Big mapping class groups and rigidity of the simple circle

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Big MCG and actions on $S^1$

\[ \Omega = \mathbb{R}^2 - K, \ K \text{ Cantor set}, \ \Gamma = \text{Mod}(\Omega). \]

**Fact:** $\Gamma$ acts **faithfully** on $S^1$.

$\implies \Gamma \hookrightarrow \text{Homeo}^+(S^1)$.

**Question 1:** Does $\Gamma$ act on $S^1$ in different ways?

Any nontrivial action with a global fixed point?

**Question 2:** Is $\Gamma$ generated by torsion?

Homeo$^+([0, 1])$ is torsion-free.

**Question 3:** Any further obstruction for $G \hookrightarrow \Gamma$?
Main results

**Question 1:** Does $\Gamma$ act on $S^1$ in different ways? No.

**Question 2:** Is $\Gamma$ generated by torsion? Yes.

**Question 3:** Any further obstruction for $G \leq \Gamma$? Yes.

**Theorem 3:** Any countable subgroup of Homeo$^+(S^1)$ embeds into $\Gamma$. Not true if uncountable, e.g. PSL$_2\mathbb{R}$.

**Theorem 2:** $\Gamma$ is normally generated by a single 2-torsion.

**Theorem 1:** $\Gamma$ acts faithfully and minimally on the simple circle $S^1_S$. Any nontrivial action of $\Gamma$ on $S^1$ is semi-conjugate to this one.

\[
\begin{align*}
S^1 & \xrightarrow{\rho(g)} S^1 \\
h \downarrow & \quad h \downarrow \\
S^1_S & \xrightarrow{g} S^1_S
\end{align*}
\]

Similar results by Mann–Wolff.
Rays

\[ S \text{ finite type} \]
\[ \text{curve graph } \mathcal{C}(S) \]
\[ \mathcal{C}(S) \text{ is hyperbolic} \]
\[ (\text{Masur–Minsky}) \]

\[ S = \Omega \]
\[ \text{ray graph } \mathcal{R} \]
\[ \mathcal{R} \text{ is hyperbolic} \]
\[ (\text{Bavard, Aramayona–Fossas–Parlier}) \]
Fix a hyperbolic structure on $\Omega$. conical cover $\Omega_C$ and conical circle $S_C^1$. 
Rays

Fix a hyperbolic structure on $\Omega$.

conical cover $\Omega_C$ and conical circle $S^1_C$.

Simple rays $= R \sqcup L \sqcup X$, $\Gamma$-invariant and closed.

$R =$ short rays, $L =$ lassos, $X =$ long rays.

one orbit one orbit uncountably many orbits, related to $\partial R$
Lemma: For any $x \in S^1_C$, the orbit closure $\overline{\Gamma x} \supset R$. 
**Lemma:** The unique minimal set $\bar{R} = R \sqcup X$ is a Cantor set on $S^1_C$, complementary intervals $\leftrightarrow L$.

**Key points:**
- each $\ell \in L$ is isolated in the simple set.
Lemma: The unique minimal set $\tilde{R} = R \sqcup X$ is a Cantor set on $S_C^1$, complementary intervals $\leftrightarrow L$.

Key points:

- each $\ell \in L$ is isolated in the simple set. $\implies R \cup X$ closed.
- the simple set is nowhere dense. $\implies R \cup X$ nowhere dense.
- can approximate each $x \in X$ by short rays.

$\implies \tilde{R} = R \cup X$, perfect.
The simple circle

Collapse complementary intervals to points $S^1_C \rightsquigarrow S^1_S$

- $\Gamma$ acts faithfully and minimally on $S^1_S$
- has an uncountable orbit $R$.
- has a **countable** orbit $L$.

$\implies$ every subgroup of $\Gamma$ has a faithful action on $S^1$ with a countable orbit.
**Theorem 3:** Any countable subgroup of $\text{Homeo}^+(S^1)$ embeds into $\Gamma$. Not true if uncountable, e.g. $\text{PSL}_2\mathbb{R}$.

**Prop.:** Every faithful action of $\text{PSL}_2\mathbb{R}$ on $S^1$ is standard.

**Tool:** The **bounded Euler class** $\text{eu}_b^\rho \in H^2_b(G)$.

\[ \rho : G \to \text{Homeo}^+(S^1), \sim \to \text{eu}_b^\rho = \rho^*\text{eu}_b. \]

- (Ghys) $\text{eu}_b^\rho$ determines the action up to semi-conjugacy.
- $\text{eu}_b^\rho = 0$ iff the action has a **global fixed point**.
- $H^2_b(G) \hookrightarrow H^2(G)$ if $G$ is uniformly perfect.

\[ \text{eu}_b^\rho \hookrightarrow \text{eu}^\rho, \text{eu}^\rho = 0 \text{ iff action lifts to } \mathbb{R}. \]
Theorem 3: Any countable subgroup of Homeo^+(S^1) embeds into $\Gamma$. Not true if uncountable, e.g. $\text{PSL}_2\mathbb{R}$.

