

# MAT235Y Problem Set 3

## Model Solutions

November 16, 2007

### 1. A trip through the mountains

Bill and Jane are attending a math conference in the mountains (For some unknown reason, math conferences tend to be held in mountainous areas in the middle of nowhere. It remains one of life's great mysteries.), taking a walk through some rough terrain. It's getting cold and dark and they decide that they should head back to the conference centre. But they need to decide on a good route. Being good mathematicians, they have approximated the terrain as a bivariate function. If  $F(x, y)$  is the function that assigns to the point  $(x, y)$  the elevation of the ground at that point, then

$$F(x, y) = 4e^{-(x/4)^2 - (y/4)^2}.$$

(a) Jane is concerned that it's getting cold. Thinking about how this might affect their trip back to the conference centre, Bill suggests modelling the temperature in the following way:

- The temperature at the point  $(0,0,0)$  is 15 degrees C. It falls by one degree for every unit of elevation.
- It is slightly warmer to the northwest: on level terrain, the temperature at elevation zero varies linearly with the  $x$  and  $y$  coordinates in such a way that it decreases by five degrees C for each unit in the positive  $x$  direction and by one degree C for each unit in the positive  $y$  direction.
- The temperature is then determined by the sum of the effects of elevation and of position.

Write down the definition of  $T(x, y)$ , which assigns to the point  $(x, y)$  the temperature at that point.

**Solution.** Given the aforementioned conditions, the effect of elevation on temperature can be modelled as

$$T_e(x, y) = 15 - F(x, y),$$

while the effect of position on temperature can be modelled as

$$T_p(x, y) = -5x - y.$$

Combining these effects, we have

$$T(x, y) = -F(x, y) - 5x - y + 15$$

or

$$T(x, y) = -4e^{-(x/4)^2 - (y/4)^2} - 5x - y + 15.$$

- (b) Now let's suppose that Bill and Jane are at point (5,-1) and walk due westwards (i.e. parallel to the  $x$  axis in the negative direction). How quickly will the temperature that they experience change as they start walking?

**Solution.** The rate at which the temperature will change is determined by the partial derivative with respect to  $x$  of  $T(x, y)$  at (5, -1). We have

$$\begin{aligned}\frac{\partial}{\partial x} T(x, y) &= -4e^{-(x/4)^2 - (y/4)^2} \left(-\frac{1}{16} 2x\right) - 5 \\ &= \frac{1}{2} x e^{-(x/4)^2 - (y/4)^2} - 5.\end{aligned}$$

Evaluating this at  $(x, y) = (5, -1)$ , we have

$$\begin{aligned}T_x(5, -1) &= \frac{1}{2} 5 e^{-(5/4)^2 - (-1/4)^2} - 5 \\ &= \frac{5}{2} e^{-26/16} - 5 \\ &\approx -4.41.\end{aligned}$$

Taking into account that Bill and Jane are walking in the *negative* direction, we see that the temperature is increasing by 4.41 degrees per unit.

- (c) Bill does not want to ascend at a rate of more than one unit for each two units of distance, nor

to descend at a rate of more than one metre for every one metre of distance. If they keep walking westwards, will they encounter difficulties with the terrain?

**Solution.** This question addresses the magnitude of the partial derivative of  $F(x, y)$  with respect to  $x$ , restricted to the line  $y = -1$ . Taking into account that they are walking in the *negative* direction, the question is whether  $F_x(x, -1)$  is outside the interval  $[-1/2, 1]$  for  $x \leq 5$ . We have

$$\frac{\partial}{\partial x} F(x, y) = -\frac{1}{2} x e^{-(x/4)^2 - (y/4)^2}.$$

Thus

$$F_x(x, -1) = -\frac{1}{2} x e^{-(x/4)^2 - 1/16}.$$

Let's first check the endpoint (5,-1):

$$\begin{aligned} F_x(5, -1) &= -\frac{1}{2} 5 e^{-(5/4)^2 - 1/16} \\ &\approx -0.59. \end{aligned}$$

Here we already see that the terrain is too steep. Thus Bill and Jane will have difficulties immediately.

- (d) In what direction should Bill and Jane begin to walk in order that the temperature initially rise most quickly?

