 3\delta) = \delta - \delta^2\]
which is a graph that is projectively equivalent to $w$ (by a factor of $\frac{1}{3m}$). Similarly, if in $w$, we fold the edge labeled $ab$ once around $b$ (without rescaling), we obtain a graph $\overline{z}$ with labels $a, b$ and $c$ and lengths $\frac{4}{3}, \frac{1}{3}$ and $\frac{1-\delta}{4}$ which a graph that is projectively equivalent to $z$ by a factor $\frac{3}{2}$. Therefore, there is a folding path from $y$ to $z$ that passes through $w$. But this is not a greedy folding path since the edge labeled $ab$ is not folded around $b$ in the segment $[y,w]$.

Let $\alpha$ be the loop representing the element $a \in \mathbb{F}_3$. Then
\[ d(x,y) = \frac{|\alpha|_y}{|\alpha|_x} = \log \frac{2(2\delta + \delta^2)}{\epsilon} \]
\[ d(x,z) = \frac{|\alpha|_z}{|\alpha|_x} = \log \frac{\delta}{\epsilon} \]
\[ d(x,w) = \frac{|\alpha|_w}{|\alpha|_x} = \log \frac{4(2 + \delta)}{\epsilon} \geq \log \frac{4}{3\epsilon}. \]
If, for example, we let $\epsilon = \delta$ then $y,z \in B_{\text{out}}(x,2)$, but $w$ can be made arbitrarily far away by making $\delta$ small. $\square$

Next we consider standard geodesic paths connecting two points which are the type of geodesics most often considered to connect two arbitrary points in CV$_n$ (not every pair of points can be connected via a folding path). The situation is improved somewhat but, by taking the ball large enough, one can construct examples where a standard geodesic that has its endpoints in a ball goes arbitrarily far from the ball.

Proposition 6.3. There exists a constant $c > 0$ such that, for every $R > 0$, there are points $x, y, z \in \text{CV}_n$ and a standard geodesic $[y,z]_{\text{std}}$ connecting $y$ to $z$ such that
\[ y,z \in B_{\text{out}}(x,R) \quad \text{and} \quad [y,z]_{\text{std}} \not\subseteq B_{\text{out}}(x,2R - c). \]
That is, the standard geodesic path can travel nearly twice as far from $x$ as $y$ and $z$ are from $x$.

Proof. As before, we construct the example in $CV_3$. Let $\psi \in \text{Out}(F_3)$ be defined as follows

$$
\begin{align*}
\psi(a) &= ab \\
\psi(b) &= a \\
\psi(c) &= c \\
\psi^{-1}(a) &= b \\
\psi^{-1}(b) &= b^{-1}a \\
\psi^{-1}(c) &= c
\end{align*}
$$

It is known (and easy to see) that, for any integer $m > 0$, the word length of $\psi^m(a)$ is $F_{m+3}$ and the word length of $\psi^m(b)$ is $F_{m+2}$, where $F_i$ is the $i$-th Fibonacci number. Similarly, the word length of $\psi^m(a)$ is $F_{m+2}$ and the word length of $\psi^m(b)$ is $F_{m+3}$. For a large integer $m > 0$, let

$$
\delta = \frac{1}{F_{m+2} + F_{m+3} + 1}
$$

and consider points $x, y, z, w \in CV_3$ which are wedges of simple loops and where the lengths and edge labels are summed up in the table.

