Reconstructing maps out of groups

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Theorem (Whittaker, '63)

M, N closed manifolds.

$\text{Homeo}(M) \cong \text{Homeo}(N) \Rightarrow M = N$ (and $\cong$ inner)

Abstract

Theorem (Fillipkiewicz, '89)

$\text{Diff}^{r}(M) \cong \text{Diff}^{s}(N) \Rightarrow M = N$, $r = s$ (and $\cong$ inner)

"Algebraic structure of $\text{Diff}^r(M)$ determines M and it's C^r structure"

Q: Is it enough to know finitely generated subgroups?

Q: (Navas '17) Fix M. $\exists$? a finitely generated subgroup of $\mathrm{Diff}^r(M)$ not isomorphic to a subgroup of $\underline{\mathrm{Diff}^s(M)}$? $s > r$

A: (Kim–Koberda '17, Mann–Wolff '19): Yes, if $\dim(M) = 1$, whenever $r < s$, in fact, there ~~exists~~ is an uncountable family of such f.g. examples for every $r \neq s$.

- True even for Hölder regularities $C^{k+\alpha}$ vs $C^{k+\beta}$ $(k \geq 1)$
- [MW] True even for $C^{k+\text{Lipschitz}}$ vs C^{k+1}
- [KK] True even if you restrict to simple groups (!)

Broader Context:

Algebraic property of group G + minor dynamical hypothesis about $G \curvearrowright$ $\longleftrightarrow$ Big Constraints on dynamics/regularity of $G \curvearrowright M$.

Ex: • G non-abelian $\Rightarrow$ for any action $G \curvearrowright \mathbb{R}$ by homeos, some $g \neq id$ has a fixed point.
[Hölder's]

• G has (T) $\Rightarrow$ any action of G on S^1 by C^2 diffeomorphisms factors thru a finite group.
[Navas '02]

Toy theorem:

$g \in \mathrm{Homeo}_*(S^1)$ commutes with $\overset{\text{rotation}}{x \mapsto x+\alpha}$, $\alpha \notin \mathbb{Q}$

$\phantom{g \in \mathrm{Homeo}_*(}\overset{\shortparallel}{\mathbb{R}/\mathbb{Z}}$

then g is a rotation: $g(x) = x+\beta$

Proof: • Suffice to show g preserves lebesgue measure,

• Suppose not, then $\exists\, y \in S^1$ s.t. $g([y, y+\varepsilon]) = [z, z+\varepsilon']$
$\varepsilon' \neq \varepsilon$.

since $x \overset{f}{\mapsto} x+\alpha$ has dense orbits

WLOG can take $[y, f^N(y)] = [y, y+N\alpha]$

but $g[y, f^N(y)] = [g(y), gf^N(y)] = [g(y), g(y)+N\alpha]$

$\square$

* This has $\overset{\text{a few}}{\text{parallels}}$ with our theorem proof, including constraints posed by commuting elements.

Recall

Thm: $\forall r$ $\exists$ f.g. $\Gamma < \mathrm{Diff}^r(\underset{}{M})$ (M: S^1, I, $\mathbb{R}$), not isomorphic to any subgroup of $\mathrm{Diff}^s(M)$; for any $s > r$.

Proof $M = S^1$

• STEP 1:

There are groups $\Gamma' < \mathrm{Diff}^\infty(S^1)$ that "act in only one way"

any $\phi: \Gamma' \to \mathrm{Diff}^r(S^1)$ is C^r-conjugate to Γ'

$\exists h \in \mathrm{Diff}^r(S^1)$ s.t. $\forall \gamma \in \Gamma'$
$h \circ \phi(\gamma) \circ h^{-1} = \gamma$

• STEP 2: Some of these Γ' are rich enough to "algebraically remember" another map:

if f, g homeos and

$\langle \Gamma', f \rangle \cong \langle \Gamma', g \rangle$ then f is C^r conjugate to g.

"RECONSTRUCTING MAPS OUT OF GROUPS"

Special case of Step 1 [To be black box]

Thm: [Ghys + ...]

$\Gamma' =$ Triangle group $\underset{\langle a,b,c \mid a^2 = b^3 = c^7 = abc = 1\rangle}{\Delta_{237}} \subset PSL(2,\mathbb{R}) \subset \mathrm{Diff}^{\infty}(S^1)$

If $\varphi : \Gamma' \longrightarrow \mathrm{Diff}^r(S^1)$ nontrivial, $r \geq 3$,

then $\varphi(\Gamma')$ is C^r-conjugate to Δ_{237}

"Acts in only one way"

Remark:
$1 < r < 3$, I or $\mathbb{R}$ instead of S^1, require work
+ results of Bonatti-Monteverde-Navas-Rivas
+ Classical dynamical arguments
(Sternberg linearization ...)

STEP 2:

"Reconstructing maps"

Def: Say $\Gamma < \mathrm{Homeo}(X)$ Recognizes maps if

$$\left.\begin{array}{l} \langle \Gamma, f\rangle \xrightarrow[\Phi]{\cong} \langle \Gamma, g\rangle \text{ and } \Phi|_\Gamma = id \\ \Phi(f) = g \end{array}\right\} \Rightarrow f = g$$

Prop: ① If Γ has small supports everywhere, then Γ recognizes homeo's of X.

