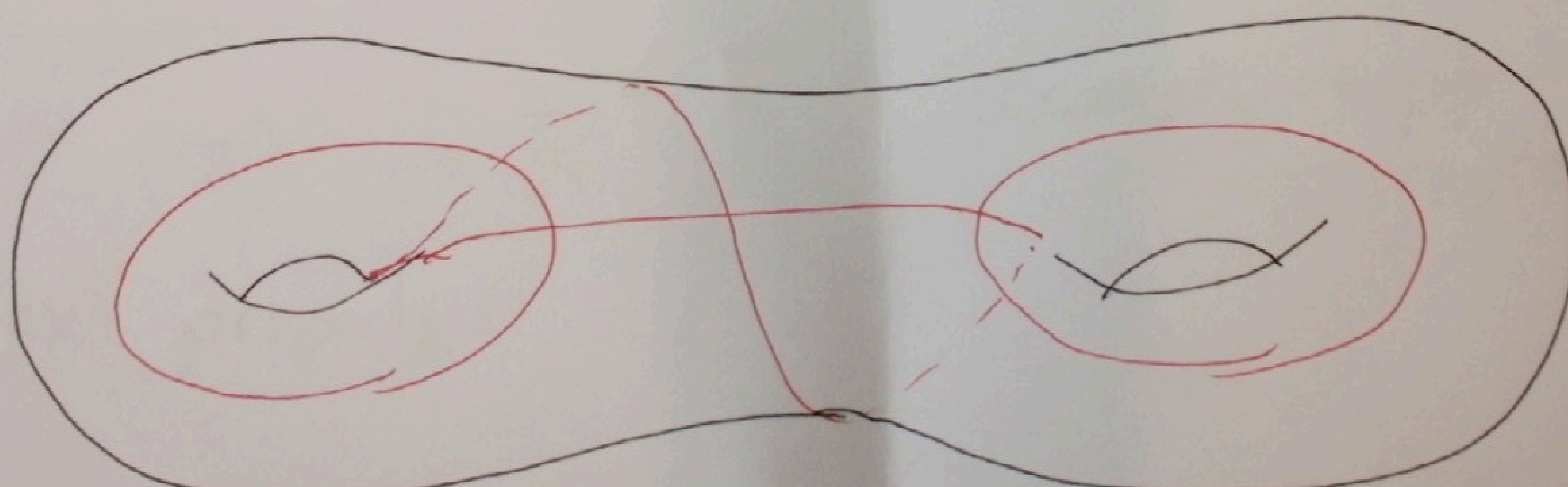
# From Curves to Currents

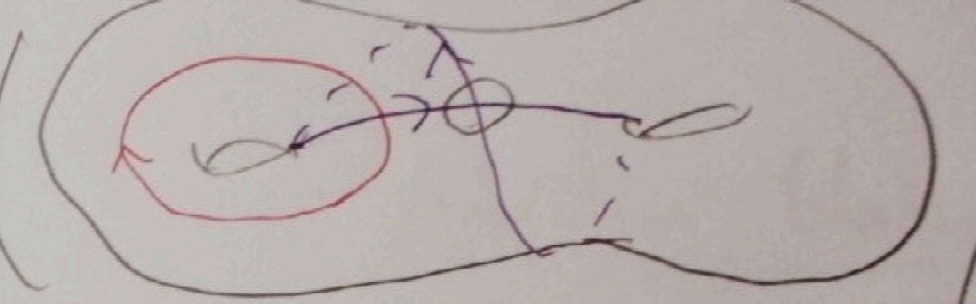
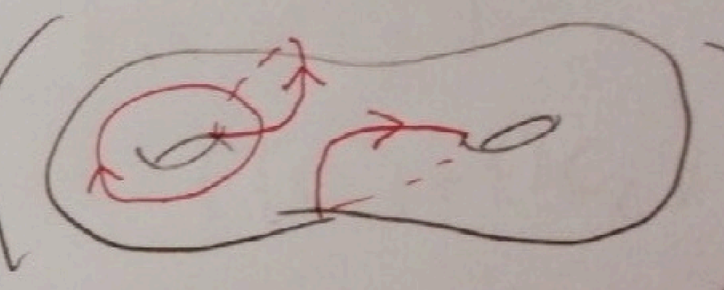
Dylan Thurston, Indiana University
Joint with D. Martínez-Granado



