UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS

**MAT 235 Y — CALCULUS II**

**TERM-TEST #2, JANUARY 24, 2008**

**LAST NAME:** ______________________________ **FIRST NAME:** ______________________________

(Please print)

**STUDENT NUMBER:** ______________________

**INSTRUCTIONS:**

- Write your name, your student number and indicate the session in which you want your test back.
- Show and explain your work in all questions. Use both sides of the paper if necessary.
- Do not tear out any page.
- Do not use pencils. Only pen written answers will be considered for remarking.
- No calculators or any aids are permitted.
- Time allowed: 110 minutes.
- This test is worth 15% of your course grade.
- Keep your student card on top of your table all the time while writing the test.

This test has 10 pages numbered from 1 to 10, plus this cover page. This test consists of 9 independent exercises.

Please select the section in which you want your paper back:

| SECTION | DAY/TIME | ROOM | INSTRUCTOR | MY SECTION |
|---|---|---|---|---|
| L–0101 | MWF/9 | SS–2135 | Y. Kim | |
| L–0201 | MWF/1 | SS–1073 | R. Pujol | |
| L–0301 | MWF/2 | RW–110 | B. Hovinen | |
| L–5101 | R/6-9 | LM–161 | R. Pujol | |

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| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | TOTAL |
|---|---|---|---|---|---|---|---|---|---|
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**Exercice 1** (10 marks)**.** Let $f(x,y,z) = xy\mathrm{e}^z$.

a) Compute the directional derivative $D_{\mathbf{u}}f(1,1,0)$, where

$$\mathbf{u} = \big\langle 2, -2, -1 \big\rangle.$$

**Answer:** $\nabla f(x,y,z) = \big\langle y\mathrm{e}^z, x\mathrm{e}^z, xy\mathrm{e}^z \big\rangle$, thus $\nabla f(1,1,0) = \big\langle 1,1,1 \big\rangle$. Hence:

$$\begin{aligned} D_{\mathbf{u}}f(1,1,0) &= \frac{1}{|\mathbf{u}|}\nabla f(1,1,0)\cdot\mathbf{u} \\ &= \frac{1}{3}\big\langle 1,1,1\big\rangle \cdot \big\langle 2,-2,-1\big\rangle \\ &= -\frac{1}{3}. \end{aligned}$$

b) Find the unit vector $\mathbf{v}_1$ for which $D_{\mathbf{v}_1}f(1,1,0)$ attains its maximum value and the unit vector $\mathbf{v}_2$ for which $D_{\mathbf{v}_2}f(1,1,0)$ attains its minimum value (you don't need to give any justification).

**Answer:**

$$\mathbf{v}_1 = \frac{\nabla f(1,1,0)}{|\nabla f(1,1,0)|} = \frac{1}{\sqrt{3}}\big\langle 1,1,1\big\rangle;$$

$$\mathbf{v}_2 = -\mathbf{v}_1 = -\frac{1}{\sqrt{3}}\big\langle 1,1,1\big\rangle.$$

**Exercice 2** (10 marks)**.** Find the following limits. You should explain your answer. In case the limit does not exist, provide the reason:

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$  b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^4+y^6}$

**Answer:** a) The limit doesn't exist: Limit along the line $y = x$ is:

$$\lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2},$$

whereas the limit along the line $y = -x$ is:

$$\lim_{x\to 0} \frac{-x^2}{x^2+x^2} = -\frac{1}{2}.$$

b) The limit exists: From the well-known inequality

$$|ab| \leq \frac{1}{2}(a^2+b^2), \text{ for all real numbers } a \text{ and } b$$

it follows (with $a = x^2$ and $b = x^3$):

$$|x^2y^3| \leq \frac{1}{2}(x^4+y^6).$$

Hence:

$$|x^3y^3| \leq \frac{|x|}{2}(x^4+y^6)$$

and

$$\frac{|x^3y^3|}{x^4+y^6} \leq \frac{|x|}{2} \underset{(x,y)\to 0}{\longrightarrow} 0.$$

Thus:

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^4+y^6} = 0.$$

**Exercice 3** (10 marks)**.** Find the linear approximation of the function $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ at $(2,\ 1)$ and use it to approximate $f(1.95, 1.09)$.

