MAT235Y Problem Set 5 Solutions

January 30, 2008

1. Let S be a sphere of radius a and centre Q. Let Γ be a plane through Q, intersecting S in a great circle C. Let l be a line through Q, orthogonal to Γ. Let R be a point on l at distance $\frac{2}{3}a$ from Q. Let P be a paraboloid whose vertex is R and whose intersection with S is the great circle C.

   Given a set $X \in \mathbb{R}^3$, the *convex hull* of X is the set of points in $\mathbb{R}^3$ which lie on some line segment whose endpoints are in X. For example, the convex hull of the sphere S is the ball whose boundary sphere is S.

   (a) *Find a formula for the volume of D, the intersection of the convex hulls of S and P, in terms of the radius a.*

   Solution. We shall embed the figure into 3-space as follows: S shall be centred at the origin and Γ shall be the xy-plane. The vertex of P shall be below the xy-plane, at the point $(0, 0, -\frac{1}{2}a)$. The region D is then the union of the convex hull of the upper hemisphere of S (i.e. the part whose points have positive z-coordinates) and the part of the convex hull of P whose points have negative z-coordinates. Recall that the volume of the sphere S is $4/3\pi a^3$, so the volume of the convex hull of its upper hemisphere is $2/3\pi a^3$. What remains is to compute the volume of the portion of the convex hull of P whose points have negative z-coordinates. With the embedding described here, P is given by the equation

$$z = \frac{1}{2a}(x^2 + y^2 - a^2)$$

   in rectangular coordinates, or

$$z = \frac{1}{2a}(r^2 - a^2)$$

in polar coordiantes. We shall now integrate with respect to polar coordinates.

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi}\int_0^a -\frac{1}{2a}(r^2-a^2)r\ dr\ d\theta \\
&= -\frac{1\pi}{a}\int_0^a (r^2-a^2)r\ dr \\
&= -\frac{1\pi}{a}\left[\frac{r^4}{4}-a^2\frac{r^2}{2}\right]_0^a \\
&= -\frac{1\pi}{a}\left(\frac{a^4}{4}-\frac{a^4}{2}\right) \\
&= \frac{\pi}{4}a^3.
\end{aligned}$$

Thus the volume in question is

$$\frac{2}{3}\pi a^3+\frac{1}{4}\pi a^3=\frac{11}{12}\pi a^3$$

(b) *Suppose that we fix* $a = 2$ *but let the position of the vertex of* P *vary: namely, that it be at distance* αa *from* Q. *What should* α *be so that the volume of* D *is* 8π*?*

Solution. In this case, using an embedding in $\mathbb{R}^3$ analogous to part (a) above, the vertex of P is $(0, 0, -\alpha a)$ and the equation defining P is

$$z=\frac{\alpha}{a}(r^2-a^2).$$

Using the same calculuation as above, we obtain the formula

$$\text{Volume of } D=\frac{\alpha\pi}{2}a^3+\frac{2}{3}\pi a^3=\frac{3\alpha+4}{6}\pi a^3.$$

Setting $a = 2$ and the volume to 8π, we obtain $8\pi = \frac{3\alpha+4}{6}\pi\cdot 8$, which we solve for α to obtain $\alpha = \frac{2}{3}$.

2. **Will Semaj miss his train?**

Semaj Neru, a University of Toronto economics student, is planning to take the train to his home town of London, Ontario. As he prepares to leave, his roommate Nairod Namdlog approaches.

"Getting ready to leave for London?" asks Nairod.

"Yeah. My train is at 4:00 PM. I'm going to take the express bus down to Union Station," replies Semaj.

Nairod, being a mathematics student with a knack for quick mental computation (and a penchant for ruining his roommate's day) replies, "Oh. More likely than not you won't make the train."

Semaj scoffs. "Of course I will. The bus stop is right outside and it's scheduled to leave in five minutes. It's 3:30 now and the journey takes twenty minutes. I'll be there five minutes early."

"Ah, but you're forgetting something: the TTC's buses are really unreliable. The actual arrival time of the express bus is normally distributed around 3:35 PM with a standard deviation of five minutes. Your bus has to arrive no later than five minutes before the train departs if you are going to catch the train. And, if you miss the 3:35 bus, you'll have to wait an hour for the next one. You'll never catch your train that way."

"Oh, crap. I hadn't thought of that," says Semaj.