# When is a functional on curves "nice"?

$\Sigma$ : surface, hyperbolic, closed

curves
↳ up to isotopy/homotopy

$f\Big($  $\Big) \geq f\Big($  $\Big)$

__types__

__Simple__ : no intersection

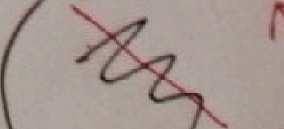
__Multi-curves__ : many connected components

__Oriented__

$f : \mathcal{C}(\Sigma) \to \mathbb{R}$ ?

__Example__ hyperbolic length

Key property:

__Smoothing__ $f\big($  $\big) \geq f\big($  $\big)$

__essential__ → cannot be removed by homotopy

# Properties of Curve Functionals



$$f : \mathcal{C}^+(\Sigma) \longrightarrow \mathbb{R}$$

oriented curves

## Homogeneity

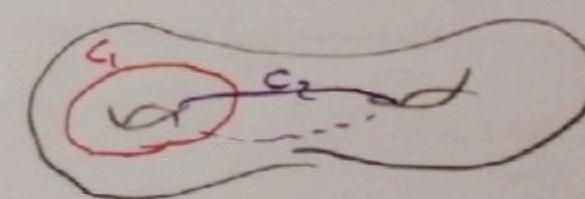
$$f(nC) = n f(C)$$


## Smoothing

$$f(\times) \geq f(\uparrow\ \uparrow)$$

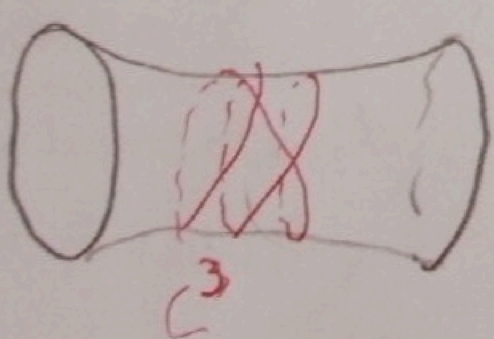
## Additive Union

$$f(C_1 \cup C_2) = f(C_1) + f(C_2)$$




## Stability

$$f(C^n) = f(nC)$$




## Quasi-Smoothing

$$f(\times) \geq f(\uparrow\ \uparrow) - K$$

universal constant

## Convex Union

$$f(C_1 \cup C_2) \leq f(C_1) + f(C_2)$$

**Thm A** If $f$ satisfies

- homogeneity
- stability
- quasi-smoothing
- convex union

on multi-curves, then it extends continuously to geodesic currents

**Thm B** If $f$ satisfies

- quasi-smoothing
- convex union

on multi-curves, then its stabilization extends continuously to geodesic currents:

$$\|f\|(c) = \lim_{n\to\infty} \frac{f(c^n)}{n}$$

**Theorem C** If $f$ satisfies

- homogeneity
- stability
- smoothing
- additive union

then it is an intersection number:

$$f(c) = i(c, \mu_f)$$

**Theorem D** (80%)

If $f$ satisfies

- homogenity
- stability
- smoothing
- convex union

then, on measured laminations, it is convex with respect to any standard (Dehn-Thurston) coordinates.

