UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
MAT 235 Y — CALCULUS II
FALL–WINTER 2007–2008
ASSIGNMENT #4, DUE ON January 10
PROBLEMS and SOLUTIONS

Note: Problems 1 and 3 are supposed to be hard. But, they are worth your effort!

1. (The gradient of a function) Consider a mountain totally covered with snow which has the height $M(x, y) = 10000 - 2x^2 - y^2$ at a point (x, y) on a map. Here the boundary of the mountain is the set $\{(x, y) \in \mathbf{R}^2 \mid 10000 - 2x^2 - y^2 = 0\}$. Suppose at the map location $(1, 1)$ (so the actual location on the mountain is $(1, 1, 9997)$) a big ice ball starts rolling down. It is a free motion which is determined totally by the slope of the surface. For example, the initial direction of the motion is completely determined by the slope of the mountain at the initial point $(1, 1)$. If you want to find the ice ball at the boundary of the mountain after certain time, where should you go to? Determine the (x, y)-coordinates of the location.

Hint: You want to find the path of the ice ball on the map and want to find the point where the path intersects the boundary of the mountain. (You only need to know the shape of path, not the actual motion in time.) For the path, first set up some differential equation by trying to relate the velocity of the path with the function $M(x, y)$. At certain moment in your calculation, you may want to use the following fact:

If $\frac{d}{ds}x(s) = Cx(s)$, C a constant, then $x(s) = x(0)e^{C\,s}$.

Solution. We know that the snow ball at (x, y) follows the steepest descent of the mountain, i.e. the direction of $-\nabla M(x, y)$. Therefore, the trajectory (on the map) of the path of the snow ball can be parametrized by $(x(s), y(s))$, $0 \leq s$, which satisfies

$$(\frac{d}{ds}x(s),\ \frac{d}{ds}y(s)) = -\nabla M(x(s), y(s)) = (4x(s),\ 2y(s)).$$

Thus we get two equations

$$\frac{d}{ds}x(s) = 4x(s),\ \frac{d}{ds}y(s) = 2y(s).$$

We now apply the fact in the hint to get

$$x(s) = x(0)e^{4s},\ y(s) = y(0)e^{2s}.$$

Using the initial condition $x(0) = 1, y(0) = 1$, we get

$$x(s) = e^{4s},\ y(s) = e^{2s}.$$

Now to find the intersection of this path with the boundary of the mountain $\{(x, y) \mid 10000 - 2x^2 - y^2 = 0\}$, we consider $10000 - 2x(s)^2 - y(s)^2 = 0$, i.e.

$$10000 - 2(e^{4s})^2 - (e^{2s})^2 = 0.$$

The last equation is a quadratic equation $10000 - 2X^2 - X = 0$ in $X = e^{4s}$, and it has the solution $X = \frac{-1\pm\sqrt{80001}}{4}$. Taking only positive value ($X = e^{4s}$), we get

$$e^{4s} = \frac{-1 + \sqrt{80001}}{4}.$$

From this, we see that the intersection point where the snowball hits the boundary is

$$\Big(\frac{-1+\sqrt{80001}}{4},\ \sqrt{\frac{-1+\sqrt{80001}}{4}}\Big).$$

This completes the solution for 1.

2. (Minimum of a function) Let $A(0,0)$, $B(6,0)$, $C(2,3)$ be three points in the xy-plane.

(a) Minimize the average distance squared from the three points A, B and C. More precisely, let $F(x,y) = d_A(x,y)^2 + d_B(x,y)^2 + d_C(x,y)^2$, where $d_A(x,y)$, $d_B(x,y)$, $d_C(x,y)$ denote the distance from (x,y) to the points A, B, and C, respectively. Find the point (x_m, y_m) where F attains its minimum.

(b) Minimize the average distance squared from the three points A, B and C, where the point (x,y) is restricted to the domain D defined as follows:

$$D := \{(x,y) \in \mathbf{R}^2 \mid (x-6)^2 + (y-3)^2 \leq 4\ ,\ x \leq 6,\ \text{and } y \leq 3\}.$$

Hint: You may want to use Lagrange multiplier method for the curved boundary of D.

