1. **Extremal length**

   \( X \) — Riemann surface, perhaps with boundary

   \[ \rho(\lambda v) = |\lambda|\rho(v) \]

   for every \( v \in TX \) and \( \lambda \in \mathbb{C} \).

   \( \Gamma \) — a set of paths or closed curves in \( X \)

   If \( \rho \) is a conformal metric and \( \gamma \in \Gamma \), then we define the **length** of \( \gamma \) with respect to \( \rho \) as

   \[ \ell(\gamma, \rho) = \int_{\gamma} \rho = \int_{\gamma} \rho(t(x)) \]

   if the integral makes sense, \( \ell(\gamma, \rho) = \infty \) otherwise, and

   \[ \ell(\Gamma, \rho) = \inf \{ \ell(\gamma, \rho) : \gamma \in \Gamma \} \]

   is the **length of the path family**.

   \[ \Gamma = \text{all paths between } p \text{ and } q \]

   \[ \ell(\Gamma, \rho) \]

   \[ d(p, q) \text{ in metric } \rho. \]

   **Def:** The **extremal length** of \( \Gamma \) is defined as

   \[ \text{EL}(\Gamma, X) = \sup_{\rho} \frac{\ell(\Gamma, \rho)^2}{\text{area}(\rho)} \]

   which gives

   \[ \text{EL}(\Gamma, X) = \frac{\ell(\Gamma, \rho)^2}{\text{area}(\rho)} = \frac{a^2}{ab} = \frac{a}{b} \]

   \[ \text{aspect ratio of the rectangle}. \]

**Key example:** \( X = \text{rectangle}, \quad \Gamma = \text{all paths joining the two vertical sides} \)

Then the metric \( \rho \) realizing the supremum in the definition of \( \text{EL}(\Gamma, X) \) is simply the Euclidean metric, which gives

\[ \text{EL}(\Gamma, X) = \frac{\ell(\Gamma, \rho)^2}{\text{area}(\rho)} = \frac{a^2}{ab} = \frac{a}{b} \]

We’ll prove this as a special case of Beurling’s criterion for the extremality of the metric \( \rho \).
2. **Beurling’s criterion**

*Def:* A metric \( \rho_0 \) on \( X \) is **evenly covered by shortest paths** of \( \Gamma \) if

- There is a non-empty subset \( \Gamma_0 \subset \Gamma \) of shortest paths, i.e., such that
  \[
  \ell(\gamma, \rho_0) = \ell(\Gamma, \rho_0)
  \]
  for every \( \gamma \in \Gamma_0 \).
- There is a measure \( \mu \) on \( \Gamma_0 \) such that
  \[
  \rho_0^2 = (\rho_0 \text{ along } \gamma) \times d\mu
  \]
  i.e., to integrate against \( \rho_0^2 \) over \( X \), we can apply an iterated integral (Fubini).

**Example:** In the rectangle example, the Euclidean metric \( \rho_0 \) is evenly covered by the horizontal paths.

\[
L \times d\mu = dy
\]

**Theorem (Beurling’s criterion):** Let \( \rho_0 \) be a conformal on \( X \) that is evenly covered by shortest paths of \( \Gamma \). Then \( \rho_0 \) realizes the supremum in the definition of \( EL(\Gamma, X) \), that is,

\[
EL(\Gamma, X) = \frac{\ell(\Gamma, \rho_0)^2}{area(\rho_0)}
\]

To prove the theorem, first note that

\[
area(\rho_0) = \int_X \rho_0^2 = \int_{\Gamma_0} \left( \int_{\gamma} \rho_0^2 \right) d\mu(\gamma)
\]

\[
= \int_{\Gamma_0} \ell(\gamma, \rho_0) d\mu(\gamma)
\]

\[
= \int_{\Gamma_0} \ell(\Gamma, \rho_0) d\mu
\]

\[
= \ell(\Gamma, \rho_0) \cdot \mu(\Gamma_0)
\]

"area = length \cdot height" if

Let \( \rho \) be any competing metric with \( 0 < area(\rho) < \infty \). We need to show

\[
\frac{\ell(\Gamma, \rho)^2}{area(\rho)} < \frac{\ell(\Gamma, \rho_0)^2}{\infty} = 0.
\]
\[
\frac{l(p, p')}{\text{area}(p)} \leq \frac{\mu(p, p')}{\text{area}(p)}.
\]

We have

\[
l(p, p) \leq l(p_0, p) \leq \frac{1}{8} |p| \quad \text{for every } x \in P_0.
\]