**Thm** (Rafi-Souto '17, following Mirzakhani)

If $f$ on multi-curves extends continuously and positively to currents and is homogeneous, and $c_0$ is fixed (filling) curve, then

$$\lim_{L\to\infty} \frac{\#\{c \mid f(c) \le L,\ c \sim c_0\}}{L^{6g-6}} = \frac{K(f)\,K(c_0)}{K(g)}$$

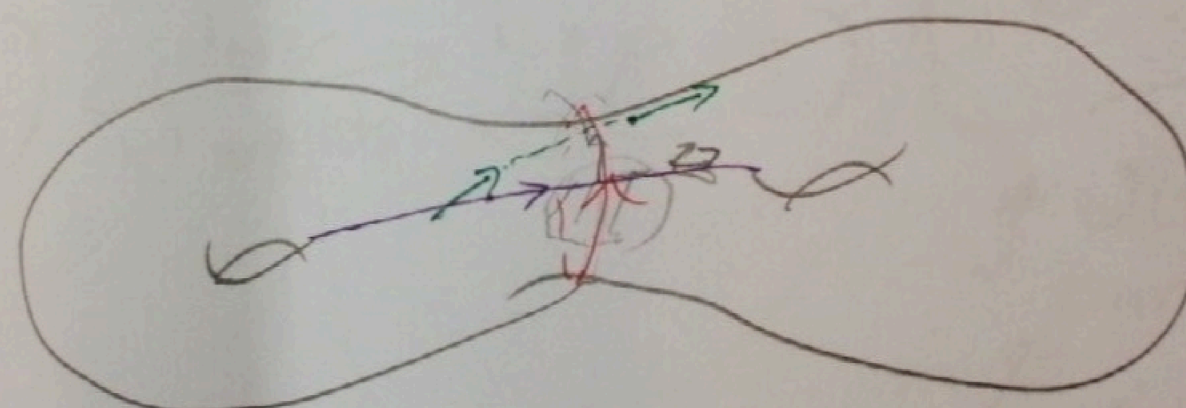
① What is that space of geodesic currents?

② Why should care?

Def 1 A *geodesic current* is a finite Borel measure on $UT\Sigma$ invariant under geodesic flow (for fixed hyperbolic structure)

3-manifold

$\varphi_t : UT\Sigma$ ↻

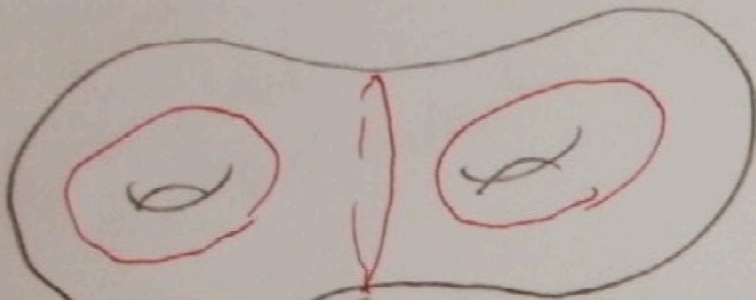


To get a current from a curve $c$

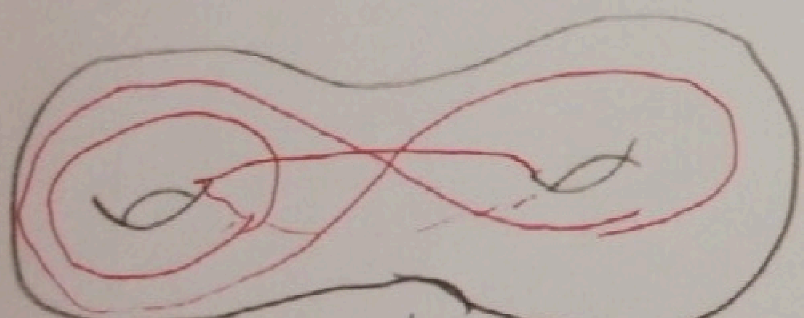
- Take geodesic representative $\gamma$
- Lift to $UT\Sigma$ $\tilde{\gamma}$
- $\mu_c(U) :=$ length of $U \cap \tilde{\gamma}$.