Solution for 2 (a). First we find the critical (stationary) points. Notice that

$$d_A(x,y)^2 = x^2 + y^2,\ d_B(x,y)^2 = (x-6)^2 + y^2,\ d_C(x,y)^2 = (x-2)^2 + (y-3)^2.$$

Thus

$$\frac{\partial}{\partial x}F = 2x + 2(x-6) + 2(x-2) = 6x - 16$$
$$\frac{\partial}{\partial y}F = 2y + 2y + 2(y-3) = 6y - 6.$$

Solving $\nabla F = 0$, we get

$$x = \frac{16}{6} = \frac{8}{3},\quad y = 1.$$

It is clear that there should be a minimum point and since the function F is smooth, the minimum point should coincide the unique critical point $(\frac{8}{3}, 1)$, i.e.

$$x_m = \frac{8}{3},\ y_m = 1.$$

This completes the solution for 2 (a).

Solution for 2 (b). It's easy to check that the only critical point $(\frac{8}{3}, 1)$ of F does not belong to D. Thus we look for the boundary points. One sees that the boundary consists of three parts:

$$B_1 = \{(x,y) :\ 4 \leq x \leq 6,\ y = 3\}$$
$$B_2 = \{(x,y) :\ x = 6,\ 1 \leq y \leq 3\}$$
$$B_3 = \{(x,y) :\ (x-6)^2 + (y-3)^2 = 4,\ 4 \leq x \leq 6,\ 1 \leq y \leq 3\},$$

where the last set B_3 is nothing but the lower left quadrand of the circle $(x-6)^2+(y-3)^2 = 4$. We find the minimums of the function F when restricted on each B_1, B_2 and B_3. Checking all such minimum points we can find the minimum of F on D. Along B_1, there is no zero point of $\frac{\partial F}{\partial x}$, and thus the minimum of F restricted to B_1 occurs at the boundary points of the segment B_1, i.e. either at $(4,3)$ or $(6,3)$. By similar reasoning the minimum of F restricted to B_2 occurs at either $(6,1)$ or $(6,3)$. To consider the function F restricted to B_3, one use Lagrange multiplier method to F with the constraint $g(x,y) = (x-6)^2 + (y-3)^2 = 4$. The gradient equation $\nabla F = \lambda \nabla g$ reads

$$\begin{aligned} 6x - 16 &= 2\lambda(x-6), \\ 6y - 6 &= 2\lambda(y-3). \end{aligned}$$

Solving this we get

$$x = \frac{8-6\lambda}{3-\lambda}, \; y = \frac{3-3\lambda}{3-\lambda}.$$

Plug this back in the constraint equation $g = 4$ we get $\lambda = 3 \pm \sqrt{34}$, and thus

$$x = \mp\frac{-10 \mp 6\sqrt{34}}{\sqrt{34}}, \quad y = \mp\frac{-6 \mp 3\sqrt{34}}{\sqrt{34}},$$

however from the condition $4 \le x \le 6$ and $1 \le y \le 3$ on B_3, we choose only

$$x = \frac{-10+6\sqrt{34}}{\sqrt{34}} = 6 - \frac{10}{\sqrt{34}}, \quad y = \frac{-6+3\sqrt{34}}{\sqrt{34}} = 3 - \frac{6}{\sqrt{34}}.$$

(Note: To find this point, one may also try to use (instead of Lagrange multiplier method) a parametrization of B_3 as

$$B_3 := \{(2\cos\theta + 6, 2\sin\theta + 3) \mid \pi \le \theta \le \frac{3\pi}{2}\}.$$

Then one can consider the function F as a function of θ.)

We now have all the possible candidates of minimum point, $(4,3)$, $(6,3)$, $(6,1)$ and $(6 - \frac{10}{\sqrt{34}}, \, 3 - \frac{6}{\sqrt{34}})$. The values of F at these points are

$$F(4,3) = 42, \; F(6,3) = 70, \; F(6,1) = 57, \; F((6 - \frac{10}{\sqrt{34}}, \; 3 - \frac{6}{\sqrt{34}}) = 61 + \frac{381}{34} - \frac{236}{\sqrt{34}}.$$

Thus, we conclude that $(6 - \frac{10}{\sqrt{34}}, \, 3 - \frac{6}{\sqrt{34}})$ is the minimum point. This completes the solution for 2 (b).

