Simple closed curves are typically non-separating on high genus surfaces

joint work with E. Goujard, P. Zograf, A. Zorich

Vincent Delecroix

CNRS - Université de Bordeaux
Multicurves and simple closed curves

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\[
\eta_{nsep,(2,0)} = \eta_{nsep,2}
\]

\[
\eta_{sep,(1,0),(1,0)} = \eta_{sep,1,1}
\]
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What is the type of the following curve?
Asymptotic counting with respect to the type

Theorem (Mirzakhani ’08)

For any type $\eta$ of multicurve, there exists a positive rational constant $c(\eta)$ such that for any metric on $S$, as $L \to \infty$ we have

$$\# \{ \text{multicurves of type } \eta \text{ and length } \leq L \} \sim B(\text{metric}) \cdot \frac{c(\eta)}{b_{g,n}} \cdot L^{6g-6},$$

where $B(\text{metric})$ is (implicitly) defined as

$$\# \{ \text{multicurves of length } \leq L \} \sim B(\text{metric}) \cdot L^{6g-6},$$

and we have $\sum_{\eta} c(\eta) = \int_{X} B(X) d\mu_{WP}(X) = b_{g,n}$. 
The flat torus case \((g, n) = (1, 1)\)

\[ T = \mathbb{R}^2 / \mathbb{Z}^2 \]
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With respect to the \(L^2\)-norm

\[ \#\{\text{multicurves of length } \leq L\} \sim \frac{\pi}{2} \cdot L^2 \]

and

\[ c(k) = \frac{1}{4k^2} \quad b_{1,1} = \frac{\pi^2}{24} \]
Counting curves

Separating vs non-separating in high genus

Theorem (Mirzakhani ’08)

\[ \#\{ \text{multicurves of type } \eta \text{ and length } \leq L \} \sim B(\text{metric}) \cdot \frac{c(\eta)}{b_g} \cdot L^{6g-6}. \]

Theorem (DGZZ’19)

For \( n = 0 \) (no puncture), as \( g \to \infty \) we have

\[ \sum_{g_1 + g_2 = g} \frac{c(\eta_{\text{sep},g_1,g_2})}{c(\eta_{\text{nsep},g})} \sim \sqrt{\frac{2}{3\pi g}} \cdot \frac{1}{4^g}. \]

<table>
<thead>
<tr>
<th>( g )</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</thead>
<tbody>
<tr>
<td>( \frac{\text{sep}}{\text{non-sep}} )</td>
<td>( \frac{1}{48} )</td>
<td>( \frac{5}{1776} )</td>
<td>( \frac{605}{790992} )</td>
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<tr>
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<td>( \simeq 0.021 )</td>
<td>( \simeq 0.0028 )</td>
<td>( \simeq 0.00076 )</td>
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Counting ribbon graphs (aka combinatorial maps)

$\mathcal{R}_{g,n}(b_1, \ldots, b_n)$: ribbon graphs of genus $g$, no vertex of degree 1, $n$ faces labeled from 1 to $n$ and perimeters $b_1, b_2, \ldots, b_n$. 
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\tilde{N}_{g,n}(b_1, b_2, \ldots, b_n) := \lim_{L \to \infty} \frac{1}{L^{6g-6+2n}} \# \mathcal{R}_{g,n}(Lb_1, Lb_2, \ldots, Lb_n).
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Theorem (Kontsevich’92, Norbury’11)

For $(b_1, \ldots, b_n)$ such that $b_1 + b_2 + \ldots + b_n \equiv 0 \mod 2$, the numbers $\tilde{N}_{g,n}(b_1, b_2, \ldots, b_n)$ coincide with a homogeneous symmetric polynomial $N_{g,n}(b_1, b_2, \ldots, b_n)$ in the $b_i^2$ of degree $6g - 6 + 2n$ with rational coefficients.
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\(C_{g,n}\): integer compositions of \(3g - 3 + n\) into \(n\) non-negative parts

For \(d = (d_1, d_2, \ldots, d_n) \in C_{g,n}\) we define the correlator \(\langle d \rangle_{g,n}\) as

\[
N_{g,n}(b_1, b_2, \ldots, b_n) =: \frac{1}{2^{5g-6+2n}} \sum_{d \in C_{g,n}} \frac{\langle d \rangle_{g,n}}{d_1!d_2! \cdots d_n!} b_1^{2d_1} b_2^{2d_2} \cdots b_n^{2d_n}.
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\]

Algebraic geometry note: the polynomials \(N_{g,n}\) are part of Kontsevich’s proof of Witten conjecture. We have

\[
\langle d \rangle_{g,n} = \int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \psi_2^{d_2} \cdots \psi_n^{d_n}
\]
Explicit formula in the unicellular case \((n = 1)\)

We have

\[
\langle 3g - 2 \rangle_{g,1} = \frac{1}{24g \cdot g!}.
\]

In other words

\[
N_{g,1}(b_1) = \frac{1}{2^{5g-6+2n}} \frac{1}{(3g - 2)!} \frac{1}{24g \cdot g!} b_1^{6g-4}.
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Note: equivalent to the Lehman-Walsh’72, Harer-Zagier’86 formulas for the exact counting of unicellular maps.
Asymptotic formula in the bicellular case \((n = 2)\)