Nairod continues, "On the other hand, the train might be delayed. In fact, its departure time is exponentially distributed with probability density function

$$f(t) = \begin{cases} 0, & t < 0 \\ 0.12 \cdot e^{-0.12 \cdot t}, & t \geq 0. \end{cases}$$

What's more, the departure times of the bus and the train are independent random variables."

What is the actual probability that Semaj will miss his train, assuming he leaves immediately? Try to state it to within 1/100 or 1% of the true probability.

Solution. First, some notation: Let s be the departure time of the bus, in minutes after 3:30 PM. Let t be the departure time of the train, in minutes after 4:00 PM. Let σ be the standard deviation of the departure time of the bus (which is 5 in this case). The joint probability density function $f(s,t)$ is then

$$f(s,t) = \begin{cases} \frac{0.12}{\sigma\sqrt{2\pi}} e^{\frac{-(s-5)^2}{2\sigma^2}} e^{-0.12t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

We now determine the region D over which to integrate f to get the required probability. First, since Semaj is leaving at 3:30 PM, the departure time of the bus must be 3:30 PM at the earliest, so $s \geq 0$ in D. Second, in order for Semaj to make the trip and the connection to his train, the train departure time must be at least twenty-five minutes after the bus departure time. That is, s and t must satisfy

$$t + 4:00 \geq s + 3:30 + 25,$$

i.e. $t \geq s - 5$. Futhermore, since the train will never depart early, we can also make $t \geq 0$. This gives the region $D = \{(s,t) \mid 0 \leq t < \infty, 0 \leq s \leq t + 5\}$.

However, if you try to integrate $f(s,t)$ over such a region, you'll end up having to find the integral of the error function. Eek! Okay, we'll change the order of integration. Observe that D can also be written

$$D = \{(s,t) \mid 0 \leq s < 5, 0 \leq t < \infty\} \cup \{(s,t) \mid 5 \leq s < \infty, s - 5 \leq t < \infty\}.$$

Thus we can write the required integral as the sum of two integrals, as follows:

$$\text{Probability} = \frac{0.12}{\sigma\sqrt{2\pi}} \left(\underbrace{\int_0^\infty \int_0^5 e^{\frac{-(s-5)^2}{2\sigma^2} - 0.12t}\, ds\, dt}_{I_1} + \underbrace{\int_5^\infty \int_{s-5}^\infty e^{\frac{-(s-5)^2}{2\sigma^2} - 0.12t}\, dt\, ds}_{I_2} \right).$$

We'll tackle each of these in turn.

Let's start with I_1, which is slightly easier. Since the integrand is the product of two functions, each of which only depends on one variable, we can represent the integral as the product of integrals of one variable.

$$\begin{aligned} I_1 &= \int_0^5 e^{\frac{-(s-5)^2}{2\sigma^2}}\, ds \cdot \int_0^\infty e^{-0.12t}\, dt \\ &= \frac{1}{0.12} \int_0^5 e^{\frac{-(s-5)^2}{2\sigma^2}}\, ds. \end{aligned}$$

The remaining integral we must express in terms of the error function. Setting $u = \frac{s-5}{\sigma\sqrt{2}}$ so that $ds = \frac{1}{\sigma\sqrt{2}}\, du$, we find that

$$\int_0^5 e^{\frac{-(s-5)^2}{2\sigma^2}}\, ds = \frac{1}{\sigma\sqrt{2}} \int_{\frac{-5}{\sigma\sqrt{2}}}^0 e^{-u^2}\, du$$

. Observe now that the integrand e^{-u^2} is even. Thus we make some modifications to the above integral to put it in terms of the error function:

$$\begin{aligned} \sigma\sqrt{2}\int_{\frac{-5}{\sigma\sqrt{2}}}^{0} e^{-u^2}\,du &= \frac{1}{\sigma\sqrt{2}}\int_{0}^{\frac{5}{\sigma\sqrt{2}}} e^{-u^2}\,du \\ &= \frac{\sqrt{\pi}}{2\sigma\sqrt{2}}\mathrm{erf}\left(\frac{5}{\sigma\sqrt{2}}\right). \end{aligned}$$

Evaluating all of this with $\sigma = 5$, we get approximately 7.117.