3. (Minimum of a function) Let $A(-1,1)$, $B(1,2)$ be two points in the xy-plane. Let L denote the x-axis. Minimize the average distance from A, B, and the line L. More precisely, let $d_A(x,y)$, $d_B(x,y)$, and $d_L(x,y)$ denote the distance (not squared) from (x,y) to the points A, B, and the line L respectively. Find the minimum point (x_m, y_m) of the function $\delta(x,y) = d_A(x,y) + d_B(x,y) + d_L(x,y)$ following the outline below.

(a) For each functions d_A, d_B, d_L, find the gradient and the set of *singular points*, i.e the set of points where the gradient is not defined. Give an interpretation of the direction and the magnitude of the gradient vector for each d_A, d_B and d_L.

(b) Use (a) to find the critical points of $\delta(x,y)$, i.e. the point where $\nabla\delta(x,y) = 0$.

(c) Check the values of $\delta(x,y)$ at its singular points. Compare them with the value (or values) of the critical points. Make conclusion.

Solution for 3 (a). Distance function $d(x, y)$ from a point (a, b) has the expression

$$d(x, y) = \sqrt{(x-a)^2 + (y-b)^2}.$$

From this expression, it is clear that the function has gradient

$$\nabla d(x, y) = \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} \langle x-a, y-b \rangle.$$

Note that the magnitude is 1 and the direction of this vector is the direction from the point (a, b) to (x, y). This result is obvious because in this direction the distance from (a, b) increase the fastest and the rate of increase of the distance function with respect to the distance the point moves should be $\frac{distance}{distance} = 1$. This result for distance function applies to d_A and d_B. Thus,

$$\nabla d_A(x, y) = \frac{1}{\sqrt{(x+1)^2 + (y-1)^2}} \langle x+1, y-1 \rangle,$$
$$\nabla d_B(x, y) = \frac{1}{\sqrt{(x-1)^2 + (y-2)^2}} \langle x-1, y-2 \rangle.$$

The singular set of d_A is $\{(-1, 1)\}$, and the singular set of d_B is $\{(1, 2)\}$.
It is obvious $d_L(x, y) = |y|$ and

$$\nabla d_L(x, y) = \langle 0, 1 \rangle \text{ for } y > 0;$$
$$\nabla d_L(x, y) = \langle 0, -1 \rangle \text{ for } y < 0.$$

The direction of ∇d_L is orthogonal to the line L. The magnitude of ∇d_L is 1.
It is also easy to see that the whole line L is the singular set of d_L.

Solution for 3 (b). The critical point of $\delta(x, y)$ is such a point (x, y) satisfying

$$0 = \nabla \delta(x,) = \nabla d_A(x, y) + \nabla d_B(x, y) + \nabla d_L(x, y).$$

Solving this equation algebraically would be very hard. Thus we proceed with a geometric reasoning. At a critical point, those three vectors ∇d_A, ∇d_B and ∇d_L sum up to a zero vector. Notice that the gradient vectors ∇d_A, ∇d_B, ∇d_L are unit vectors. To make zero vector from sum of three unit vectors, the only possible case is that these three vectors form 120° with consecutive ones. (Draw the figure!) Note ∇d_L is either $\langle 0, 1 \rangle$ or $\langle 0, -1 \rangle$. Therefore, the only possible directions for ∇d_A and ∇d_B at a critical point are

$$\pm\langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle, \quad \pm\langle \tfrac{1}{2}, -\tfrac{\sqrt{3}}{2} \rangle$$

By the property of the direction of the gradient of a distance function from a given point (see Solution for 3 (a)), we see that the critical point should lie in the union of lines

$$\{(-1, 1) + t(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) \mid -\infty < t < \infty\}$$

and also simultaneously in

$$\{(1, 2) + s(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) \mid -\infty < s < \infty\}$$

(To understand this, draw a figure in the xy-plane.) Therefore the possible critical points are the intersection points of these sets, i.e

$$P_1 = (\frac{\sqrt{3}}{6},\ \frac{3}{2} + \sqrt{3}), \quad P_2 = (-\frac{\sqrt{3}}{6},\ \frac{3}{2} - \sqrt{3}).$$

Examining the gradient vectors of d_A, d_B, and d_L, one can conclude that the point $P_2 = (-\frac{\sqrt{3}}{6},\ \frac{3}{2} - \sqrt{3})$ is the unique critical point of $\delta(x, y)$. This completes the solution for 3 (b).