**Answer:** At the point $(2, 1)$, the linear approximation of $f$ is:

$$f(x, y) \simeq f(2, 1) + (x - 2)\frac{\partial f}{\partial x}(2, 1) + (y - 1)\frac{\partial f}{\partial y}(2, 1).$$

The partial derivatives of $f$ at $(x, y)$ are given by:

$$\frac{\partial f}{\partial x}(x, y) = -\frac{x}{\sqrt{20 - x^2 - 7y^2}}, \qquad \frac{\partial f}{\partial y}(x, y) = -\frac{7y}{\sqrt{20 - x^2 - 7y^2}},$$

hence:

$$\frac{\partial f}{\partial x}(2, 1) = -\frac{2}{3}, \qquad \frac{\partial f}{\partial y}(2, 1) = -\frac{7}{3}.$$

Hence the linear approximation of $f$ at $(2, 1)$ is:

$$f(x, y) \simeq 3 - \frac{2}{3}(x - 2) - \frac{7}{3}(y - 1).$$

An approximation of $f(1.95, 1.09)$ is thus:

$$f(1.95, 1.09) \simeq 3 + \frac{0.1}{3} - 7 \times 0.03 \simeq 3 + 0.03 - 0.21 \simeq 2.82.$$

**Exercice 4** (10 marks)**.** Find equations of the tangent plane and the normal line to the given surface at the specified point:

$$x^2 - 2y^2 + z^2 + yz = 2, \text{ at the point } (2, 1, -1).$$

**Answer:** We differentiate the relation $x^2 - 2y^2 + z^2 + yz = 2$ with respect to $x$ and $y$ (where $z$ is considered as a function of $x$ and $y$):

$$\frac{\partial}{\partial x}: \qquad 2x + 2z\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = 0;$$
$$\frac{\partial}{\partial y}: \qquad -4y + 2z\frac{\partial z}{\partial y} + z + y\frac{\partial z}{\partial y} = 0.$$

Hence, at a point $(x, y)$:

$$\frac{\partial z}{\partial x}(x, y) = -\frac{2x}{y + 2z}$$
$$\frac{\partial z}{\partial y}(x, y) = -\frac{4y - z}{y + 2z}$$

and at the point $(2, 1)$ when $z = -1$:

$$\frac{\partial z}{\partial x}(2, 1) = 4$$
$$\frac{\partial z}{\partial y}(2, 1) = -5$$

The equation of the tangent plane at $(2, 1, -1)$ is:

$$(z + 1) = 4(x - 2) - 5(y - 1), \text{ or } 4x - 5y - z = 4.$$

Equation of the normal line at $(2, 1, -1)$:

$$\mathbf{r}(t) = t\langle, 4, -5, -1\rangle + \langle 2, 1, -1\rangle \qquad \text{(parametric);}$$

$$\frac{x - 2}{4} = \frac{1 - y}{5} = -z - 1 \qquad \text{(symmetric).}$$

**Exercice 5** (20 marks)**.** Let $f(x, y) = xy - \mathrm{e}^x y^2$.

a) Compute the differential of the function $f$.

**Answer:** The partial derivatives of $f$ at a point $(x, y)$ are:

$$\frac{\partial f}{\partial x} = y - \mathrm{e}^x y^2, \qquad \frac{\partial f}{\partial y} = x - 2y\mathrm{e}^x.$$

Hence the differential of $f$ is:

$$df = y(1 - y\mathrm{e}^x)\, dx + (x - 2y\mathrm{e}^x)dy.$$

b) Show that the critical points of the function $f$ are $(0, 0)$ and $\left(2, \mathrm{e}^{-2}\right)$.

**Answer:** $(x, y)$ is a critical point of $f$ if and only if $df = 0$. Now,

$$df = 0 \iff \begin{cases} y(1 - y\mathrm{e}^x) = 0 \\ x - 2y\mathrm{e}^x = 0 \end{cases} \iff \begin{cases} y = 0 \quad \text{or} \quad y = \mathrm{e}^{-x} \\ y = \frac{1}{2} x \mathrm{e}^{-x} \end{cases} \iff x = y = 0 \quad \text{or} \quad x = 2 \text{ and } y = e^{-2}$$

Hence the 2 critical points of $f$ are $(0, 0)$ and $(2, \mathrm{e}^{-2})$.

c) What is the nature of the critical points of $f$ (local minimum, local maximum or saddle point)?