Now we proceed to I_2. Evaluating the inner integral, we obtain

$$\begin{aligned} I_2 &= \int_5^\infty \int_{s-5}^\infty e^{\frac{-(s-5)^2}{2\sigma^2}-0.12t}\,dt\,ds \\ &= \frac{1}{0.12}\int_5^\infty e^{\frac{-(s-5)^2}{2\sigma^2}-0.12(s-5)}\,ds \\ &= \frac{1}{0.12}\int_5^\infty e^{\frac{-1}{2\sigma^2}(s^2-4s-5)}\,ds. \end{aligned}$$

To express this in terms of the error function, we must complete the square of the exponent, from which we will obtain a constant which we can pull in front of the integral.

$$\begin{aligned} I_2 &= \frac{1}{0.12}\int_5^\infty e^{\frac{-1}{2\sigma^2}(s^2-4s+4-4-5)}\,ds \\ &= \frac{1}{0.12}\int_5^\infty e^{\frac{-1}{2\sigma^2}((s-2)^2-9)}\,ds \\ &= \frac{e^{\frac{9}{2\sigma^2}}}{0.12}\int_5^\infty e^{\frac{-(s-2)^2}{2\sigma^2}}\,ds. \end{aligned}$$

Now we apply the same type of transformation as with I_1 above to obtain

$$I_2 = \frac{e^{-\frac{9}{2\sigma^2}}\sqrt{\pi}}{0.12\cdot 2\sigma\sqrt{2}}\left(\frac{1}{2}+\mathrm{erf}\left(\frac{2}{\sigma\sqrt{2}}\right)\right).$$

Evaluating this with $\sigma = 5$, we get approximately 1.032.

Putting everything together, we obtain our final answer of 0.078, or about 8% (i.e. not bloody likely!).

3. **Ellipsoidal coordinates**

In this problem you will find a formula for integrating a function over region bounded by an ellipsoid $\left\{(x,y,z) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right\}$ for $a,b,c \in \mathbb{R}$. In keeping with the tradition that better coordinates make for easier computations, you will do so by defining a new coordinate system in $\mathbb{R}^3$, the *ellipsoidal* *coordinates.*

(a) *Write a parameterization for the ellipse* $\left\{(x,y) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$. *(Hint: Modify the parameterization of the circle of radius* $a>0$.*)*

Solution. The desired parameterization is

$$\theta \mapsto (a\cos\theta, b\sin\theta).$$

The reader may easily verify that the equation defininining the aforementioned ellipse is satisfied.

(b) For $r>0$ consider the *dilation* $E_r := \left\{(x,y) \mid \frac{x^2}{(ra)^2}+\frac{y^2}{(rb)^2}=1\right\}$ of the aforementioned ellipse. Define E_0 to be the origin $(0,0)$ (a "degenerate ellipse," for the purposes of this exercise). The set $\{E_r \mid r \geq 0\}$ is a family of ellipses (including one "degenerate" one) covering the whole plane $\mathbb{R}^2$. Define a coordinate system (r,θ) on $\mathbb{R}^2$ in which the first coordinate r specifies on which ellipse E_r of the aforementioned family that point is and the second coordinate θ specifies where on the ellipse E_r the point is (using the paramterization in part (a)). Write a change of coordinates map to convert from (r,θ) to ordinary rectangular coordinates (x,y).

Solution. The desired change of coordinates map is

$$\begin{aligned} x &= ra\cos\theta, \\ y &= rb\sin\theta. \end{aligned}$$

(c) Fix $a,b,c \in \mathbb{R}$. Define a family of ellipsoids $\{\tilde{E}_\rho, \rho \geq 0\}$ via

$$\tilde{E}_\rho := \begin{cases} \{0,0,0\}. & \text{if } \rho = 0, \\ \left\{(x,y,z) \mid \frac{x^2}{(\rho a)^2}+\frac{y^2}{(\rho b)^2}+\frac{z^2}{(\rho c)^2}=1\right\}, & \text{if } \rho > 0. \end{cases}$$

Define a coordinate system on $\mathbb{R}^3$ with coordinates (ρ,θ,ϕ) such that ρ specifies on which ellipsoid $\tilde{E}_\rho$ the point is and θ and ϕ specify where on the ellipsoid $\tilde{E}_\rho$ the point is. Write the change of coordinates map to change from these *ellipsoidal coordinates* to rectangular coordinates. (Hint: Think of spherical coordinates.)

