UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
MAT 235 Y - CALCULUS II
TEST #1. NOVEMBER 1, 2007.

INSTRUCTIONS: Show all your work in all questions. Use both sides of the papers, if necessary. Do not tear out any pages. Do not use pencils. Only pen written answers will be considered for remarking. No calculators or any other aids are permitted. Write your name and your student number on the front page of each of your examination booklets. This test is worth 15% of your course grade. Duration: 110 minutes.

1. (10 marks) Let C be the curve defined by the parametric equations $x = 5 + t^2$ and $y = 8t^3 - t^4$.

   Find the values of t for which the curve C is concave upward.

Solution:

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{24t^2 - 4t^3}{2t} = 12t - 2t^2$ (undefined at $t = 0$),

$\frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt} = \frac{12-4t}{2t} = \frac{2(3-t)}{t}$ and $\frac{d^2y}{dx^2} > 0$ implies $0 < t < 3$.

The curve C is concave upward only when $0 < t < 3$.

2. (15 marks) Find the length of the loop of the curve $x = 3t - t^3$, $y = 3t^2$.

Solution: (Notice that this is one of the recommended exercises. Section 10.2, #48).

First we find the values of t at the point where the curve intersects itself.

Suppose that $(x(a), y(a)) = (x(b), y(b))$, with $a \neq b$.

Then, from $y(a) = y(b)$ we obtain $3a^2 = 3b^2$, and $b = -a$ with $a \neq 0$ (because $a \neq b$).

Now, from $x(a) = x(b)$ and $b = -a$ we obtain $3a - a^3 = -3a + a^3$, and $a = \pm\sqrt{3}$ (because $a \neq 0$).

That is, the values of t at the start and the end of the loop are $t = -\sqrt{3}$ and $t = \sqrt{3}$, respectively.

The length of the loop is

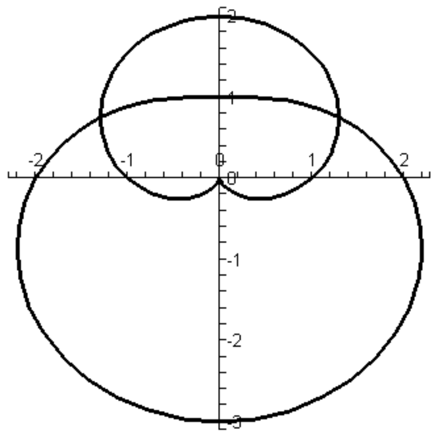
$$L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(3-3t^2)^2 + (6t)^2}\, dt$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(3+3t^2)^2}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} (3+3t^2)\, dt = \Big[3t + t^3\Big]_{-\sqrt{3}}^{\sqrt{3}} = 12\sqrt{3}\ .$$

3. (15 marks) Find the area of the region that lies inside $r = 1 + \sin\theta$ but outside $r = 2 - \sin\theta$.

Solution:

A rough sketch of the graphs of the polar curves shows that they intersect only at two points. From $1 + \sin\theta = 2 - \sin\theta$ we obtain $\sin\theta = 1/2$, and $\theta = \pi/6$ or $\theta = 5\pi/6$.



Then, the area A inside $r = 1 + \sin\theta$ but outside $r = 2 - \sin\theta$ can be calculated as

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left((1+\sin\theta)^2 - (2-\sin\theta)^2\right) d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (6\sin\theta - 3)\, d\theta = \frac{1}{2}\left[-6\cos\theta - 3\theta\right]_{\pi/6}^{5\pi/6} = 3\sqrt{3} - \pi .$$

4. Let $A(1,2,3)$, $B(2,2,2)$ and $C(0,4,5)$.

a) (5 marks) Find the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution:

Let $\mathbf{v} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{AC}$, and let θ denote the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Then $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}$, where $\mathbf{v} = (1, 0, -1)$ and $\mathbf{w} = (-1, 2, 2)$.

Now, $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|} = \dfrac{-3}{3\sqrt{2}} = \dfrac{-\sqrt{2}}{2}$ and $\theta = 3\pi/4$.

b) (5 marks) Find the area of the triangle ABC .

Solution:

Let S denote the area of the triangle ABC , then $S = \dfrac{1}{2}|\mathbf{v}\times\mathbf{w}|$.

Now, $\mathbf{v}\times\mathbf{w} = (2, -1, 2)$, $|\mathbf{v}\times\mathbf{w}| = 3$ and $S = 3/2$.

5. Let $P(4,2,3)$, $Q(5,3,1)$, $R(3,-5,5)$ and let L be the line that passes through the points P and Q.

a) (5 marks) Find parametric equations for the line that passes through the point R and is parallel to the line L.