**Answer:** We first compute the second-order partial derivatives of $f$ at a point $(x, y)$:

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -y^2\mathrm{e}^x; \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) = 1 - 2y\mathrm{e}^x; \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = -2\mathrm{e}^x.$$

At $(0,0)$ and $(2, \mathrm{e}^{-2})$ respectively:

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 0; \qquad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 1; \qquad \frac{\partial^2 f}{\partial y^2}(0,0) = -2;$$

$$\frac{\partial^2 f}{\partial x^2}(2,1) = -\mathrm{e}^{-2}; \qquad \frac{\partial^2 f}{\partial x \partial y}(2,1) = -1; \qquad \frac{\partial^2 f}{\partial y^2}(2,1) = -2\mathrm{e}^2.$$

We compute the determinants for the second derivative test:

$$\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1 < 0 \qquad \begin{vmatrix} -\mathrm{e}^{-2} & -1 \\ -1 & -2\mathrm{e}^2 \end{vmatrix} = 1.$$

Hence the point $(0,0)$ is a saddle point, and the point $(2, \mathrm{e}^{-2})$ is a local maximum.

d) Use the method of Lagrange multipliers to find the minimum and the maximum value of $f$ on the set

$$S = \{(x,y) \mid 0 \le x \le 1,\ \mathrm{e}^x y^2 = 1\}.$$

**Answer:** We need to find the minimum and maximum of $f$ under the constraint $g(x,y) = \mathrm{e}^x y^2 - 1$ (with $0 \le x \le 1$). The partial derivatives of $g$ are:

$$\frac{\partial g}{\partial x} = y^2\mathrm{e}^x, \qquad \frac{\partial g}{\partial y} = 2y\mathrm{e}^x.$$

Let $\lambda$ be the Lagrange mutliplier. We need to solve the system:

$$\begin{cases} \mathrm{e}^x y^2 = 1 \\ y - \mathrm{e}^x y^2 = \lambda y^2 \mathrm{e}^x \\ x - 2y\mathrm{e}^x = 2\lambda y \mathrm{e}^x \end{cases} \iff \begin{cases} \mathrm{e}^x y^2 = 1 \\ 1 - y\mathrm{e}^x = \lambda y \mathrm{e}^x \\ x - 2y\mathrm{e}^x = 2\lambda y \mathrm{e}^x \end{cases} \iff \begin{cases} \mathrm{e}^x y^2 = 1 \\ 1 = (1+\lambda) y \mathrm{e}^x \\ x = 2(1+\lambda) y \mathrm{e}^x = 2 \end{cases} \iff \begin{cases} x = 2 \\ y = \pm 1/\mathrm{e} \end{cases}$$

But $x = 2$ is outside the $x$-range. Hence, the minimum and maximum values of $f$ on $S$ are on the end points of $S$. There are $4$ such points:

$$(0,1),\ (0,-1),\ \left(1, \frac{1}{\sqrt{\mathrm{e}}}\right),\ \left(1, -\frac{1}{\sqrt{\mathrm{e}}}\right).$$

The values of $f$ at these points are:

$$f(0,1) = 1 \qquad f(0,-1) = -1 \qquad f\left(1, \frac{1}{\sqrt{\mathrm{e}}}\right) = \frac{1}{\sqrt{\mathrm{e}}} - 1 \qquad f\left(1, -\frac{1}{\sqrt{\mathrm{e}}}\right) = -\frac{1}{\sqrt{\mathrm{e}}} - 1$$

Hence $f$ attains its minimum value on $S$ at the point $\left(1, -\frac{1}{\sqrt{\mathrm{e}}}\right)$ and its minimum value is $-\frac{1}{\sqrt{\mathrm{e}}} - 1$;

and $f$ attains its maximum value on $S$ at the point $\left(1, \frac{1}{\sqrt{\mathrm{e}}}\right)$ and its maximum value is $\frac{1}{\sqrt{\mathrm{e}}} - 1$

**Exercice 6** (10 marks)**.** Evaluate the following integral by reversing the order of integration:

$$\int_0^1 \int_{3y}^3 \mathrm{e}^{x^2}\ dx\,dy.$$

**Answer:** Let $D$ be the domain of integration:

$$D = \{(x,y) \,|\, 0 \le y \le 1 \text{ and } 3y \le x \le 3\}.$$

As we need to change the order of integration, we write $D$ as:

$$D = \{(x,y) \,|\, 0 \le x \le 3 \text{ and } 0 \le y \le x/3\}.$$

Hence:

$$\int_0^1 \int_{3y}^3 \mathrm{e}^{x^2}\ dx\,dy = \int_0^3 \int_0^{x/3} \mathrm{e}^{x^2}\ dy\,dx = \int_0^3 \frac{x}{3}\mathrm{e}^{x^2}\ dx = \frac{1}{6}\left[\mathrm{e}^{x^2}\right]_0^3 = \frac{\mathrm{e}^9 - 1}{6}.$$

