Actions of Homeo and Diff groups on manifolds

Lei Chen @Caltech

Joint with Kathryn Mann
Ghys in 1990s has studied “extension problem” and proved the following:

\[
\text{Diff}(D^3) \xrightarrow{\pi_1} \text{Diff}(S^1) \text{ has no sections.}
\]

A section of \( \pi_1 \) can be considered as extensions of elements in \( \text{Diff}(S^1) \) that preserves group structure.

His argument:

If \( \exists \) a section \( \Rightarrow \)

1. \( \exists \) a global fixed point (Smith theory)
2. Take a derivative at 0
\[
\text{Diff}(S^1) \xrightarrow{D} \text{GL}_n \mathbb{R}
\]
not hard to show \( D \) is trivial
3. Thurston Stability (Reeb St)
   trivial derivative \( \Rightarrow \) torsion free

\( \Rightarrow \) contradiction

“A flat \( S^1 \)-bundle” is not always the boundary of a flat \( D^2 \)-bundle.
Topologically, the extension exists.

\[ \text{Homeo}(U^3) \rightarrow \text{Homeo}(S') \quad \text{has a section.} \]

"Coring-off"

\[ f \in \text{Homeo}(S') \]

\[ p(f) (r, \theta) = (r, \theta_0) \]

Naturally, we ask a question

Is this "coring-off" the only extension?
I asked Katie about it and she gave me a paper of Militon.

Militon’s example

"Militon’s action"

\[ \text{Homeo}(s') \rightarrow \text{Homeo}(A^2) \]

\[ A^2 = S' \times [0,1] \]

\[ p(f)(\Theta, r) = (f(\Theta), \tilde{f}(\Theta + r) - \tilde{f}(\Theta)) \]

\( \tilde{f}, \tilde{\Theta} \) are lifts to universal cover \( IR \rightarrow s' \)

Now the \( t \)-coordinate also change!
Militon’s Theorem

fully classified actions of $\text{Homeo}(S') \simeq \mathbb{A}^2$

$K \subseteq [0,1]$ closed subset

$\text{Homeo}(S') \simeq S^1 \times K$ as coning

complement of $S^1 \times K \subseteq S^1 \times [0,1]$

by Militon action

This construction apparently doesn’t extend to higher dimension.

Bigger goal

Can we classify all $\text{Homeo}(M)$, $\text{Diff}(M)$ action on $N$?

for $M$, $N$ two manifolds.

In this talk, $\text{Homeo}(M)$, $\text{Diff}(M)$ denote id

component. We will not talk about mapping

class group.
Basic Properties of Homeo(M) and Diff(M)

1. Locally generated and locally contractible
   \[ \text{Diff} \rightarrow \text{Banach manifold structure} \]
   \[ \text{Homeo} \rightarrow \text{Kirby–Edward's result} \]

2. Fragmentation property (Kirby—Edward)
   \[ \forall g \in G \quad \text{a cover } \{ \text{U}_a \} \text{ of } M \]
   \[ \Rightarrow g = \prod g_i \text{ s.t. } \text{supp}(g_i) \subseteq \text{U}_a \text{ some } \]
   \[ \text{Diff} \rightarrow \text{partition of unity} \]
   \[ \text{Homeo} \rightarrow \text{Kirby–Edward} \]

3. Simplicity (Thurston, Epstein, Herman, Yoccoz, Mather)
   \[ v + \text{dim}(M) + 1 \Rightarrow \text{Diff}^r(M) \text{ is simple} \]
   \[ \gamma \text{ is regularity.} \]
4. Automatic continuity (Rosendal-Solecki, Rosendal, Mann, Hurtado)

\((RS, R, M) \xrightarrow{\text{Homeo}(M)} G \xrightarrow{p} G \text{ separable} \Rightarrow p \text{ is continuous}\)

\((H) \xrightarrow{\text{Diff}^\infty(M)} \xrightarrow{\text{Diff}^\infty(N)} \text{ always continuous}\)

5. Uniqueness (Whittaker, Filipkewicz)

Any isomorphism of \(\text{Diff}^\infty(M)\) and \(\text{Diff}^\infty(N)\)

is induced by a diffeomorphism of \(f\)

\(M \xrightarrow{f} N\).

