Homeomorphic subsurfaces and the omnipresent arcs

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joint with Federica Fanoni and Tyrone Ghaswala
A surface $\Sigma$ is of infinite-type if $\pi_1(\Sigma)$ is not finitely generated.

The Loch Ness Monster $L_1$

The punctured Monster $L_1^\circ$

The Ladder $L_2$
The Cantor Tree
An end is where the surface “goes off to infinity”.

*Admissible chain:* $U_1 \supset U_2 \supset \ldots$

$U_i \subset \Sigma$ noncompact, and $\partial U_i$ a separating curve for every compact $K \subset \Sigma$, $U_n \cap K = \emptyset$, for large $n$.

$U_1 \supset U_2 \supset \ldots \sim V_1 \supset V_2 \supset \ldots$

if $\forall n \exists N$ such that $U_n \subset V_N$ and vice versa.

An *end* is $e = [U_1 \supset U_2 \supset \ldots]$.

If every $U_i$ has genus then $e$ is *nonplanar*, and *planar* otherwise.
Does it make sense to say one surface is bigger than another?

\[ \Sigma_i^o \subset \Sigma_2 \text{ and } \Sigma_2 \not\subset \Sigma_i^o \]
Given two infinite-type surfaces, when can one be realized as a subsurface of the other?

Restrict focus to cutting along arcs or curves.
\[ L_1^0 \subset L_1 \]
$L_1 \subset L_2$
$L_2 \subset L_1$
$L_n \subset L_m$ for any $n, m \in \mathbb{N}$
If $\alpha$ is a separating arc, call a component $S$ of $\Sigma \setminus \alpha$ a one-cut subsurface.

If $\Sigma$ is finite-type then $S \not\cong \Sigma \iff \alpha$ is essential.

If $S \cong \Sigma$, call it a homeomorphic one-cut subsurface.
Are there infinite-type surfaces containing (essential) homeomorphic one-cut subsurfaces?

**Theorem (Fanoni-Ghaswala-M)**

A surface is infinite-type $\iff$ it contains essential one-cut homeomorphic subsurfaces.
Proof by pictures

Case 1- There is an isolated nonplanar end.

Case 2- Infinitely many isolated planar ends.

Case 3- Finite genus, all planar ends nonisolated.

Case 4- Infinite genus, all nonplanar ends nonisolated.
Finite-type: $\alpha$ is essential $\Leftrightarrow$ it intersects every homeomorphic one-cut subsurface.

Infinite-type: Not true!

$\alpha$ is omnipresent if it is 2-ended and it intersects every homeomorphic one-cut subsurface.
Every 2-ended arc in $L_3$ is omnipresent.

$\alpha$ not omnipresent $\Rightarrow$

There exists a one-cut subsurface $S$ of $L_3 \setminus \alpha$ such that $S \cong L_3$. 
The Cantor Tree has no omnipresent arcs.
\[ \text{Map}(\Sigma) = \text{Homeo}^+(\Sigma)/\text{isotopy} \]

A finite orbit end is an end with finite \( \text{Map}(\Sigma) \)-orbit.

If an arc connects two finite-orbit ends, then it is omnipresent.

The converse is not always true!
An end $e$ is stable if $e = [U_1 \supset U_2 \supset \ldots]$, where $U_i \cong U_{i+1}$ for all $i \in \mathbb{N}$.

A surface is stable if every end is stable.

**Theorem (Fanoni-Ghaswala-M)**

For stable surfaces, an arc is omnipresent if and only if it connects two finite orbit ends.
The arc and curve graph $\mathcal{AC}(\Sigma)$

Vertices: isotopy classes of arcs and curves
Edges: pairs of disjoint arcs and curves
Bavard, Aramayona-Fossas-Parlier

Let $P$ be the set of isolated planar ends.
$A(\Sigma, P)$ is the full subgraph whose vertices are arcs with endpoints in $P$.
If $|P| < \infty$ then $A(\Sigma, P)$ is connected, infinite diameter, and $\delta$-hyperbolic.

Rasmussen

Let $g$ be the genus of $\Sigma$.
$\mathcal{N}(\Sigma)$ is the full subgraph of nonseparating curves.
If $0 < g < \infty$ then $\mathcal{N}(\Sigma)$ is connected, infinite diameter, and $\delta$-hyperbolic.

Durham-Fanoni-Vlamis

Let $\Sigma$ have at least four finite orbit ends,
There exists a graph of curves; connected, infinite diameter, and $\delta$-hyperbolic.
Theorem (Fanoni-Ghaswala-M)

If \( \Sigma \) is stable with at least three finite orbit ends, then the omnipresent arc graph \( \Omega(\Sigma) \) is connected, \( \delta \)-hyperbolic, and infinite diameter.
Infinitely intersecting unicorns

A *unicorn* of $\alpha, \beta$ is an arc given by $a \cup b$, where $a \subset \alpha$, $b \subset \beta$, and $p = a \cap b$ is the unique corner.

If $\alpha, \beta$ are 2-ended, a 2-ended unicorn always exists.
Let $A_0(\alpha, \beta)$ be the set of 2-ended unicorns. If $|\alpha \cap \beta| = \infty$ then $A_0(\alpha, \beta)$ is almost connected.

Let $A_1(\alpha, \beta)$ be the 1-nbhd of $A_0(\alpha, \beta)$. Then $A_1(\alpha, \beta)$ is connected.
Is $\Omega(L_2)$ connected?
Guessing geodesics lemma

For all $x \in [\alpha, \beta]$ there exists $y \in [\alpha, \gamma] \cup [\gamma, \beta]$ such that

$$d(x, y) < \delta.$$ 

For all $x \in A_1(\alpha, \beta)$ there exists $y \in A_1(\alpha, \gamma) \cup A_1(\gamma, \beta)$ such that

$$d(x, y) < M.$$
$A(\Sigma, P)$
\mathcal{N}(\Sigma)