Big mapping class groups acting on homology

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\( S \) surface \( (\text{conn, oriented}, \exists S = d) \)

\[ \text{Finite-type} \]

The \( S \) f.g.

Ends(\( S \)) = space of ends

Mapping class group: \( \text{MCG}(S) := \text{Homeo}^+(S)/\text{homotopy} \)

\( \text{f.g.} \)

\( \text{discrete} \)

\( \text{not f.g.} \)

\( \text{not discrete} \)
**Homology**

$H_1(S; \mathbb{R})$: vector space generated by $\{[\alpha] \mid \alpha \text{ curve}\}$

$H_1(S, \mathbb{Z})$: $\mathbb{Z}$-span of $\{[\alpha] \mid \alpha \text{ curve}\}$ (simple closed, oriented)

Remark: $\alpha$ sep. cutting off $X \cap \alpha$ (\text{=}) $[\alpha] = 0$

Algebraic intersection form $i: H_1(S; \mathbb{R}) \times H_1(S; \mathbb{R}) \to \mathbb{R}$

$$\alpha, \beta \to i([\alpha], [\beta]) = \sum_{p \in \alpha \cap \beta} p \in \alpha \cap \beta$$

Rule: $\alpha$ sep. $\Rightarrow i([\alpha], \cdot) = 0$ $[\alpha]$ isotropic

Remark: $i$ is symplectic if and only if $S$ has at most one end.

$GL(H_1(S; \mathbb{R}))$

Action of $\text{MCG}(S)$ on homology $\sim \rho_S: \text{MCG}(S) \to \text{Aut}(H_1(S; \mathbb{Z}); i)$

Question: what is the image of $\rho_S$? i.e., which $\varphi \in GL(H_1)$ come from a mapping class?
A classical result

**Theorem** (Burkhardt)

If $S$ is closed of genus $g$, $\rho_S$ is surjective onto $\text{Aut}(H_1(S; \mathbb{Z}); i) \simeq \text{Sp}(2g; \mathbb{Z}).$

"Proof:" $\psi \in \text{Sp}(2g; \mathbb{Z})$

\[ \alpha_i, \beta_i \text{ symplectic basic} \]

Only $\alpha_i$ & $\beta_i$ intersect, in 1 pt

\[ \varphi(\alpha_i), \varphi(\beta_i) \]

$\alpha_i', \beta_i'$ curves w/ same int. pattern

$S \setminus \bigcup_{i} U \alpha_i \cup U \beta_i \simeq S^2 \setminus g \simeq S \setminus \bigcup_{i} U \alpha_i' \cup U \beta_i'$

\[ \alpha_i \rightarrow \alpha_i', \beta_i \rightarrow \beta_i' \]

\[ \varphi: S \rightarrow S \text{ homeo} \]

$F = \varphi$

D
The Loch Ness monster

**Theorem** (F.–Hensel–Vlamis)
If $S$ is the Loch Ness monster, $\rho_S$ is surjective onto $\text{Aut}(H_1(S;\mathbb{Z});\hat{i}) \simeq \text{Sp}$(N;Z).

**Problem:**

**Solution:** [Richards] $\psi \in \text{Sp}(N,\mathbb{Z})$ 

$\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$ $\varphi$ a exhaustion $B_2 = \Sigma_2$

Construct subs. $A_n, B_n, \overline{f_n}: A_n \rightarrow B_n$

- $n$ odd $A_n \supseteq \Sigma_n$ $\Rightarrow A_n, B_n$ exh.
- $n$ even $B_n \supseteq \Sigma_n$

$\overline{f_n} |_{A_{n+1}} = \overline{f_{n-1}} \Rightarrow \overline{f} = \lim_{n\rightarrow \infty} \overline{f_n}$

$\overline{(f_n)}_* = \psi |_{H_n(A_n,\mathbb{Z})} \Rightarrow \overline{f_*} = \psi$
The general case

What if $\hat{\imath}$ is not symplectic?

$\exists \phi \in \text{Aut}(\mathfrak{H}(S;\mathbb{Z}))$

$\phi([\alpha]) = [\beta]$

but $\mathcal{F} \notin \text{MCG}$:

$\mathcal{F}_s = \phi$

$\rightarrow$ need to:

(1) detect ends

(2) deal w/ sep. curves

Ends

$X$ flare surface if $\partial X$ is a single separating curve and $X$ is unbounded.

$\mathcal{F} := \{H_1(X;\mathbb{Z}) \mid X$ flare surface$\}$ w/ partial order $\leq$

$\leq$

< $[\alpha]$ | $\alpha$ curves, $\alpha \leq X$

Fact: ends $\leftrightarrow$ ultrafilters fn $(\mathcal{F}, \leq)$

$e \leftrightarrow \mathcal{F}_e = \{ \mathcal{V} \in \mathcal{F} \mid \exists \text{ flare } \mathcal{V} = H_1(X;\mathbb{Z})$ & $e \in \text{Ends}(X) \}$
Separating curves

\( \gamma \) separating curve \( \leadsto \mathcal{L}(\gamma) := \{ e \in \text{Ends}(S) \mid e \text{ is to the left of } \gamma \} \)

\[ \mathcal{L}(\alpha) = \{ x \} \]
\[ \mathcal{L}(\beta) = \{ p, q \} \]

Lemma

\( \alpha, \beta \) separating. Then \( [\alpha] = [\beta] \) if and only if \( \mathcal{L}(\alpha) = \mathcal{L}(\beta) \).

\( \xrightarrow{\text{X q p}} \) \( \implies [\alpha] = [\beta] \)
\( \xrightarrow{\text{no end in } X} \) \( \implies \mathcal{L}(\alpha) = \mathcal{L}(\beta) \)

\[ C = [\gamma], \gamma \text{ sep. } \implies \mathcal{L}(C) = \mathcal{L}(\gamma) \]
The general case

**Theorem** (F.–Hensel–Vlamis)
Let $S$ be either:

- a finite-type surface with at least 4 punctures, or
- an infinite-type surface different from the Loch Ness monster or the once-punctured Loch Ness monster.

Then:

- $\varphi \in \text{Aut}(H_1(S; \mathbb{Z}); \hat{1})$ preserving $\mathcal{F} \Rightarrow$ it preserves
  \[
  \{[\alpha] \mid \alpha \text{ separating}\}
  \]
  and it induces a homeomorphism
  \[
  f_{\varphi} : \text{Ends}(S) \to \text{Ends}(S);
  \]

- the image of $\rho_S$ is
  \[
  \left\{ \varphi \in \text{Aut}(H_1(S; \mathbb{Z}); \hat{1}) \mid \begin{array}{c}
    \varphi \text{ preserves } \mathcal{F} \text{ and } \\
    \exists \alpha \text{ separating}, [\alpha] \neq 0 : \\
    \mathcal{L}(\varphi([\alpha])) = f_{\varphi}(\mathcal{L}([\alpha]))
  \end{array} \right\}.
  \]