Integrate over \( P_0 \) ⇒

\[
l(p, p) \cdot \mu(p_0) \leq \int_{P_0} \left( \frac{1}{8} |p| \right) d\mu(x)
\]

\[
= \int_{P_0} \int_{x} \frac{1}{8} \frac{p}{p_0} \cdot \frac{p_0^2}{p} \, d\mu(x)
\]

\[
= \int_{x} \frac{1}{8} \frac{p}{p_0} \cdot \frac{p_0^2}{p} = \frac{1}{8} \int_{x} |p| \cdot p_0
\]

\[
\leq \sqrt{\int_{x} |p|^2} \cdot \sqrt{\int_{x} p_0^2}
\]

\[
= \sqrt{\text{area}(p)} \cdot \sqrt{\text{area}(p_0)}
\]

\[
\Rightarrow \quad \frac{l(p, p')^2}{\text{area}(p)} \leq \frac{\text{area}(p_0)}{\mu(p_0)^2}
\]

\[
= \frac{\text{length} \cdot \text{height}}{\text{height}^2} = \frac{\text{length}}{\text{height}} = \frac{\text{length}^2}{\text{length} \cdot \text{height}} = \frac{l(p, p')^2}{\text{area}(p)}
\]
More examples:  

0) The rectangle example.  

1) $X$ any Riemann surface, $\Gamma = [\alpha]$ where $\alpha \subset X$ is an essential simple closed curve.  

Then a theorem of Jenkins/Strebel says that there is a quadratic differential $q$ that makes $X$ isometric to a Euclidean cylinder modulo some gluings along the boundary.

The induced conformal metric is evenly covered by shortest curves in $[\alpha]$ so that  

$$EL([\alpha], X) = \frac{\text{circumference}^2}{\text{area}} = \frac{\text{circumference}}{\text{height}} = \frac{1}{\text{modulus}}$$  

by Beurling's criterion.
A special case of this is when \( X \) is a flat torus. Then \( EL([\alpha], X) = \text{length}^2 / \text{area} \) for the flat metric.

2) \( X = \mathbb{RP}^2 = S^2 / \text{antipodal map} \), \( \Gamma = \text{all non-contractible curves in} \ X \).

Then the spherical metric is evenly covered by quotients of great circles, which are shortest in \( \Gamma \).

By Beurling’s criterion,
\[
EL(\Gamma, X) = \frac{\pi^2}{2\pi} = \frac{\pi}{2}.
\]

3. **Systolic inequalities**

\((M, g) \) — compact Riemannian \( n \)-manifold

Systole: \( \text{sys}(M, g) = \text{length of shortest non-contractible curve in} \ (M, g) \)

Systolic ratio: \( \text{SR}(M, g) = \frac{\text{sys}(M, g)^n}{\text{vol}(M, g)} \) (invariant under scaling)

**Question** (Berger, Gromov): Given a smooth manifold \( M \), which Riemannian metrics on \( M \) maximize the systolic ratio?

**Answer known only if \( M \in \{ 2 \text{ – torus, projective plane, Klein bottle} \} \)**

Relationship with extremal length:

We can stratify the space of Riemannian metrics on \( M \) by conformal classes:

\( \sup\{\text{SR}(M, g) : g \text{ is a Riemannian metric on} \ M\} = \sup\{\sup\{\text{SR}(M, g) : g \in c\} : c \text{ is a conformal class of metrics on} \ M\} \)

so we can try to solve the problem for each conformal class, then maximize over moduli space.

For a fixed conformal structure \( X \) (or conformal class \( c \)) on \( M \) we have

\[
\sup\{\text{SR}(M, g) : g \in c\} = \sup_{g \in c} \text{sys}(M, g)^n = \sup_{\rho} \left( \frac{\text{vol}(\Gamma_{\text{ess}})}{\text{vol}(\rho)} \right)^n = \mathbb{L}\left( \Gamma_{\text{ess}}, X \right).
\]

where \( \Gamma_{\text{ess}} = \text{all non-contractible curves in} \ X \).
\[ l(\text{Res}, p) = \inf \{ \text{length}(\gamma, p) : \gamma \in \text{Res} \} \]

\[ = \inf \text{ length of non-cont. curves } \]

\[ = \text{sys}(X, p). \]

\[ \text{EL sys} = \inf_{\alpha} \text{EL}(\alpha, X) \]

\[ = \inf_{\alpha} \sup_{p} \frac{\ell^2}{\text{area}} \]

\[ \max \text{ SR} = \sup_{p} \inf_{\alpha} \frac{\ell^2}{\text{area}} \]