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{label} & \text{length} & \text{label} & \text{length} & \text{label} & \text{length} \\
\hline
\text{Edge 1} & a & \delta & \psi^m(a) & \delta & \psi^m(a) & F_{m+3} \delta & a & \frac{1}{3} \\
\hline
\text{Edge 2} & b & \delta & \psi^m(b) & \delta & \psi^m(b) & F_{m+2} \delta & b & \frac{1}{3} \\
\hline
\text{Edge 3} & c & 1-2\delta & c & 1-2\delta & c & \delta & c & \frac{1}{3} \\
\hline
\end{array}
$$

If we let $\overline{\phi}: w \to \overline{\sigma}$ that maps the edge of $w$ labeled $\psi^m(a)$ to an edge path containing $F_{m+3}$ edges and maps the edge of $w$ labeled $\psi^m(b)$ to an edge path containing $F_{m+2}$ edges. The graph $z$ is obtained from $\overline{\sigma}$ by scaling by a factor $\frac{1}{3\delta}$. Hence, $w$ can be connected to $z$ using a folding path and tension graph of $\overline{\phi}: w \to z$ is all of $w$. The map from $y$ to $w$ is scaling two of the edges and contracting the third. Hence, the standard geodesic from $y$ to $z$ passes through $w$.

Next, we compute the distance from $x$ to these points. Let $\alpha$ be the loop representing the element $a \in F_3$ and $\beta$ be the loop representing $b \in F_3$. The loop $\alpha$ has a combinatorial length $F_{m+2}$ (which is the word length of $\phi^{-m}(a)$) in both $y$ and $w$ and $\beta$ has a combinatorial length $F_{m+3}$ (which is the word length of $\phi^{-m}(b)$) in both $y$ and $w$. In particular,

$$
|\beta|_w \geq F_{m+3} \cdot (F_{m+2} \delta)
$$
because the geodesic representative of \( \beta \) in \( w \) consists of \( F_{m+3} \) edges each having a length of at least \( F_{m+2} \delta \). We have

\[
d(x, y) = \log \frac{|\beta|_y}{|\beta|_x} = \log \frac{F_{m+3} \delta}{\delta} = \log F_{m+3}
\]

\[
d(x, z) = \log \frac{|\alpha|_z}{|\alpha|_x} = \log \frac{|\beta|_y}{|\beta|_x} = \log \frac{1/3}{\delta} = \log \frac{1}{3\delta}
\]

\[
d(x, w) \geq \log \frac{|\beta|_y}{|\beta|_x} > \log \frac{F_{m+3} \cdot (F_{m+2} \delta)}{\delta} = \log (F_{m+3} F_{m+2}).
\]

We now set \( R = \log F_{m+3} \) which is larger than \( \log \frac{1}{3\delta} \). Then, \( y, z \in B_{\text{out}}(x, R) \). There is a constant \( c \), (slightly larger than the golden ratio) so that

\[
\log(F_{m+2} F_{m+3}) \geq 2 \log(F_{m+3}) - c = 2R - c
\]

which implies \( w \notin B_{\text{out}}(x, 2R - c) \). This finishes the proof.

The most well-behaved geodesic often considered is a greedy folding path. In fact, as mentioned in the introduction, the lengths of curves are quasi-convex function of distance along a greedy folding path. However, we show that a greedy folding path with end point inside of a ball may exit the ball.

**Proposition 6.4.** For \( n \geq 4 \) and every \( R > 0 \), there are points \( x, y, z \in CV_{n+2} \) where \( y \) and \( z \) are connected by a greedy folding path \( [y, z]_{\text{gf}} \) such that

\[
y, z \in B_{\text{out}}(x, R) \quad \text{but} \quad [y, z]_{\text{gf}} \notin B_{\text{out}}(x, R).
\]

**Proof.** Let \( x, y, z, w \in CV_{n+2} \) be four graphs that are each a bouquet of \( n + 2 \) simple loops. Consider \( F_{n+2} \) as being generated by \( a, b \) and \( c_i \), for \( i = 1, \ldots, n \). The lengths and the labels of these graphs are described in the table below where \( \epsilon \) is a small positive number.