Solution. The desired change of coordinates map is

$$\begin{aligned} x &= a\rho \sin\phi \cos\theta, \\ y &= b\rho \sin\phi \sin\theta, \\ y &= c\rho \cos\phi. \end{aligned}$$

(d) Write a formula for integrating $f(x, y, z)$ over the region bounded by the ellipsoid $\tilde{E}_1$ (in the notation of part (c)).

Solution. Denote by G the aforementioned change of coordinates map. We first compute the Jacobian

$$\begin{aligned}
\mathrm{Jac}(G) &= \det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{pmatrix} \\
&= \det \begin{pmatrix} a\sin\phi\cos\theta & b\sin\phi\sin\theta & c\cos\phi \\ a\rho\cos\phi\cos\theta & b\rho\cos\phi\sin\theta & -c\sin\phi \\ -a\rho\sin\phi\sin\theta & b\rho\sin\phi\cos\theta & 0 \end{pmatrix} \\
&= abc\rho^2 \det \begin{pmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{pmatrix} \\
&= abc\rho^2 \left(\cos\phi \det \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta \\ -\sin\phi\sin\theta & \sin\phi\cos\theta \end{pmatrix} + \sin\phi \det \begin{pmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta \\ -\sin\phi\sin\theta & \sin\phi\cos\theta \end{pmatrix} \right) \\
&= abc\rho^2 \left(\cos\phi(\cos\phi\sin\phi\cos^2\theta + \cos\phi\sin\phi\sin^2\theta) + \sin^3\phi(\cos^2\theta + \sin^2\theta) \right) \\
&= abc\rho^2 \sin\phi.
\end{aligned}$$

Thus the integration formula is

$$\iiint_D f(x, y, z)\ dV = \iiint_D f(a\rho\sin\phi\cos\theta, b\rho\sin\phi\sin\theta, c\rho\cos\theta) abc\rho^2 \sin\phi\ d\rho\ d\phi\ d\theta.$$

(e) *As a simple example, use part(d) to derive a formula for the volume of region bounded by the ellipsoid* $\tilde{E}_1$.

Solution. In this case, we are integrating the function $f(x,y,z)=1$ over the region $\tilde{E}_1$, which, in our coordinates is just $\{(\rho,\theta,\phi \mid 0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\pi\}$. So we compute

$$\begin{aligned}\iiint_D 1\ dV &= \iiint_D abc\rho^2\sin\phi\ d\rho\ d\phi\ d\theta\\ &= abc\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\sin\phi\ d\rho\ d\phi\ d\theta\\ &= abc\int_0^{2\pi} d\theta\int_0^{\pi}\sin\phi\ d\phi\int_0^1\rho^2\ d\rho\\ &= \frac{4}{3}\pi abc.\end{aligned}$$

4. **Moments and image recognition**

In this exercise you will learn that moments are of interest to other people besides just physicists! In particular, they can be used in computers for image recognition. For this, we need to give a more general definition than what you have seen.

Let $f(x,y)$ be a function defined on some closed, bounded set $D\subseteq\mathbb{R}^2$. For integers $p,q\ge 0$, the p,q-*moment*, denoted $\mu_{p,q}$, is

$$\mu_{p,q}=\iint_D x^p y^q f(x,y)\ dA.$$

In this exercise, you will show that a continuous function f is determined by its moments. That is, if two continuous functions f and g have the same moments $\mu_{p,q}$ for all $p,q\ge 0$, then $f=g$.

The idea with image recognition is that, if a function (say, representing an image) is determined by its moments, then its parameters can be stored as moments rather than as pixel data. The computer can compute the moments of the input data and match them to the models which it has stored. This has some advantages. For example, the first- and second-order moments encode the position (i.e. centre of mass) and rotation (i.e. moments of inertia) of an object, so, after computing those moments for a function, the coordinates can be changed to give the object a prescribed position and rotation, thus making recognition position- and rotation-independent.

We will proceed in stages.