**Solution.** The direction in which temperature rises most quickly is given by the gradient of  $T(x, y)$ . We have

$$\begin{aligned} \nabla T(x, y) &= \frac{\partial}{\partial x} T(x, y) \vec{i} + \frac{\partial}{\partial y} T(x, y) \vec{j} \\ &= \left( \frac{1}{2} x e^{-(x/4)^2 - (y/4)^2} - 5 \right) \vec{i} + \left( \frac{1}{2} y e^{-(x/4)^2 - (y/4)^2} - 1 \right) \vec{j}. \end{aligned}$$

Thus we have

$$\nabla T(5, -1) = \left( \frac{5}{2} e^{-26/16} - 5 \right) \vec{i} + \left( -\frac{1}{2} e^{-26/16} - 1 \right) \vec{j} \approx -4.41 \vec{i} - 1.10 \vec{j}.$$

So Bill and Jane should walk in the direction  $(-4.41, -1.10)$ , approximately west-southwest.

## 2. A mad scientist's architectural creation

The mad scientist Dr. Nenivoh, working on contract with Vandelay Industries, Inc. just designed a new Museum of Geometric Paradoxes, to be filled with exhibitions of surfaces that can't actually exist in our universe. In a feat of brilliance, he decided that the roof profile of the museum should be the set of points satisfying the equation  $x^2 = y^2z^2 - z^3$ , where  $z$  is the vertical axis and  $x$  and  $y$  are the horizontal axes. The coordinates are such that the ground is the plane  $z = -5$ . His assistant, Dr. Oicer, then reminded him that their clients wanted an antenna pointed straight up at a place where it would be orthogonal to the surface of the building.

(a) Is there any suitable location for the antenna?

**Solution.** The question essentially asks whether there is a point  $P = (x_0, y_0, z_0)$  on the given surface that has a horizontal tangent plane. Let  $F(x, y, z) = x^2 - y^2z^2 - z^3$ . It is clear that  $F$ ,  $F_x$ ,  $F_y$ , and  $F_z$  exist and are continuous everywhere. Furthermore, supposing  $P$  exists,  $F_z(x_0, y_0, z_0) \neq 0$  since otherwise the surface would have a vertical tangent at that point. Thus we may apply the implicit function theorem so that, near  $P$ ,  $F(x, y, z) = 0$  is the graph  $z = f(x, y)$  of some function  $f$ . This allows us to use implicit differentiation.

First, let us identify points where  $F_z(x, y, z) = 0$ , since those points cannot be candidates. We have

$$\frac{\partial}{\partial z}F(x, y, z) = -2zy^2 - 3z^2 = -z(2y^2 + 3z).$$

Hence the loci  $\{z = 0\}$  and  $\{z = -\frac{2}{3}y^2\}$  are not allowed.

Now, let us differentiate the equation  $F(x, y, z) = 0$  with respect to  $x$  and  $y$ :

$$\begin{aligned} 0 &= 2x - 2y^2z \frac{\partial z}{\partial x} - 3z^2 \frac{\partial z}{\partial x}, \\ 0 &= -2yz^2 - 2y^2z \frac{\partial z}{\partial y} - 3z^2 \frac{\partial z}{\partial y}. \end{aligned}$$

The condition that the tangent plane be horizontal requires that  $\frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y}$ . Solving for  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ , we obtain

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{2x}{2y^2z + 3z^2}, \\ \frac{\partial z}{\partial y} &= \frac{-2yz^2}{2y^2z + 3z^2}. \end{aligned}$$

Hence  $x = 0$ . If  $x = 0$ , then the condition  $F(x, y, z) = x^2 - y^2z^2 - z^3 = x^2 - z^2(y^2 + z)$  implies

that  $z = 0$  or  $z = -y^2$ . But  $z$  may not be zero, as stated above. This leaves  $z = -y^2$ . But the condition  $2yz^2 = 0$  implies that  $y = 0$  or  $z = 0$ . If  $z = -y^2$ , then both  $y$  and  $z$  vanish as soon as one of them does, and we again are left with the disallowed case  $z = 0$ . Thus we conclude that there is no point on the roof where the antenna may be mounted.

- (b) Upon reviewing the specifications, Vandelay Industries determined that the antenna should in fact not point straight up but should instead be 30 degrees from the vertical, oriented in the positive  $y$  direction. It should still be situated so as to be orthogonal to the surface of the roof. Where on the surface should the antenna go?