1-1 correspondence

\((M, r) \rightarrow \text{Diff}^r(M)\)

It's possible to study \((M, r)\) property by studying its transformation \(g \circ \text{Diff}^r(M)\)
Example Zoo

Homeo(M)

1. Coning-off
   $\text{Homeo}(S^n) \vee D^{n+1}$

2. Suspension
   $\text{Homeo}(S^n) \vee S^{n+1}$

3. Double suspension
   $\text{Homeo}(H^n) \vee S^{n+2}$

   Cannon's thin: $\Sigma^2H^n$ for $H^n$ homology sphere is $= S^{n+2}$.

4. $M, M \times M, M \times M \times K$

   $M \times M \times K / \sim$

   $(x, x, t) \sim (x, x, t')$

Diff(M) $\cong$ N

$M, TM, PM$

$FM, G_{K}(M)$

$\text{Jet}^r(M)$

Observation:

- $\dim N \geq \dim M$

  (conj of Ghys)

Hurtado 2015

$\text{Diff}(M) \xrightarrow{\text{p}} \text{Diff}(N)$

$p$ non trivial

$\Rightarrow \dim N \geq \dim M$
Orbit Classification Theorem (C—Mann)

\( X \) finite dim CW complex and \( M \) closed manifold

Any continuous action of \( \text{Homeo}(M) \times X \), every orbit is homeomorphic to \( \text{Conf}_n(M) \)

\( \text{Conf}_n(M) \) unoriented configuration space

\[
\left\{ \{x_i, \ldots, x_n\} \mid x_i \neq x_j, \ x_i \in M \right\} \subseteq \text{Sym}^n(M)
\]

For \( \text{Diff}^r(M) \), every orbit is \( \text{Jet}^r_n(M) / A \)

\( A \leq \text{Jet}^r \) a lie subgroup.

You may wonder which cover can appear as orbits:

Eg: \( \text{Homeo}(S^1) \) has no action on \( IR \)

but \( \text{Homeo}(S_g) \) acts on \( \tilde{S_g} \) universal cover.

\( g \geq 2 \)

This question is closely related to \( \Pi_1(\text{Homeo}(M)) \)
Applications

1. Ghys’ dimension growth conjecture
   
   \[ \text{pf: OCT + invariance of domain} \]
   
   \[ \text{no injective cont map from } \mathbb{R}^m \to \mathbb{R}^n \text{ if } m < n. \]

2. When \( M, N \) have the same dimension.

   \[ \Rightarrow \exists \text{ an embedding of } \mathbb{R}^n \hookrightarrow N. \]

3. When \( \dim(M) + 1 = \dim(N) \) and “extension problem”

   ① \( \exists \) extension for Diff case

   ② When \( \pi_1(M) = 1 \), \( \exists \) extension for Homeo case

   \[ \text{if only if } M \cong S^n. \]

Open problem: Does \( \text{Homeo}(H^g) \to \text{Homeo}(S^g) \) have a section?
Outline the proof of Orbit Classification Theorem

\[ G = \text{Homeo}(M) \]

**Def:** \( A \leq G \) a closed subgp

we say \( A \leq G \) has finite codim if

\[ G/A \text{ can be embedded in finite-dim CW complex.} \]

Example:

\[ G \succeq N \leq n \in N \]

\( \text{Stab}(n) \leq G \) satisfies that

\[ G/\text{Stab}(n) \cong O(n) \leq N \]

\( \Rightarrow \text{Stab}(n) \) is a finite codim subgp.
Thm. (C.-Mann) OCT2

If $A \leq G$ is finite $\Rightarrow$

$\exists x_1, \ldots, x_n \in M \text{ s.t. }$

$\text{Stab}(x_1, \ldots, x_n)_0 \leq A \leq \text{Stab}(x_1, \ldots, x_n)$

Identity component

in general $\text{Stab}(x_1, \ldots, x_n)$ is not connected.