Solution for 3 (c). The singular set of the function $\delta(x, y)$ is the union of the singular sets of d_A, d_B and d_L, i.e.

$$\{(-1, 1)\} \cup \{(1, 2)\} \cup \{(x, 0) \mid -\infty < x < \infty\},$$

where $\cup$ denote the union of sets. Now, one can easily check

$$\begin{aligned} \delta(-1, 1) &= 1 + \sqrt{5}, \\ \delta(1, 2) &= 2 + \sqrt{5}, \\ \delta(x, 0) &= d_A(x, 0) + d_B(x, 0) = \sqrt{(x+1)^2 + 1} + \sqrt{(x-1)^2 + 4} \end{aligned}$$

Note that at the critical point $P_2 = (-\frac{\sqrt{3}}{6},\ \frac{3}{2} - \sqrt{3})$, the value

$$\delta(-\frac{\sqrt{3}}{6},\ \frac{3}{2} - \sqrt{3})\ >\ \delta(-\frac{\sqrt{3}}{6},\ 0) \geq\ \text{(minimum of } \delta(x, 0) \text{ on the } x\text{-axix)}\ .$$

This last fact can be easily checked by drawing the three points A, B, P_2 and the line L. From P_2 along the vertical line $(-\frac{\sqrt{3}}{6}, t)$, $\frac{3}{2} - \sqrt{3} \leq t \leq 0$, toward the line L(= the x-axis). It is clear the all distances d_A, d_B and d_L decrease until the segment touches the line L (= x-axis). Therefore, to find the minimum point it suffices to compare the values of $\delta(x, y)$ at $A = (-1, 1)$, $B = (1, 2)$ with the minimum of $\delta(x, 0)$ along the x-axis. Let

$$f(x) = \delta(x, 0) = d_A(x, 0) + d_B(x, 0).$$

The minimum point of f should be a critical point (f is differentiable everywhere). However, trying to find the zero of

$$\frac{df}{dx} = \frac{x+1}{\sqrt{(x+1)^2 + 1}} + \frac{x-1}{\sqrt{(x-1)^2 + 4}}.$$

seems not obvious. There is at least one way of doing it algebraically: try to solve

$$\frac{x+1}{\sqrt{(x+1)^2 + 1}} = -\frac{x-1}{\sqrt{(x-1)^2 + 4}},$$

by squaring both sides. To show an alternative method, we again follow the geometric reasoning as we did in Solution 2 (b). Suppose $(x_0, 0)$ be a critical point of f, i.e. $\frac{d}{dx}f(x_0) = 0$. We want to draw some geometric property of this point $(x_0, 0)$. At $(x_0, 0)$, the two unit vectors ∇d_A and ∇d_B should be have the opposite projection to the x-axis. In other words, for the two line segments l_A from $(x_0, 0)$ to A and l_B from $(x_0, 0)$ to B have the same angle (but in opposite side) from the x-axis. The broken trajectory from A to B following these line segments resembles the motion of a rigid ball which goes from A to B bouncing from the wall L (= x-axis). We reflect the picture with respect to the x-axis, then the broken trajectory is reflected to a straight line from A to the point $B' = (1, -2)$, the reflected image of B. Of course the point $(x_0, 0)$ remains fixed under this reflection and so $(x_0, 0)$ is nothing but the intersection point of between the x-axis and the line segment from A to B'. This intersection point is easily found to be $(-\frac{1}{3}, 0)$. Thus the minimum of f occurs at $x = -\frac{1}{3}$ (one can check that $-\frac{1}{3}$ is indeed the zero of $\frac{df}{dx}$ and its value is $f(-\frac{1}{3}) = \sqrt{13}$. Compare this value $\sqrt{13}$ with $\delta(-1, 1) = 1 + \sqrt{5}$, $\delta(1, 2) = 2 + \sqrt{5}$, the smallest one is $1 + \sqrt{5}$. Thus, we conclude that the point $A = (-1, 1)$ is the minimum point (x_m, y_m) of $\delta(x, y)$. This completes the solution for 3 (c).