$\hookrightarrow \forall$ open $U \subset X$ $\exists \gamma \in \Gamma$ supported on U (ptwise Fix $X - U$)

② If Γ has the contraction property, then Γ recognizes maps with non-total support

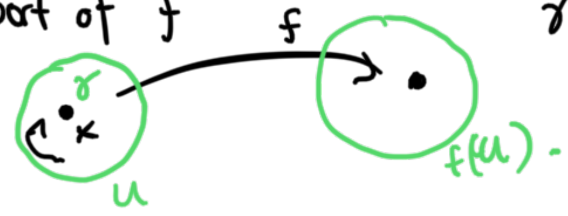
$\forall$ open $U \subset X$
$\exists \gamma \in \Gamma$ s.t. $\gamma(X-U) \subset U$.

(Eg. Cocompact Fuchsian gp on ...

Proof ①:

First recognize support of f

· If $f(x) \neq x$,



$\gamma \circ f \neq f \circ \gamma$

$[f, \gamma] \neq id.$

· If $x \notin Supp(f)$, then f fixes nbhd of x, so for any small U containing x, any γ supported on U

$[f, \gamma] = id.$

This shows $supp(f) = supp(g)$ if $\langle \Gamma, f \rangle \underset{\Phi}{\cong} \langle \Gamma, g \rangle$

$\Gamma \xrightarrow{id} \Gamma$

$f \longrightarrow g$

To get $f = g$,

suppose for contradiction $f(x) \neq g(x)$.

(WLOG $f(x) \neq x$)

WLOG $\gamma(x) \neq x$

· $f\gamma^{-1}f^{-1}\gamma$ supported in $U \cup f(U)$

· $g\gamma^{-1}g^{-1}\gamma$ has support in $gU \cup U$

Contradiction w/ earlier step! □

Case ② (contracting property) is more interesting, but same outline: First recognize support.

· $f(x) \neq x \Rightarrow f$ and $\gamma f \gamma^{-1}$ don't commute

contraction to nbhd of x

· $x \notin supp(f) \Rightarrow$ " " do commute.

Proof of Main theorem, easy case:

Let $\Gamma = \langle \underline{\Delta}_{237}, \underline{f} \rangle$

Diffeo that is C^5, not C^6, non-total support.

Claim: Γ is not isomorphic to a group of C^6 diffeos.

Proof: Suppose it were, via some $\Phi : \Gamma \to \mathrm{Diff}^6(S^1)$

- $\boxed{\text{GHYS}}$ $\Rightarrow$ After conjugating by C^6 diffeo
  $\underline{\Phi|_{\Delta_{237}} = \mathrm{id}}$.
- Δ_{237} has contracting property.
- $\underline{\text{Prop} \Rightarrow \Phi(f)} = f$ C^5 not C^6 contradiction.

C^6

$\square$

The good part:

This is a general strategy, didn't use $M = S^1$ except for Black box. Potential to generalize!

The hard part:

Generalizing the black box.

- Ghys' argument uses uniqueness of projective structures on $\mathbb{R}P^1$ (C^3 comes from Schwarzian)
- To reduce from C^3 or work with I, $\mathbb{R}$, need a lot of work

- Totally open in higher dimensions:
  Find a group that "acts in only one way"

JUST A FEW DETAILS:

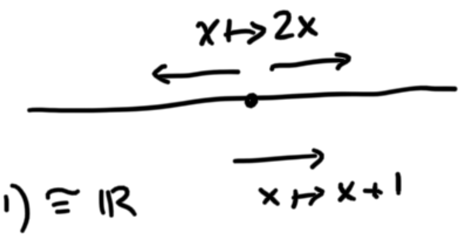
- What's "intermediate regularity"?
  Imagine diffeo fixing 0, with r^{th} derivative

$$x \mapsto x^\alpha \text{ near } 0 \qquad \alpha \in (0,1)$$

$$\text{or } \quad x \mapsto x^\alpha \log\left(\tfrac{1}{x}\right)^\beta \qquad \beta \in \mathbb{R}$$

- What's Γ' (group that acts in only one way)
  when $M = [0,1]$?

$$\Gamma'' \cong \langle \underset{x \mapsto 2x}{BS(1,2)}, \underset{x \mapsto \lambda x}{\text{irrational}} \rangle$$



$(0,1) \cong \mathbb{R}$

<u>Prop</u>: If $\Gamma'' \to \mathrm{Diff}^r[0,1]$ is C^0-conjugate to the standard action, then it is C^r-conjugate.

- BMNR: action of $\underset{\underset{a}{x \mapsto 2x}}{BS(1,2)}$ has derivative 2 at its fixed point

- Sternberg linearization:
  hyperbolic fixed point for $a \rightsquigarrow$ locally conjugate to $x \mapsto 2x$,
  + regularity of conjugacy.

- Irrational $\lambda \rightsquigarrow$ conjugacy works for Γ''
  ([illegible] theorem)

(toy theorem)
$\approx$
+ uniqueness
statement from Sternberg

OK, but still need C^0-conjugacy to get started.

For this put $\Gamma'' \subset \Gamma' = \langle \Gamma'', f \rangle$

$\uparrow$ with small support

& argue using
classification of $BS(1,2)$ actions + map recognition