Now $\text{OCT}_2 \implies \text{OCT}_1$

$G/\text{Stab}(x_1, \ldots, x_n) \cong \text{Conf}_n(M)$

$G/\text{Stab}(x_1, \ldots, x_n) \quad \text{cover}$

$G/A$ is an intermediate cover.
Recall Milton’s example

\[ \text{Homeo}(s') \sqcup A^2 = s' \times [0,1] \]

\[ \text{PConf}_{2}(s') = s' \times (0,1) \]

Milton example is the compactified action

\[ \text{Homeo}(s') \sqcup \text{PConf}_{2}(s') \cup s' \times \{0,1\} \]
How to prove OCT2 \( G = \text{Homeo}(M) \).

**Proof:**

1. Let \( B \in M \) be a ball

   \[
   GB = \{ f \in G \mid \text{supp}(f) \subseteq B \}
   \]

Claim: If \( A \subseteq G \) finite codim \( \Rightarrow \) \( A \supseteq GB \) for some \( B \).

**Proof:** If not \( B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots \)

\[ A \not\supseteq GB_i \Rightarrow GB_i / ANGB_i \subseteq G/A \]

contributes a copy of \( R \) in \( G/A \).

Some argument with simplicity of \( GB_i \)

\[ GB_i \times \cdots \times GB_n / AN(GB_i \times \cdots \times GB_n) \] has a copy of \( IR^n \)

\[ \Rightarrow \text{blow up the dim of } G/A. \] \( \Box \)
② Consider the action of $A \vartriangleleft M$

$$\text{Orbit}(A) = A x \quad \forall x \in M$$

Claim: $A x$ is either finite or infinite $\forall x \in M$.

(finite complement)

$\text{Pf:}$ If not, $\exists$ two infinite orbits

$$a_1, a_2, a_3, \ldots$$

$$\Rightarrow \mathbb{R}^\infty \rightarrow \mathbb{G}/A \quad \text{is almost injective blowup \lim}$$

③ $\text{① + ② } \Rightarrow \exists \text{ finite set } S \subseteq M$

$s.t. \quad f \in \text{Homeo}(M-S)_c \subseteq \tilde{A}$

(fragmentation) $A \cong G \backslash B \subset M-S$

$$\{B_i\} \text{ cover } M-S$$

$$\Rightarrow S = \text{Stab}(S) = \text{Homeo}(m-S)_c \subseteq \tilde{A} \subseteq \text{Stab}(S)$$
In low dimensional case 

\[ \dim N < 2 \dim M \quad P : \text{Homeo}(M) \rightarrow N \]

Every orbit is either a point or a cover of \( M \).

\[ N - \text{Fix}(\rho) \leq \text{global fixed point} \quad n \in N - \text{Fix}(\rho) \]

\[ \Rightarrow \quad \text{Apply OCT2 to Stab}(n) \]

\[ M \subseteq \text{Stab}(n) \quad \text{Set} \quad \text{Stab}(n) = \text{Stab}(m) \]

**Thm (C.-Mann)** It is a flat bundle.

**Diff case.** Flat bundle Thm can be used to prove extension problem.

1. \( \exists \) global fixed point by flat bundle Thm
2. the derivative is trivial
3. Thurston type argument.
What's next?

1. How orbits can be glued together?
   - Generalisation of flat bundle Thm

2. Shape of global Fixed Points.

Thm(C) If \( \text{Homeo}(S^n) \) \( \cong \mathbb{R}^{n+k} \) \( k < n \)

\[ N = S^n \times K_1, \; \sim \]

\( K_1 \) is a \( \mathbb{Z} \)-homology manifold with boundary \( K_0 \)

\( (m, x) \sim (n, y) \) \( \forall \; x = y \in K_0 \)

Ex: \( S^n \times \mathbb{L}^n \)/\( \sim \) = \( S^{n+m} \)

\( S^n \times K_1 / \sim \) is always a manifold.
Obstructing global fixed points in a special case

Thm (C.): If Homeo(M) contains a free p-torsion and M is not \( \mathbb{Z}/p \)-homology sphere

\[ \Rightarrow N = M \times K \text{, } 2 \text{ Homeo}(M). \]

Cor: Homeo(S'x M) acts nontrivially on \( N^K \), ken always without fixed points

\[ \Rightarrow N^K \text{ is a cover of } S'x M^n. \]

Future goal: Show the above Thm without condition on M.

THANKS for listening!