Let us introduce

\[
h(d) = \frac{1}{24^g \cdot g!} \cdot \frac{(6g - 1)!!}{\prod_{i=1}^{n} (2d_i + 1)!!}
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$$h(d) = \frac{1}{24^g \cdot g!} \cdot \frac{(6g - 1)!!}{\prod_{i=1}^n (2d_i + 1)!!}$$

Theorem (DGZZ’19)

For any $d \in C_{g,2}$ we have

$$1 - \frac{2}{6g - 1} \leq \frac{\langle d \rangle_{g,2}}{h(d)} \leq 1.$$
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*For any \(d \in C_{g,2}\) we have \(1 - \frac{2}{6g - 1} \leq \langle d \rangle_{g,2} \leq \frac{h(d)}{h(d)} \leq 1.\)*

Note: generalized in Aggarwal’20 for correlators with \(n \geq 3\).
From simple multicurves to stable graphs

stable graph:

Decorated graph dual to a multicurve and forgetting the embedding in the surface
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\[ \{ \text{topological types of simple multicurves} \} \sim \{ \text{stable graphs} \} \]
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\[ \{ \text{topological types of simple multicurves} \} \approx \{ \text{stable graphs} \} \]
\[ \{ \text{topological types of multicurves} \} \approx \{ \text{weighted stable graphs} \} \]
The coefficient $c(\eta)$ and Kontsevich polynomials $N_{g,n}$

For each stable graph $\Gamma$ (dual to a multicurve $\eta$) we associate a polynomial with variables $(b_e)_{e\in E(\Gamma)}$ and define

$$P_\Gamma(b) = A_{g,n} \frac{1}{2|V(\Gamma)|-1} \cdot \frac{1}{|\text{Aut}(\Gamma)|} \cdot \prod_{e\in E(\Gamma)} b_e \cdot \prod_{v\in V(\Gamma)} N_{g,v,n_v}(b_v).$$

where $A_{g,n} = \frac{2^{2g-3+n}}{(6g-6+2n) \cdot (6g-7+2n)!}$. 

Theorem (Mirzakhani '08, DGZZ '19)

For $\eta$ is a simple multicurve and associated stable graph $\Gamma$ we have $c(\eta) = \sum P_\Gamma(b)$ where $Y: k \prod_{i=1}^{\prod m_i} \mapsto k \prod_{i=1}^{\prod m_i!}$.
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$$c(\eta) = \mathcal{Y}(P_\Gamma) \quad \text{where} \quad \mathcal{Y} : \prod_{i=1}^k b_i^{m_i} \mapsto \prod_{i=1}^k m_i!.$$
From correlators to $c(\eta)$

$c(\eta)$ for simple closed curves

**Theorem (Mirzakhani ’08, DGZZ ’19)**

$$c(\eta) = \mathcal{V}(P_\Gamma) \quad \text{where} \quad \mathcal{V} : \prod_{i=1}^{k} b_i^{m_i} \mapsto \prod_{i=1}^{k} m_i!.$$

Non-separating curve

$$c(\eta_{nsep,g}) = \frac{1}{A_{g,n}} \frac{1}{2} \mathcal{V} \left( b_{N_{g-1,2}}(b,b) \right)$$

Separating curve

$$c(\eta_{sep,g_1,g_2}) = \frac{1}{A_{g,n} \text{Aut}} \mathcal{V} \left( b_{N_{g_1,1}}(b) N_{g_2,1}(b) \right)$$
Mirzakhani’s curve counting theorem

\{\text{multicurves on } S\} = \{\text{integral points in } \mathcal{ML}(S)\}
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It follows that as \( L \to \infty \)

\{\text{all multicurves of length } \leq L\} \sim B(\text{metric}) \cdot L^{6g-6}. 
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The fact that for each type of multicurves \( \eta \) its proportion \( c(\eta)/b_g \) exists and is positive relies on the ergodic action of \( \text{MCG}(S) \) on \( \mathcal{ML}(S) \) (Masur’85).
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The explicit formula for \( c(\eta) \) can be proven via Weil-Petersson volumes (Mirzakhani’08) or square-tiled surface counting (DGZZ’19).
Witten conjecture (Kontsevich theorem) states that the correlators $\langle d \rangle_{g,n}$ satisfy recurrence relations.
Asymptotics of 2-correlators

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One can then prove by induction

\[
1 - \frac{2}{6g - 1} \leq \frac{\langle d \rangle_{g,2}}{h(d)} \leq 1.
\]
Recall that 1-correlators and 2-correlators are respectively the coefficients of $N_{g,1}(b_1)$ and $N_{g,2}(b_1, b_2)$. 
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From the formulas $c(\eta_{nsep,g}) = \frac{1}{A_{g,n}} \frac{1}{2} \mathcal{Y}(bN_{g-1,2}(b, b))$ and $c(\eta_{sep,g_1,g_2}) = \frac{1}{A_{g,n} \text{Aut}} \frac{1}{\mathcal{Y}(bN_{g_1,1}(b)N_{g_2,1}(b))}$, we deduce asymptotics for $c(\eta_{nsep,g})$ and $c(\eta_{sep,g_1,g_2})$. 
Further remarks

- (weak) generalization to multicurves with more components using Aggarwal’20 (DGZZ’20)
- for generic hyperbolic metric, the separating systole has order $2 \log(g)$ (Mirzakhani’13, Nie-Wu-Xue’20)