Prop.: Every faithful action of $\text{PSL}_2\mathbb{R}$ on $S^1$ is standard.

Proof sketch:

- any $g \in \text{PSL}_2\mathbb{R}$ is a commutator, so uniformly perfect.
- $H^2(\text{PSL}_2\mathbb{R}; \mathbb{Z}) \cong \mathbb{Z}$, generated by $\text{eu}^{std}$.
- $\text{eu}^{\rho} = \lambda \cdot \text{eu}^{std}$, $\lambda = \pm 1$ (torsion+rigid subgroup).

Then show the action $\rho$ is transitive, so no countable orbit.
Countable subgroups

**Theorem 3:** Any countable subgroup of $\text{Homeo}^+(S^1)$ embeds into $\Gamma$. Not true if uncountable, e.g. $\text{PSL}_2\mathbb{R}$.

**Proof:** Denjoy’s blow-up construction + suspension.
**Theorem 1:** $\Gamma$ acts faithfully and minimally on the simple circle $S^1_S$. Any nontrivial action of $\Gamma$ on $S^1$ is semi-conjugate to this one (up to a change of orientation).

**Proof sketch:** Fix an action $\rho$ without fixed points.

$r$ short ray, $\Gamma_r := \text{Stab}(r)$.

**Step 1:** Each $\Gamma_r$ acts with a global fixed point.

Fix $r_0$, pick $P(r_0) \in \text{Fix}(\Gamma_{r_0})$. Let $P(r) = \rho(g).P(r_0)$ if $r = g.r_0$.

$$
\begin{array}{c}
R \xrightarrow{P} S^1_S \\
\downarrow_{\Gamma\text{-equivariant}} \\
S^1 \\
\end{array}
$$

fixed by $\Gamma_r = g\Gamma_{r_0}g^{-1}$

**Step 2:** $P$ preserves the circular order and is injective.
Step 1 details

$r$ short ray, $\Gamma_r := Stab(r)$, $\Gamma_{(r)} \leq \Gamma_r$ ($id$ on a nbhd of $r$).

**Step 1:** Each $\Gamma_r$ acts with a global fixed point.

- Any circle action of $\Gamma_{(r)}$ has a fixed point ($H^2_b(\Gamma_{(r)}) = 0$).

  uniformly perfect (suspension trick) $g = a_g(ha_g^{-1}h^{-1})$

  $H^k(\Gamma_{(r)}) = 0$ (Mather’s suspension argument)

\[
\begin{align*}
  g & \quad \cdots \quad h \quad h \quad \cdots \\
  g h h^{-1} & \quad \cdots \quad a_g = g \cdot (hgh^{-1}) \cdot (h^2gh^{-2}) \cdots
\end{align*}
\]
$r$ short ray, $\Gamma_r := \text{Stab}(r)$, $\Gamma_{(r)} \leq \Gamma_r$ (id on a nbhd of $r$).

**Step 1:** Each $\Gamma_r$ acts with a global fixed point.

- Any circle action of $\Gamma_{(r)}$ has a fixed point ($H^2_b(\Gamma_{(r)}) = 0$).
- $\Gamma_r = \langle \Gamma_{(r)}, \Gamma_{(s)} \cap \Gamma_r \rangle$ if $\text{end}(r) \neq \text{end}(s)$.
  - $\Gamma_{(s)} \cap \Gamma_r$ preserves $\text{Fix}(\Gamma_{(r)})$ and has fixed points.

- $\Gamma = \langle \Gamma_r, \Gamma_s \rangle$ if $\text{end}(r) \neq \text{end}(s)$.

**Cor:** $\Gamma$ is generated by elements supported in disks.

Use this + suspension to prove Theorem 2.
Theorem 2: \( \Gamma \) is normally generated by a single 2-torsion.

Proof: The suspension trick.
Step 2: $P$ preserves the circular order and is injective.

- $P(r) \neq P(s)$ if $\text{end}(r) \neq \text{end}(s)$ since $\Gamma = \langle \Gamma_r, \Gamma_s \rangle$.
- $\text{Or}(P(r_1), P(r_2), P(r_3)) = 1$ if $\text{Or}(r_1, r_2, r_3) = 1$ and $r_1, r_2, r_3$ are disjoint with distinct endpoints.

goal: remove “disjoint”.
The filtration

Induction using a filtration associated to an “equator” \( \gamma \).

**equator**: embedded circle containing \( K \).

\( R_0(\gamma) \subset R_1(\gamma) \subset \cdots \subset R \), each is a Cantor set.

each step adds Cantor to each complementary interval.
The induction