4.

(a) **(local minimum, local maximum, and saddle point)** For the function $f(x, y) = x^4 - xy + y^4$, find all the critical points (the points where the gradient vanishes). Determine for each critical point whether it is a local minimum, a local maximum or a saddle point.

(b) **(absolute maximum and minimum)** Find the absolute maximum and minimum values of f in (a) if there are any. Justify your answer.

(c) **(Least squares)** What numbers x, y come closest to satisfying the three equations $x - y = 1$, $2x + y = -1$, $x + 2y = 1$? Square and add the errors, $(x - y - 1)^2 + A + B$. Then minimize. (Find appropriate A and B, and then finish the minimization.)

Solution for 4 (a). To find critical points, solve

$$\nabla f(x, y) = \langle 4x^3 - y, 4y^3 - x \rangle = 0.$$

Then, $4(4x^3)^3 = x$ and thus $x = \pm\frac{1}{2}, 0$. Similarly, $y = \pm\frac{1}{2}, 0$. (One can also get this by using symmetry of the equation $\nabla f = 0$ with respect to x and y.) By $4x^3 - y = 0$ we see that both x and y have the same sign. Therefore, all the critical points are $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$. The discriminant

$$\begin{aligned} D &= f_{xx} f_{yy} - f_{xy}^2 = 12x^2 12y^2 - (-1)^2 \\ &= 144x^2y^2 - 1. \end{aligned}$$

For the critical point $(0, 0)$, $D(0, 0) = -1 < 0$. Thus $(0, 0)$ is a saddle point. For both critical points $(\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$,

$$D = 144\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{2}\Big)^2 - 1 > 0.$$

Thus, the critical points $(\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$ are local minimum.

Solution for 4 (b). From (a), the critical point $(0, 0)$ is neither local maximum nor local minimum. Thus, we exclude this point for absolute maximum and minimum. The other critical points $(\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{2})$ are local minimum and

$$f(\frac{1}{2}, \frac{1}{2}) = -\frac{1}{8} = f(-\frac{1}{2}, -\frac{1}{2}).$$

As $(x, y) \to (\pm\infty, \pm\infty)$, $f(x, y) \to +\infty$ since the powers x^4, y^4 increases much faster than xy. Therefore, there is no absolute maximum of this function. The same asymptotic behavior tells us that f should have the absolute minimum: if (x, y) is far away enough from $(0, 0)$, then $f(x, y)$ is big enough, for example, $f(x, y) \geq 0$. Therefore, both local minima $(\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{2})$ should be the absolute minima!

Solution for 4 (c). $A = (2x+y+1)^2$ and $B = (x+2y-1)^2$. Thus, we want to minimize

$$f(x,y) = (x-y-1)^2 + (2x+y+1)^2 + (x+2y-1)^2.$$

Take the gradient

$$\begin{aligned}&\nabla f(x,y)\\ &= \langle 2(x-y-1) + 4(2x+y+1) + 2(x+2y-1),\\ &\quad -2(x-y-1) + (2x+y+1) + 4(x+2y-1)\rangle\\ &= \langle 12x+6y+1, 4x+11y-1\rangle.\end{aligned}$$

To find the critical point (which is the unique minimum point for the squared sum f), solve $\nabla f = 0$, and get

$$x = -\frac{17}{108},\ y = \frac{4}{27}.$$

This completes the solution for 4 (c).

5. (Lagrange Multipliers) For (a) & (b) use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraints.

(a) $f(x,y) = \frac{1}{x} + \frac{1}{y}$. Constraint: $\frac{1}{x^2} + \frac{1}{y^2} = 1$.

(b) $f(x,y,z) = x^2 + 2y^2 + 3z^2$. Constraint: $x+y+z = 1$ and $x - y + 2z = 2$.

