UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
MAT 235 Y — CALCULUS II
FALL–WINTER 2007–2008
ASSIGNMENT #5, DUE ON FEBRUARY 14
COVER PAGE

INSTRUCTIONS: Submit this page as your cover page. Use different pages for each of the assigned problems. Staple all your papers and clearly write your name and student number in each of the pages. Show and explain all the steps in each of your solutions. Marks will be deduced for messy or unclear solutions. Late submissions will not be accepted.

NAME: ________________________________ , ______________________
(family name, please print) (given name)

STUDENT NUMBER: ___________________________

SECTION: (please choose one)

SECTION	DAY/TIME	ROOM	INSTRUCTOR	MY SECTION
L–0101	MWF/9	SS–2135	Y. Kim	
L–0201	MWF/1	SS–1073	R. Pujol	
L–0301	MWF/2	RW–110	B. Hovinen	
L–1501	R/6-9	LM–161	F. Recio	

PLEASE DO NOT WRITE HERE

Question 1 (out of 3)	Question 2 (out of 4)	Question 3 (out of 4)	Question 4 (out of 4)	TOTAL (out of 15)

MAT235Y Problem Set 5

Due 14 February 2008

1. Let S be a sphere of radius a and centre Q. Let Γ be a plane through Q, intersecting S in a great circle C. Let l be a line through Q, orthogonal to Γ. Let R be a point on l at distance $\frac{1}{2}a$ from Q. Let P be a paraboloid whose vertex is R and whose intersection with S is the great circle C.

   Given a set $X \in \mathbb{R}^3$, the *convex hull* of X is the set of points in $\mathbb{R}^3$ which lie on some line segment whose endpoints are in X. For example, the convex hull of the sphere S is the ball whose boundary sphere is S.

   (a) Find a formula for the volume of D, the intersection of the convex hulls of S and P, in terms of the radius a.

   (b) Suppose that we fix $a = 2$ but let the position of the vertex of P vary: namely, that it be at distance αa from Q. What should α be so that the volume of D is 8π?

2. **Will Semaj miss his train?**

   Semaj Neru, a University of Toronto economics student, is planning to take the train to his home town of London, Ontario. As he prepares to leave, his roommate Nairod Namdlog approaches.

   "Getting ready to leave for London?" asks Nairod.

   "Yeah. My train is at 4:00 PM. I'm going to take the express bus down to Union Station," replies Semaj.

   Nairod, being a mathematics student with a knack for quick mental computation (and a penchant for ruining his roommate's day) replies, "Oh. More likely than not you won't make the train."

   Semaj scoffs. "Of course I will. The bus stop is right outside and it's scheduled to leave in five minutes. It's 3:30 now and the journey takes twenty minutes. I'll be there five minutes early."

"Ah, but you're forgetting something: the TTC's buses are really unreliable. The actual arrival time of the express bus is normally distributed around 3:35 PM with a standard deviation of five minutes. Your bus has to arrive no later than five minutes before the train departs if you are going to catch the train. And, if you miss the 3:35 bus, you'll have to wait an hour for the next one. You'll never catch your train that way."

"Oh, crap. I hadn't thought of that," says Semaj.

Nairod continues, "On the other hand, the train might be delayed. In fact, its departure time is exponentially distributed with probability density function

$$f(t) = \begin{cases} 0, & t < 0 \\ 0.12 \cdot e^{-0.12 \cdot t}, & t \geq 0. \end{cases}$$

What's more, the departure times of the bus and the train are independent random variables."

The question is: What is the actual probability that Semaj will miss his train, assuming he leaves immediately? Try to state it to within 1/100 or 1% of the true probability.

Note 1: See Section 15.5 in your textbook for basic probability theory. Recall that the probability density function of the normal distribution with mean μ and standard deviation σ is

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}.$$

You will need to express the integral of $g(x)$ in terms of the *error function*

$$\text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt.$$

See the Wikipedia page `http://en.wikipedia.org/wiki/Error_function` for more information and some formulae with which to approximate the error function.

Note 2: The characters presented here are fictional. Any resemblence to actual University of Toronto mathematics graduate students is a purely speculative and utterly ridiculous conspiracy theory.

3. **Ellipsoidal coordinates**

In this problem you will find a formula for integrating a function over region bounded by an ellipsoid $\left\{(x,y,z) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right\}$ for $a,b,c \in \mathbb{R}$. In keeping with the tradition that better coordinates make for easier computations, you will do so by defining a new coordinate system in $\mathbb{R}^3$, the *ellipsoidal coordinates.*

(a) Write a parameterization for the ellipse $\left\{(x,y) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$. (Hint: Modify the parameterization of the circle of radius $a>0$.)