$U \subset UT\Sigma$ open

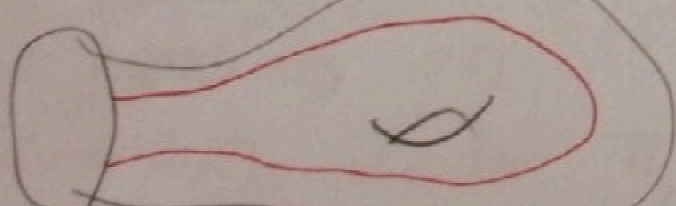
# $\Sigma$ closed, hyperbolic surface



simple multi-curves

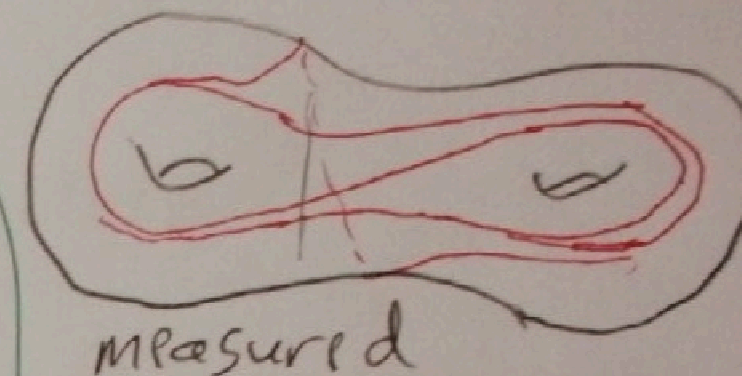


general multi-curves

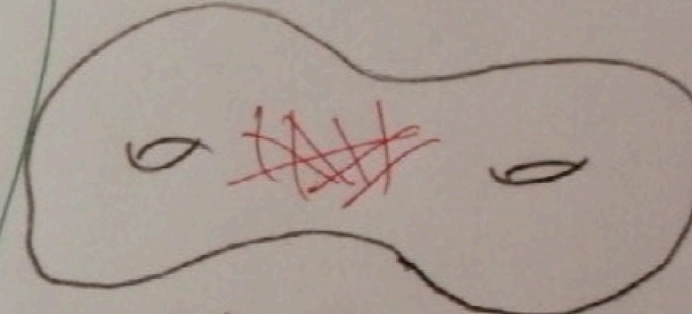


$$\begin{tikzcd} \mathcal{S}(\Sigma) \arrow[r, "\text{dense}", "\text{w/ weights}"'] \arrow[d, hook] & \mathcal{ML}(\Sigma) \arrow[d, hook, "\substack{\text{nice space} \\ \subset \mathbb{R}^{6g-6}}"] \\ \mathcal{C}(\Sigma) \arrow[r, "\text{dense}", "\text{w/ weights}"'] & \mathcal{GC}(\Sigma) \end{tikzcd}$$

completion



measured laminations (foliations)



geodesic currents

space of measures

Space is independent of hyperbolic structure

Not obvious

$c$ curve

$\mu_c$ associated geodesic current

$$\mu_c(UT\Sigma) = \text{hyp. length of } c \text{ wrt given structure}$$

To see this:

geodesic flow

$$\begin{tikzcd}
 & \mathbb{R} \circlearrowright UT\mathbb{H}^2 \circlearrowleft \pi_1(\Sigma) \arrow[dl, "/\pi_1(\Sigma)"'] \arrow[dr, "/\mathbb{R}"] & \\
\circlearrowright UT\Sigma & & \text{geodesics} \circlearrowleft \pi_1(\Sigma)
\end{tikzcd}$$

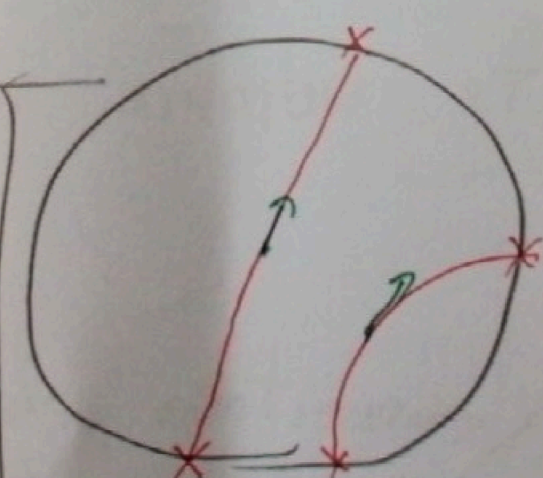
$UT\mathbb{H}^2 = PSL_2(\mathbb{R})$

geodesic flow $\circlearrowright UT\Sigma$

$\text{geodesics} = (S^1 \times S^1 \setminus \Delta) \circlearrowleft \pi_1(\Sigma)$