(a) *We will start by showing that if all moments of a continuous function f are zero, then f is the zero function. Show that this is the case if f is known to be a polynomial $\sum_{i,j}a_{ij}x^iy^j$. (Hint: Remember that integration is linear over functions, i.e. $\iint_D ag(x,y)+bh(x,y)\ dA = a\iint_D g(x,y)\ dA + b\iint_D h(x,y)\ dA$.)*

Solution. We have in any case that $(f(x,y))^2 \geq 0$ for all x, y, with equality if and only if $f(x,y) = 0$. Given that f is a polynomial (and a fortiori continuous), this means that $f(x,y) = 0$ if and only if $\iint_D f(x,y)^2 \, dA = 0$. Now write $f(x,y) = \sum_{i,j} a_{ij} x^i y^j$. Since integration is linear, we have

$$\begin{aligned} \iint_D f(x,y)^2 \, dA &= \sum_{i,j} a_{ij} \iint_D x^i y^j f(x,y) \, dA \\ &= \sum_{i,j} a_{ij} \mu_{i,j}. \end{aligned}$$

Now, if $\mu_{i,j} = 0$ for *all* i, j, then a fortiori $\mu_{i,j} = 0$ for all i, j appearing in the sum above. Hence the sum $\sum_{i,j} a_{ij} \mu_{i,j}$ must also be zero, and $f(x,y) = 0$ as well.

(b) *This part is a more challenging optional exercise. We include it for completeness, but you may skip it without penalty. If you complete it, we'll add one mark to your mark for the assignment, up to a maximum of 15/15.*

We now use a big theorem in analysis: the Stone-Weierstraß Theorem. It says the following: If f is a continuous function defined on a closed, bounded set $D \in \mathbb{R}^2$, then, for any $\epsilon > 0$, there exists a polynomial function g such that $|g(x,y) - f(x,y)| < \epsilon$. That is, you can approximate any continuous function arbitrarily well with a polynomial. Show how to use this and part (a) above to show that any continuous function defined on D whose moments are all zero must be identically zero on D.

Solution. Suppose that $f(x,y) \neq 0$ but that all of its moments $\mu_{i,j}$ are zero. Since D is closed and bounded, it has finite area, say k. Furthermore, since $f(x,y)$ is continuous, we can integrate f^2 over D and get $K := \iint_D (f(x,y))^2 \, dA$. Recall that the integral of the constant function $g(x,y) = \delta$ over D is just δk. Furthermore, if $g(x,y)$ is a function such that $0 < g(x,y) < \delta$ for all points (x,y) on D, then $\iint_D g(x,y) \, dA < \delta k$, too. Now, using the Stone-Weierstraß Theorem, pick some polynomial function $h(x,y)$ such that $|h(x,y) - f(x,y)| < \min\left\{1, \frac{K}{k}\right\}$ for all points (x,y) in D. Then $\iint_D |h(x,y) - f(x,y)| \, dA < K$. Futhermore, since $|h(x,y) - f(x,y)| < 1$, $(h(x,y) - f(x,y))^2 < |h(x,y) - f(x,y)|$ and so $\iint_D (h(x,y) - f(x,y))^2 \, dA < \iint_D |h(x,y) - f(x,y)| \, dA < K$. We then have that

$$\begin{aligned}\iint_D (h(x,y)-f(x,y))^2\ dA &= \iint_D h^2(x,y)-2h(x,y)f(x,y)+f^2(x,y)\ dA\\ &= \iint_D h^2(x,y)\ dA - 2\iint_D h(x,y)f(x,y)\ dA + \iint_D f^2(x,y)\ dA\\ &= \iint_D h^2(x,y)\ dA + \iint_D f^2(x,y)\ dA \quad \text{(Since all } \mu_{i,j}=0)\\ &\geq \iint_D f^2(x,y)\ dA \quad \text{(Since } (h(x,y))^2 \geq 0)\\ &= K \quad \text{(By definition).}\end{aligned}$$

So, we simultaneously have $\iint_D (h(x,y)-f(x,y))^2\ dA < K$ and $\iint_D (h(x,y)-f(x,y))^2\ dA \geq K$, which implies that $K < K$, which is nonsense. The only possibilities are that $f(x,y) = 0$ everywhere or that some moments $\mu_{i,j}$ are nonzero. This proves the claim.

(c) Use part (b) (and don't worry if you didn't do part (b)) to show the main result: if two continuous functions f and g have the same moments $\mu_{p,q}$ for all $p,q \geq 0$, then $f = g$.

Solution. If we have two continuous functions f and g, then the moments $\mu_{i,j}$ of the difference $f-g$ are just the differences $\mu^f_{i,j} = \mu^g_{i,j}$, where $\mu^f_{i,j}$ is the i,j-moment of f and $\mu^g_{i,j}$ is the i,j-moment of g. Those moments are all the same if and only if $\mu_{i,j} = 0$ for all i,j. By part (b), $\mu_{i,j} = 0$ for all i,j if and only if $f-g=0$, i.e. if and only if $f = g$.