(b) For $r>0$ consider the *dilation* $E_r := \left\{(x,y) \mid \frac{x^2}{(ra)^2}+\frac{y^2}{(rb)^2}=1\right\}$ of the aforementioned ellipse. Define E_0 to be the origin $(0,0)$ (a "degenerate ellipse," for the purposes of this exercise). The set $\{E_r \mid r \geq 0\}$ is a family of ellipses (including one "degenerate" one) covering the whole plane $\mathbb{R}^2$. Define a coordinate system (r,θ) on $\mathbb{R}^2$ in which the first coordinate r specifies on which ellipse E_r of the aforementioned family that point is and the second coordinate θ specifies where on the ellipse E_r the point is (using the paramterization in part (a)). Write a change of coordinates map to convert from (r,θ) to ordinary rectangular coordinates (x,y).

(c) Fix $a,b,c \in \mathbb{R}$. Define a family of ellipsoids $\{\tilde{E}_\rho, \rho \geq 0\}$ via

$$\tilde{E}_\rho := \begin{cases} \{0,0,0\}. & \text{if } \rho = 0, \\ \left\{(x,y,z) \mid \frac{x^2}{(\rho a)^2}+\frac{y^2}{(\rho b)^2}+\frac{z^2}{(\rho c)^2}=1\right\}, & \text{if } \rho > 0. \end{cases}$$

Define a coordinate system on $\mathbb{R}^3$ with coordinates (ρ,θ,ϕ) such that ρ specifies on which ellipsoid $\tilde{E}_\rho$ the point is and θ and ϕ specify where on the ellipsoid $\tilde{E}_\rho$ the point is. Write the change of coordinates map to change from these *ellipsoidal coordinates* to rectangular coordinates. (Hint: Think of spherical coordinates.)

(d) Write a formula for integrating $f(x,y,z)$ over the region bounded by the ellipsoid $\tilde{E}_1$ (in the notation of part (c)).

(e) As a simple example, use part(d) to derive a formula for the volume of region bounded by the ellipsoid $\tilde{E}_1$.

4. **Moments and image recognition**

In this exercise you will learn that moments are of interest to other people besides just physicists! In particular, they can be used in computers for image recognition. For this, we need to give a more general definition than what you have seen.

Let $f(x,y)$ be a function defined on some closed, bounded set $D \subseteq \mathbb{R}^2$. For integers $p, q \geq 0$, the *p, q-moment*, denoted $\mu_{p,q}$, is

$$\mu_{p,q} = \iint_D x^p y^q f(x,y)\ dA.$$

In this exercise, you will show that a continuous function f is determined by its moments. That is, if two continuous functions f and g have the same moments $\mu_{p,q}$ for all $p, q \geq 0$, then $f = g$.

The idea with image recognition is that, if a function (say, representing an image) is determined by its moments, then its parameters can be stored as moments rather than as pixel data. The computer can compute the moments of the input data and match them to the models which it has stored. This has some advantages. For example, the first- and second-order moments encode the position (i.e. centre of mass) and rotation (i.e. moments of inertia) of an object, so, after computing those moments for a function, the coordinates can be changed to give the object a prescribed position and rotation, thus making recognition position- and rotation-independent.

We will proceed in stages.

(a) We will start by showing that if all moments of a continuous function f are zero, then f is the zero function. Show that this is the case if f is known to be a polynomial $\sum_{i,j} a_{ij} x^i y^j$. (Hint: Remember that integration is linear over functions, i.e. $\iint_D ag(x,y) + bh(x,y)\ dA = a \iint_D g(x,y)\ dA + b \iint_D h(x,y)\ dA$.)

(b) *This part is a more challenging optional exercise. We include it for completeness, but you may skip it without penalty. If you complete it, we'll add one mark to your mark for the assignment, up to a maximum of 15/15.*

We now use a big theorem in analysis: the Stone-Weierstraß Theorem. It says the following: If f is a continuous function defined on a closed, bounded set $D \in \mathbb{R}^2$, then, for any $\epsilon > 0$, there exists a polynomial function g such that $|g(x,y) - f(x,y)| < \epsilon$. That is, you can approximate any continuous function arbitrarily well with a polynomial. Show how to use this and part (a) above to show that any continuous function defined on D whose moments are all zero must be identically zero on D.

(c) Use part (b) (and don't worry if you didn't do part (b)) to show the main result: if two continuous functions f and g have the same moments $\mu_{p,q}$ for all $p, q \geq 0$, then $f = g$.