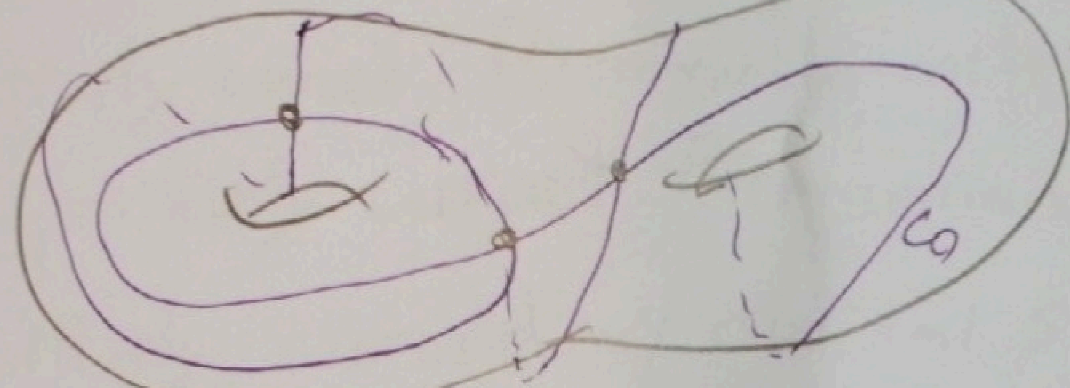
**Def 2** Geodesic currents are measures on $(S^1 \times S^1 \setminus \Delta)$ invariant under $\pi_1(\Sigma)$

locally finite, Radon



## Why to care?

- Counting problems




$\# \left\{ \begin{array}{l} \text{curves } c \\ \text{related } c_0 \\ \text{by mapping classing group} \\ f(c) \le L \end{array} \right\}$

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## Many examples!

Almost all examples satisfy (quasi-) smoothing

$i(c, c)$ = twice number of self-intersections of $c$

$\sqrt{i(c,c)} \rightsquigarrow$ extends contiauously by Bonahon
not a consequence of our theorem

## Functionals extending to currents

**Examples**

- Hyperbolic length (Bonahon '88)
- Intersection number (Bonahon '88)
- Length on negatively curved Riemannian metric (Otal '90)
- Length on non-positively curved Riemannian metric (Croke-Fathi-Feldman '92)
- Lenth on negatively curved w/ conical singularities (Hersonsky-Paulin '97)
- Length on flat metric w/ conical singularities (Bankovic-Leininger '18)
- Length for quadratic differentials (Duchin-Leininger-Rafi '10)
- Length w.r.t. simple generators for $\pi_1$ (Erlandsson '19)
- Stable lengths for co-compact actions (Bonahon '91)
- Stable length w.r.t. arbitrary generators for $\pi_1$ (Erlandsson-Parlier-Souto '20)