**Solution.** As with part (a) above, we can use the implicit function theorem and implicit differentiation. We'll skip the first details and go right to the formulae

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{2x}{2y^2z + 3z^2}, \\ \frac{\partial z}{\partial y} &= \frac{-2yz^2}{2y^2z + 3z^2}.\end{aligned}$$

The instructions on the position of the antenna imply that  $\frac{\partial z}{\partial x} = 0$  and  $\frac{\partial z}{\partial y} = \arctan -30^\circ = -\frac{1}{\sqrt{3}}$ .

So the equations become

$$\begin{aligned}0 &= \frac{2x}{2y^2z + 3z^2}, \\ -\frac{1}{\sqrt{3}} &= \frac{-2yz^2}{2y^2z + 3z^2}.\end{aligned}$$

Put another way,

$$\begin{aligned}x &= 0, \\ 2y^2z + 3z^2 - 2\sqrt{3}yz^2 &= 0.\end{aligned}$$

In particular, we get  $0 = z(2y^2 - \sqrt{3}2yz + 3z)$ . But, as above,  $z = 0$  is not allowed, so  $2y^2 - 2\sqrt{3}yz + 3z = 0$ . Furthermore, since  $x = 0$  and  $F(x, y, z) = 0$ , we have  $z^2(y^2 - z) = y^2z^2 - z^3 = 0$ , and since again  $z \neq 0$ ,  $y^2 = z$ . Hence  $0 = 2z - 2\sqrt{3}yz + 3z = 5z + 2\sqrt{3}yz = z(5 + 2\sqrt{3}y)$ . Once again appealing to the fact that  $z \neq 0$ , we have  $y = -\frac{2\sqrt{3}}{5}$ . Since  $y^2 = z$ ,  $z = -\frac{12}{5}$ . Thus the required point is

$$\left(0, -\frac{2\sqrt{3}}{5}, -\frac{12}{5}\right)$$

### 3. The heat equation

In this problem we will study the evolution of the temperature in a one-dimensional rod with time. Suppose that the rod is embedded in  $\mathbb{R}$  as the interval  $[0, 1]$ . Denote by  $u(t, x)$  the temperature at time  $t$  of the point  $x$ . We can determine  $u$  using a partial differential equation that relates  $u_t := \frac{\partial}{\partial t}u$  with  $u_{xx} := \frac{\partial^2}{\partial x^2}u$ . Namely,

$$u_t = ku_{xx}. \quad (1)$$

The parameter  $k$  is a constant depending on the material, called *thermal conductance*.

- (a) Let  $u(0, x) = \sin(\pi x)$ . Describe in words how the temperature is distributed in the body at time  $t = 0$ .

**Solution.** The temperature appears as the hump of a sine wave, namely it is higher in the centre of the rod and tapers to zero at the ends.

- (b) Verify that  $u(t, x) = \sin(\pi x)e^{-\pi^2 kt}$  is a solution to equation (1) and that  $u(0, x) = \sin(\pi x)$ .

**Solution.** First, the easy part:

$$u(0, x) = \sin(\pi x)e^{-\pi^2 k \cdot 0} = \sin(\pi x)$$

. We have that

$$u_t = \sin(\pi x)(-\pi^2 k)e^{-\pi^2 kt} = -\pi^2 k \sin(\pi x)e^{-\pi^2 kt}.$$

Furthermore

$$u_x = \pi \cos(\pi x)e^{-\pi^2 kt},$$

so

$$u_{xx} = -\pi^2 \sin(\pi x)e^{-\pi^2 kt}.$$

The reader may now easily see that  $u_t = ku_{xx}$ , as required.

- (c) Define  $F(t, x) := |u_x(t, x)|$ . Using the solution  $u$  above, write down a formula for  $F(t, x)$ .

**Solution.** Using the above calculations, we have

$$\begin{aligned} F(t, x) &= |u_x(t, x)| \\ &= |\pi \cos(\pi x) e^{-\pi^2 kt}|. \end{aligned}$$

(d) Describe (with justification) the behaviour of  $F$  as  $t \rightarrow \infty$ . Explain in words what this means.