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>label</td>
<td>length</td>
<td>label</td>
<td>length</td>
<td>label</td>
</tr>
<tr>
<td>Edge 1</td>
<td>( a )</td>
<td>( \epsilon )</td>
<td>( ab^2 )</td>
<td>( 1/(2n + 4) )</td>
</tr>
<tr>
<td>Edge 2</td>
<td>( b )</td>
<td>( (1-\epsilon)/2 )</td>
<td>( b )</td>
<td>( 1/(2n + 4) )</td>
</tr>
<tr>
<td>Edge i</td>
<td>( c_i )</td>
<td>( (1-\epsilon)/2n )</td>
<td>( c_i b )</td>
<td>( 2/(2n + 4) )</td>
</tr>
</tbody>
</table>

Note that, the greedy folding path from \( y \) to \( z \) passes through \( w \). In fact, \( [y, z]_{\text{gf}} \) consists of two subsegments, in the first part \( c_i b \) and \( ab^2 \) wrap around \( b \) simultaneously to reach \( w \), and in the second part the edge labeled \( ab \) wraps around \( b \) to reach \( z \). The distance
\[ d(y, w) = \log \frac{2n+4}{n+3} \] and the associated stretch factors of edges are

\[
\begin{align*}
\lambda(ab^2) &= \frac{3/(n+3)}{3/(2n+4)} = \frac{2n+4}{n+3} \\
\lambda(b) &= \frac{1/(n+3)}{1/(2n+4)} = \frac{2n+4}{n+3} \\
\lambda(ci) &= \frac{2/(n+3)}{2/(2n+4)} = \frac{2n+4}{n+3}
\end{align*}
\]

are all the same. Likewise, the distance \( d(w, z) = \log \frac{n+3}{n+2} \) and associated stretch factors of edges are

\[
\begin{align*}
\lambda(ab) &= \frac{2/(n+2)}{2/(n+3)} = \frac{n+3}{n+2} \\
\lambda(b) &= \frac{1/(n+2)}{1/(n+3)} = \frac{n+3}{n+2} \\
\lambda(ci) &= \frac{1/(n+2)}{1/(n+3)} = \frac{n+3}{n+2}
\end{align*}
\]

which again are the same for every edge.

Next, we measure distances from the center of the ball \( x \). Let \( \alpha \) be the loop associated to the element \( a \in \mathbb{F}_3 \). For \( \epsilon \) small enough, all three distances are realized by the stretch factor associated to \( \alpha \). That is,

\[
\begin{align*}
d(x, y) &= \log \frac{|\alpha|_y}{|\alpha|_x} = \log \frac{5/2n+4}{\epsilon} = \log \frac{5}{2n\epsilon + 4\epsilon} \\
&= \log \frac{5}{2n\epsilon + 4\epsilon} \\
d(x, z) &= \log \frac{|\alpha|_z}{|\alpha|_x} = \log \frac{1/n+2}{\epsilon} = \log \frac{1}{n\epsilon + 2\epsilon} \\
&= \log \frac{1}{n\epsilon + 2\epsilon} \\
d(x, w) &= \log \frac{|\alpha|_w}{|\alpha|_x} = \log \frac{3/n+3}{\epsilon} = \log \frac{3}{n\epsilon + 3\epsilon}
\end{align*}
\]

But, for all \( n \geq 4 \), we have

\[
\frac{3}{n\epsilon + 3\epsilon} > \max \left( \frac{1}{n\epsilon + 2\epsilon}, \frac{5}{2n\epsilon + 4\epsilon} \right)
\]

Thus, if we set \( R = \frac{1}{n\epsilon + 2\epsilon} \), we have an example of a greedy folding path with end point in \( B_{\text{out}}(x, R) \) that travels outside the ball. \( \square \)

7. **In-coming balls**

In contrast with out-going balls, we prove that in-coming balls are not weakly quasi-convex:

**Theorem 7.1.** For any constant \( R > 0 \), there are points \( x, y, z \in \mathbb{C}V_n \) such that, \( y, z \in B_{\text{in}}(x, 2) \) but, for any geodesic \( [y, z] \) connecting \( y \) to \( z \),

\[
[y, z] \not\subset B_{\text{in}}(x, R).
\]
Proof. We show that there exists a family of balls and pairs of points \( y_m \) and \( z_m \) in these balls such that the geodesic connecting \( y_m \) to \( z_m \) is unique and it travels arbitrarily far away from the center of the corresponding ball. Since the geodesic is unique, this can be restated as: every geodesic connecting \( y_m \) to \( z_m \) travels arbitrarily far from the center of the balls.