Solution:

A vector director of the line L is $\mathbf{v} = (5,3,1) - (4,2,3) = (1,1,-2)$.

Therefore, parametric equations for the line that passes through the point R and is parallel to the line L are:

$x = 3 + t$, $y = -5 + t$ and $z = 5 - 2t$.

b) (5 marks) Find an equation of the plane that passes through the midpoint of the segment QR and is perpendicular to the line L.

Solution:

The coordinates of the midpoint of the segment QR are $(4,-1,3)$.

The vector $\mathbf{v} = (1,1,-2)$ is a vector normal to our plane. Therefore, an equation of the plane that passes through the midpoint of the segment QR and is perpendicular to the line L is $x + y - 2z = -3$.

c) (5 marks) Find the coordinates of the point of the line L which is closest to the point R.

Solution:

Parametric equations for the line L are: $x = 4 + t$, $y = 2 + t$ and $z = 3 - 2t$.

An equation of the plane that passes through R and is perpendicular to the line L is: $x + y - 2z = -12$.

The point of the line L which is closest to R is just the intersection point of the line L and the plane $x + y - 2z = -12$. At this intersection point, $(4+t) + (2+t) - 2(3-2t) = -12$.

Then, $t = -2$, $x = 2$, $y = 0$ and $z = -1$.

The coordinates of the point of the line L which is closest to the point R are $(2,0,-1)$.

6. (15 marks) Find an equation of the plane that contains the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$.

Solution: (Notice that this is one of the recommended exercises. Section 12.5, #38)

The vectors $(1,0,-1)$ and $(0,1,2)$ are normal to the planes $x - z = 1$ and $y + 2z = 3$, respectively.

Therefore, the vector $(1,0,-1) \times (0,1,2) = (1,-2,1)$ is a vector parallel to our plane.

The vector $(1,1,-2)$ is also parallel to our plane because it is normal to the plane $x + y - 2z = 1$.

Then, $(1,-2,1) \times (1,1,-2) = (3,3,3)$ is a vector normal to our plane.

Now, the point $(1,3,0)$ is a point in our plane because it is one of the points on the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$. Finally, an equation for our plane is: $x + y + z = 4$.

7. (10 marks) Let $\mathbf{v}(t)$ and $\mathbf{a}(t)$ respectively denote the velocity and acceleration vectors of a particle that moves through space. Suppose that $\mathbf{v}(1) = \mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{a}(t) = 4t\,\mathbf{i} + 2\,\mathbf{j} - 3t^2\,\mathbf{k}$.
Find the velocity and speed of the particle when $t = 2$.

Solution:

From $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt$, we obtain $\mathbf{v}(t) = (2t^2 + C_1, 2t + C_2, -t^3 + C_3)$.

Now, $\mathbf{v}(1) = (0, 1, 5)$ implies $C_1 = -2$, $C_2 = -1$ and $C_3 = 6$.

Therefore $\mathbf{v}(t) = (2t^2 - 2, 2t - 1, -t^3 + 6)$.

The velocity when $t = 2$ is $\mathbf{v}(2) = (6, 3, -2) = 6\,\mathbf{i} + 3\,\mathbf{j} - 2\,\mathbf{k}$ and the speed is $|\mathbf{v}(2)| = 7$.

8. (10 marks) Let C be the curve of intersection of the cylinder $x^2 - 2x + y^2 = 3$ and the plane $z = y + 1$.
Find the radius of the osculating circle of the curve C at the point $(1, 2, 3)$.

Solution:

The first equation can be rewritten as $(x-1)^2 + y^2 = 4$. This equation suggests the parametrization $x = 1 + 2\sin t$, $y = 2\cos t$ and $z = 1 + 2\cos t$ with $0 \le t \le 2\pi$.

Making $\mathbf{r}(t) = (1 + 2\sin t, 2\cos t, 1 + 2\cos t)$ we get $\mathbf{r}'(t) = (2\cos t, -2\sin t, -2\sin t)$ and $\mathbf{r}''(t) = (-2\sin t, -2\cos t, -2\cos t)$.

Notice now that $(1, 2, 3) = \mathbf{r}(0)$ and also $\mathbf{r}'(0) = (2, 0, 0)$ and $\mathbf{r}''(0) = (0, -2, -2)$.

The radius ρ of the osculating circle of the curve C at the point $(1, 2, 3)$ is just the inverse of the curvature of C when $t = 0$.

Therefore, $\rho = \dfrac{|\mathbf{r}'(0)|^3}{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}$, where $\mathbf{r}'(0) \times \mathbf{r}''(0) = (2, 0, 0) \times (0, -2, -2) = (0, 4, -4)$.

Finally, $\rho = \dfrac{2^3}{4\sqrt{2}} = \sqrt{2}$.