**Exercice 7** (10 marks)**.** Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $z^3 - z\mathrm{e}^x = y^2$.

**Answer:**

$\dfrac{\partial}{\partial x}$: $\quad 3z^2\dfrac{\partial z}{\partial x} - \mathrm{e}^x\dfrac{\partial z}{\partial x} - z\mathrm{e}^x = 0.$

$\dfrac{\partial}{\partial y}$: $\quad 3z^2\dfrac{\partial z}{\partial y} - \mathrm{e}^x\dfrac{\partial z}{\partial y} = 2y.$

Hence:

$$\frac{\partial z}{\partial x} = \frac{z\mathrm{e}^x}{3z^2 - \mathrm{e}^x} \qquad \frac{\partial z}{\partial y} = \frac{2y}{3z^2 - \mathrm{e}^x}$$

**Exercice 8** (10 marks)**.** Compute

$$\iint_D x^2y\ dA$$

where $D$ is the bounded set bounded by the parabolas $y = 1 - x^2$ and $y = 2x^2 - 2$.

**Answer:** The set $D$ has the form:

$$D = \left\{(x, y) \,\middle|\, -1 \leq x \leq 1 \text{ and } 2x^2 - 2 \leq y \leq 1 - x^2\right\}$$

(it is a "type I" set). Hence:

$$\begin{aligned}
\iint_D x^2y\, dA &= \int_{-1}^{1} \int_{2x^2-2}^{1-x^2} x^2y\, dy\, dx \\
&= \int_{-1}^{1} \frac{1}{2}x^2 \left[(1 - x^2)^2 - (2x^2 - 2)^2\right]\ dx \\
&= \frac{1}{2} \int_{-1}^{1} x^2 \big[-3x^4 + 6x^2 - 3\big]\, dx \\
&= \frac{3}{2} \int_{-1}^{1} \big[-x^6 + 2x^4 - x^2\big]\, dx \\
&= 3 \int_{0}^{1} [-x^6 + 2x^4 - x^2]\, dx \qquad \text{(since the function we're integrating is even)} \\
&= 3\left(-\frac{1}{7} + \frac{2}{5} - \frac{1}{3}\right) \\
&= -\frac{3}{7} + \frac{6}{5} - 1 \\
&= \frac{-15 + 42 - 35}{35} \\
&= -\frac{8}{35}.
\end{aligned}$$

**Exercice 9** (10 marks)**.** Let $0 \le a < 1$ and define

$$S_a = \{\,(x,y) \mid y \ge 0,\ a^2 \le x^2 + y^2 \le 1\,\}.$$

Compute the position $(\bar{x}_a, \bar{y}_a)$ of the center of mass of $S_a$ (with constant density function), and the limit

$$\lim_{a\to 1} \bar{y}_a.$$

**Answer:** We express $S_a$ in polar coordinates:

$$S_a = \big\{(r,\theta) \,\big|\, a \le r \le 1 \text{ and } 0 \le \theta \le \pi\big\}.$$

Let $\rho$ be the density.

The total mass is:

$$m = \rho A(S_a) = \rho \frac{\pi}{2}\big(1 - a^2\big).$$

From the symmetry of the problem, $\bar{x}_a = 0$.

We now compute $\bar{y}_a$, the $y$-coordinate of the center of mass:

$$\begin{aligned}
\bar{y}_a &= \frac{1}{m}\iint_{S_a} y\rho\, dA = \frac{\rho}{m}\int_0^\pi \int_a^1 r\sin\theta r\, dr\, d\theta \\
&= \frac{\rho}{m}\int_0^\pi \frac{1}{3}\big(1-a^3\big)\sin\theta\, d\theta \\
&= \frac{2\rho}{3m}\big(1-a^3\big) \\
&= \frac{4\rho\big(1-a^3\big)}{3\rho\pi\big(1-a^2\big)} = \frac{4\big(1-a^3\big)}{3\pi\big(1-a^2\big)} \\
&= \frac{4}{3\pi}\frac{1+a+a^2}{1+a}.
\end{aligned}$$

Hence,

$$\lim_{a\to 1} \bar{y}_a = \frac{2}{\pi}.$$

Alternatively you can use l'Hospital's rule to compute this last limit:
From

$$\bar{y}_a = \frac{4\big(1-a^3\big)}{3\pi\big(1-a^2\big)}$$

we get:

$$\lim_{a\to 1} \bar{y}_a = \lim_{a\to 1} \frac{4\big(-3a^2\big)}{3\pi\big(-2a\big)} = \frac{2}{\pi}.$$