Fix an integer \( m > 0 \) and, as usual, let \( a, b \) and \( c \) be generators for \( \mathbb{F}_3 \). Examples in higher dimension can be adapted from this example by adding loops on which the map is identity along the path. Let \( x = x_m, y = y_m \) and \( z = z_m \) be roses with labels and lengths specified in the table.

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>( y )</th>
<th>( w )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>label</td>
<td>length</td>
<td>label</td>
<td>length</td>
</tr>
<tr>
<td>Edge 1</td>
<td>( a )</td>
<td>( \frac{1}{2} - \frac{1}{m} )</td>
<td>( ab^m )</td>
<td>( \frac{m+1}{2m+4} )</td>
</tr>
<tr>
<td>Edge 2</td>
<td>( b )</td>
<td>( \frac{1}{m} )</td>
<td>( b )</td>
<td>( \frac{1}{2m+4} )</td>
</tr>
<tr>
<td>Edge 3</td>
<td>( c )</td>
<td>( \frac{1}{2} )</td>
<td>( cb^m a )</td>
<td>( \frac{m+2}{2m+4} )</td>
</tr>
</tbody>
</table>

Note that, \( w \) is obtained from \( z \) by wrapping the edge labeled \( ab^m \) around the edge labeled \( b m \)–times and then scaling by a factor \( \frac{2m+4}{m+4} \). Throughout this portion, the illegal turn \( \langle ab^m, b \rangle \) is the only illegal turn. Similarly, \( z \) is obtained from \( w \) wrapping the edge labeled \( ab^m \) around the edge labeled \( a \) once, then around the edge labeled \( b \) \( m \)–times and finally scaling by \( \frac{m+4}{3} \). Again, during each sub-segment, there is exactly one non-yo-yo illegal turn; first \( \langle cb^m a, a \rangle \) and next \( \langle cb^m, b \rangle \). The illegal turn is never a yo-yo since there is no cut edge in the graphs along the paths. The loop labeled \( b \) in \( y \) is legal throughout and hence is maximally stretched from \( y \) to \( w \) and from \( w \) to \( z \). Therefore \( w \) lies on a rigid folding path from \( y \) to \( z \). By Theorem 1.5 the folding path is the unique (up to re-parametrization) geodesic from \( y \) to \( z \).

We now compute distance to the center of the ball. For large enough \( m \), we have

\[
d(y, x) = \log \left| \frac{|cb^m a|_x}{|cb^m a|_y} \right| = \log \frac{\frac{1}{2} + 1 + \frac{1}{2} - \frac{1}{m}}{\frac{m+2}{2m+4}} = \log \frac{4m^2 + 6m - 4}{m^2 + 2} < \log 5 < 2
\]

\[
d(w, x) = \log \left| \frac{|a|_x}{|a|_w} \right| = \log \frac{\frac{1}{2} - \frac{1}{m}}{\frac{1}{m+4}} = \log \frac{m^2 + 2m - 8}{2m} \geq \log \frac{m}{2}
\]

\[
d(z, x) = \log \left| \frac{|c|_x}{|c|_z} \right| = \log \frac{\frac{1}{3}}{\frac{1}{3}} = \log \frac{3}{2} < 2.
\]

That is, \( y, z \in B_{in}(x, 2) \) and the distance \( d(w, x) \) can be made to be arbitrarily large. \( \square \)

References


CONVEXITY OF BALLS IN OUTER SPACE


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