additive union

- Extremal length

convex union

$f(D) = i(C, D)$ satisfies smoothing so extends

$f(C) = i(C, \mu)$ satisfies smoothing so extend

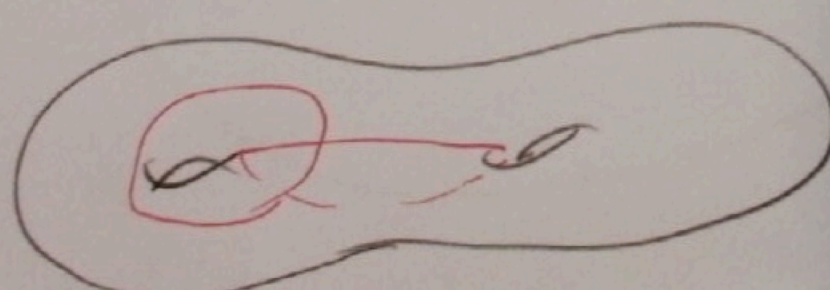
current

$i(\mu_1, \mu_2)$ defined continuous in each variable

does not imply $i(\cdot, \cdot)$ is continuous as a 2-variable function

$\sqrt{i(c, c)}$ does satisfy smoothing

not concave union



$i(C_1, C_1) = 0$

$i(C_2, C_2) = 0$

$i(C_1 \cup C_2, C_1 \cup C_2) = 2$

## Classical Theorem

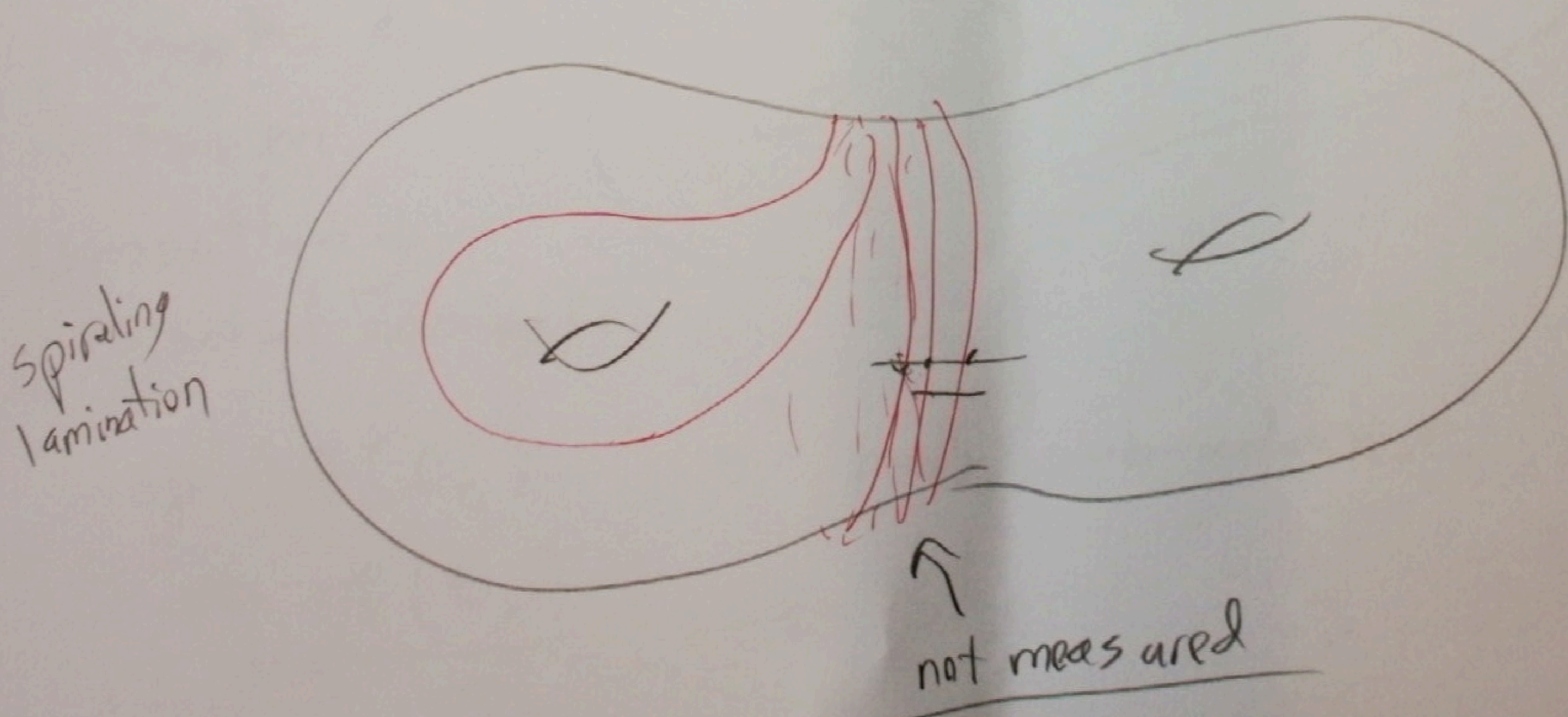
$f : \mathbb{Q}^n \to \mathbb{R}$

that is convex:

$$f(v_1 + v_2) \leq f(v_1) + f(v_2)$$

$$f(k \cdot v_1) = k\, f(v_1)$$

then $f$ extends continuously to a convex fcn on $\mathbb{R}^n$



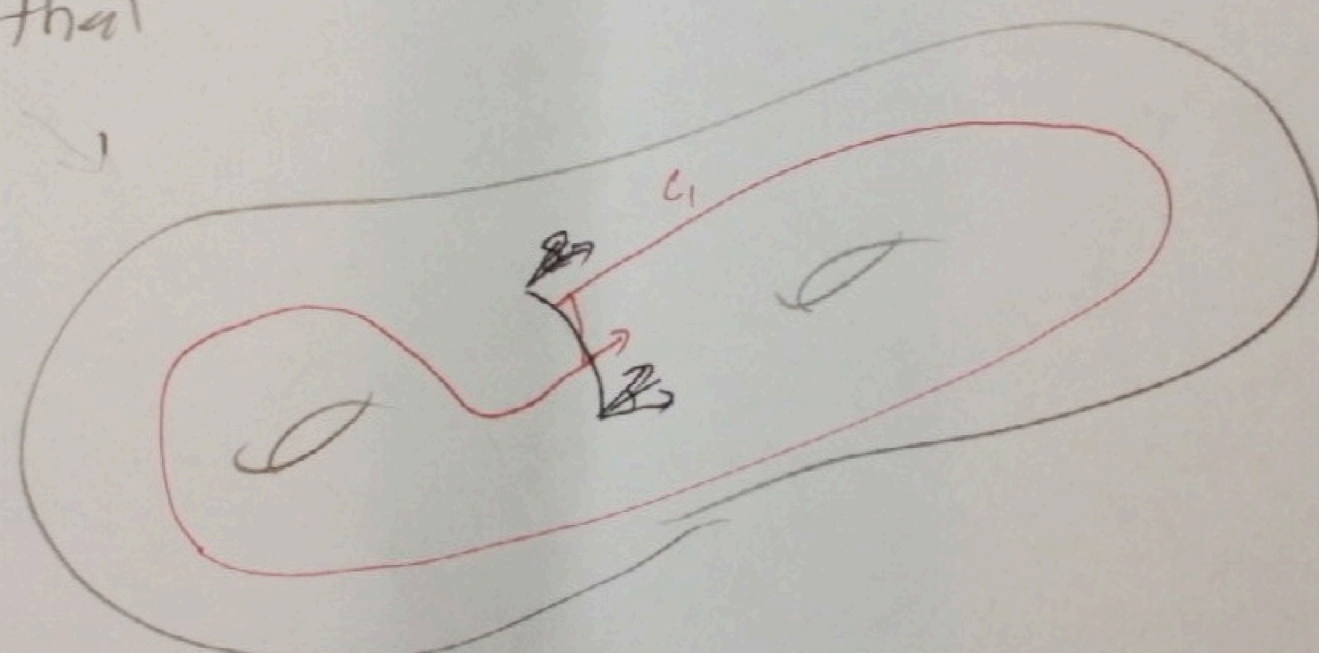
Proof idea  Define $f(\mu)$

Approximate $\mu$ by closed curves

use ~~first~~ n-th return map of geodesic flow

curves $c_n$ so $\lim_{n\to\infty} \frac{c_n}{n} = \mu$

- Arrange so that $c_n \cup c_m \cup K$ smooths to $c_{n+m}$ (smoothing essential crossings)



$\Rightarrow f(c_n)$ is nearly sub-additive

$\Rightarrow \lim \frac{f(c_n)}{n}$ exists