(c) A package in the shape of a rectangular box can be mailed by the Toronto-Imaginary Post Service if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108 cm. Find the dimensions of the package with largest volume that can be mailed.

Solution for 5 (a). The best way to deal with this question is to change the variables. Let $a = \frac{1}{x}$ and $b = \frac{1}{y}$. Then we rewrite

$$F(a,b) = a+b. \text{ Constraint: } G(a,g) = a^2+b^2 = 1.$$

Since the circle $a^2+b^2=1$ is closed and bounded, the function F restricted to this circle has both maximum and minimum. Lagrange multiplier method leads to the equation

$$\nabla F = \lambda \nabla G,$$

i.e.

$$-1 = -2\lambda a, \quad 1 = 2\lambda b.$$

Simplify these and get

$$a = \frac{1}{2\lambda} = b.$$

Plug-in this to $G = 1$, and get

$$\lambda = \pm\frac{\sqrt{2}}{2}, \text{ and thus } a = \pm\tfrac{1}{\sqrt{2}} = b.$$

We so have two critical points $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. They are either max. or min. points. Compute

$$F(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \sqrt{2}, \quad F(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -\sqrt{2}.$$

Thus,$\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is the maximum point and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ is the minimum point. (But, in the (a, b)-coordinates!) Back to the original coordinate $(x, y) = (\sqrt{2}, \sqrt{2})$ is the maximum with the value $f\sqrt{2}, \sqrt{2}) = \sqrt{2}$, and $(x, y) = (-\sqrt{2}, -\sqrt{2})$ is the minimum with the value $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.

Solution for 5 (b). Let $g(x, y, z) = x + y + z$ and $h(x, y, z) = x - y + 2z$. Lagrange multiplier method leads to the following equation:

$$\nabla f = \lambda \nabla g + \mu \nabla h,$$

which implies

$$x = \frac{\lambda + \mu}{2}, \quad y = \frac{\lambda - \mu}{4}, \quad z = \frac{\lambda + 2\mu}{6}.$$

Plug-in these last equations to the constraint equations $g = 1$ and $h = 2$, then we get

$$\frac{\lambda + \mu}{2} + \frac{\lambda - \mu}{4} + \frac{\lambda + 2\mu}{6} = 1,$$
$$\frac{\lambda + \mu}{2} - \frac{\lambda - \mu}{4} + \frac{\lambda + 2\mu}{3} = 2.$$

Solve this system of equations to get

$$\lambda = \frac{6}{23}, \quad \mu = \frac{30}{23}.$$

Thus,

$$x = \frac{18}{23}, \quad y = -\frac{6}{23}, \quad z = \frac{11}{23}.$$

Thus the point $(\frac{18}{23}, -\frac{6}{23}, \frac{11}{123})$ is the unique critical point of the function f restricted to the constraint (in this case planes) $g = 1$ and $h = 2$. This restricted function should have a minimum point (since f is bounded from below $f \geq 0$) and such a minimum point should be a critical point. Thus we conclude $(\frac{18}{23}, -\frac{6}{23}, \frac{11}{123})$ is indeed the minimum point. (Note: There is no maximum point from this conclusion. Notice that the intersection of these two constraint planes is a line and along this line the function can increase to $+\infty$.)

Solution for 5 (c). Let x, y, z denote the length, width, and height, respectively. The constraint is $x + 2y + 2z \leq 108$. It is obvious that the volume xyz is largest when the equality $x + 2y + 2z = 108$ holds in the constraint. Thus, we have the maximization problem of

$$f(x, y, z) = xyz, \quad \text{Constraint: } g(x, y, z) = x + 2y + 2z = 108.$$

Lagrange multiplier method leads

$$\nabla f = \lambda \nabla g,$$

i.e.

$$yz = \lambda, \quad xz = 2\lambda, \quad xy = 2\lambda.$$

From this, we see $x = 2y = 2z$ (by dividing the equations with the other). Back to the constraint $g = 108$, $x = 2y = 2z = 36$. The maximum volume is then $36 \cdot 18 \cdot 18 = 11664$.