**Solution.** As  $t \rightarrow \infty$ ,  $e^{-\pi^2 kt} \rightarrow 0$  while  $\pi \cos(\pi x)$  remains bounded for all values of  $x$ . Thus, for any  $x$ ,  $\lim_{t \rightarrow \infty} F(t, x) = 0$  and hence, as a function  $\lim_{t \rightarrow \infty} F(t, x) = 0$ . The quantity  $F(t, x)$  measures how quickly the heat in the rod changes from one point to the next at a given point in time. The observation that  $\lim_{t \rightarrow \infty} F(t, x) = 0$  is just saying that, as time evolves, the heat diffuses through the rod and becomes ever more evenly distributed.

#### 4. Reflection properties

(a) Prove the following reflection property of prolate spheroids: Given an ellipsoid embedded in space as  $\left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1 \right\}$  with  $c > a$  (note the middle term!), show that there exist two points  $F_1$  and  $F_2$  (the *foci*) on the interior of the ellipsoid such that a particle ejected from  $F_1$  in any direction will bounce off the surface of the ellipsoid in such a manner that it will hit  $F_2$ .

**Solution.** Let  $E$  be the ellipsoid  $\left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1 \right\}$ . If we slice  $E$  by intersecting it with, say, the  $x-z$  plane, we get the ellipse  $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ . This ellipse satisfies the usual elliptic reflection property. Thus, if the points  $F_1$  and  $F_2$  exist, it is clear what they must be: setting  $b^2 := c^2 - a^2$ , the points are  $F_1 := (0, 0, -b)$  and  $F_2 := (0, 0, b)$ . Let us now prove that those two points work.

Fix a point  $P = (x, y, z)$  on  $E$ . It suffices to show that the lines  $\overline{PF_1}$  and  $\overline{PF_2}$  make the same angle with the plane tangent to  $E$  and that the plane containing  $P$ ,  $F_1$ , and  $F_2$  contains the vector normal to  $E$  at  $P$ . Let us first calculate the vector normal to  $E$  at  $P$ . Assume first that

$$\frac{\partial}{\partial z} \left( \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} - 1 \right) \neq 0,$$

so that we may apply the implicit function theorem, viewing  $z$  as a function of  $x$  and  $y$ . We have

$$\begin{aligned}\frac{\partial}{\partial x} \left( \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} - 1 \right) &= \frac{2x}{a^2} + \frac{2z}{c^2} \frac{\partial z}{\partial x} \\ \frac{\partial}{\partial y} \left( \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} - 1 \right) &= \frac{2y}{a^2} + \frac{2z}{c^2} \frac{\partial z}{\partial y}.\end{aligned}$$

Setting the above equal to zero and solving for  $\frac{\partial z}{\partial x}$ , respectively  $\frac{\partial z}{\partial y}$ , we have

$$\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{c^2}{a^2} \frac{x}{z}, \\ \frac{\partial z}{\partial y} &= -\frac{c^2}{a^2} \frac{y}{z}.\end{aligned}$$

Hence a normal vector at  $P$  is

$$\left( \frac{c^2}{a^2} \frac{x}{z}, \frac{c^2}{a^2} \frac{y}{z}, 1 \right).$$

After scaling by  $\frac{z}{c^2}$ , we obtain a normal form

$$\vec{n} = \left( \frac{x}{a^2}, \frac{y}{a^2}, \frac{z}{c^2} \right). \quad (2)$$

Note that (2) is valid at any point  $(x, y, z)$ . The reader may easily verify that it indeed gives the normal vector at any point  $(x, y, z)$  on  $E$ , even those for which the partial derivative with respect to  $z$  vanishes. Just observe that at least one partial derivative must not vanish and thus the implicit function theorem can be applied to treat one of the variables as a function of the other two. It doesn't matter which variable is chosen – any choice, properly renormalized, will yield the above normal vector.

Now let  $\vec{v}_1 := P\vec{F}_1 = (x, y, z + b)$  and let  $\vec{v}_2 := P\vec{F}_2 = (x, y, z - b)$ . Denote the angle between  $\vec{v}_1$  and  $n$  by  $\theta_1$  and denote the angle between  $\vec{v}_2$  and  $n$  by  $\theta_2$ . We need to show that  $\theta_1 = \theta_2$ . Recall the angle formula for dot products

$$\cos \theta_1 = \frac{\vec{v}_1 \cdot \vec{n}}{\|\vec{v}_1\| \|\vec{n}\|}.$$

Thus we want that

$$\frac{\vec{v}_1 \cdot \vec{n}}{\|\vec{v}_1\| \|\vec{n}\|} = \frac{\vec{v}_2 \cdot \vec{n}}{\|\vec{v}_2\| \|\vec{n}\|}.$$

We compute

$$\begin{aligned}
\vec{v}_1 \cdot \vec{n} &= \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} + \frac{zb}{c^2} \\
&= 1 + \frac{zb}{c^2}, \\
\vec{v}_2 \cdot \vec{n} &= \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} - \frac{zb}{c^2} \\
&= 1 - \frac{zb}{c^2}.
\end{aligned}$$

Here we have used the fact that  $P$  lies on  $E$ , and thus satisfies its defining equation.

Rewriting slightly, we want to show that

$$(c^2 + zb)\sqrt{x^2 + y^2 + (z - b)^2} = (c^2 - zb)\sqrt{x^2 + y^2 + (z + b)^2}.$$

We'll first square both sides and write down the difference of the left and the right (which we want to show is zero):

$$D := (c^2 + zb)^2(x^2 + y^2 + (z - b)^2) - (c^2 - zb)^2(x^2 + y^2 + (z + b)^2).$$

Regrouping the terms gives

$$D = ((c^2 + zb)^2 - (c^2 - zb)^2)(x^2 + y^2) + (c^2 + zb)^2(z - b)^2 - (c^2 - zb)^2(z + b)^2.$$

There are now two terms that are differences of two squares, so we can factor and simplify

$$\begin{aligned}
D &= ((c^2 + zb) + (c^2 - zb))((c^2 + zb) - (c^2 - zb))(x^2 + y^2) + (c^2 - zb)^2(x^2 + y^2 + (z + b)^2) \\
&= 4c^2zb(x^2 + y^2) + ((c^2 + zb)(z - b) + (c^2 - zb)(z + b))((c^2 + zb)(z - b) - (c^2 - zb)(z + b)) \\
&= 4c^2zb(x^2 + y^2) + (2c^2z - 2zb^2)(2z^2b - 2bc^2) \\
&= 4c^2zb(x^2 + y^2) + 4bz(c^2 - b^2)(z^2 - c^2) \\
&= 4bz(c^2(x^2 + y^2) + (c^2 - b^2)(z^2 - c^2)).
\end{aligned}$$

Recall now that  $a^2 = c^2 - b^2$ . Furthermore, since  $P$  lies on  $E$ , we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1,$$

or, rewriting slightly,

$$c^2(x^2 + y^2) + a^2z^2 = a^2c^2.$$

Putting these facts together, we see that

$$\begin{aligned} D &= 4bz(c^2(x^2 + y^2) + a^2z^2 - a^2c^2) \\ &= 4bz(a^2c^2 - a^2c^2) \\ &= 0. \end{aligned}$$

Going backwards, it follows that

$$(c^2 + zb)\sqrt{x^2 + y^2 + (z - b)^2} = \pm(c^2 - zb)\sqrt{x^2 + y^2 + (z + b)^2}.$$

But  $b < c$  (by the way  $b$  was defined) and  $|z| < c$  (as is generally true for an ellipsoid – just look at the defining equation for a moment!), so  $c^2 - zb$  and  $c^2 + zb$  are both positive. Square roots are positive as well, so both sides of the equation are positive and indeed

$$(c^2 + zb)\sqrt{x^2 + y^2 + (z - b)^2} = (c^2 - zb)\sqrt{x^2 + y^2 + (z + b)^2},$$

as required.

One other item remains to be shown: That the plane containing  $P$ ,  $F_1$ , and  $F_2$  (hence the vectors  $\vec{v}_1$  and  $\vec{v}_2$ ) also contains the normal vector  $\vec{n}$ . This part is easy: denoting for a moment the components of  $P$  as  $(r, s, t)$ , the equation of the aforementioned plane is  $sx - ry = 0$  (as the reader may easily verify) and the normal vector is  $(\frac{r}{a^2}, \frac{s}{a^2}, \frac{t}{c^2})$ . Clearly the components of the normal vector satisfy the equation of the plane.

- (b) Does the same property hold for arbitrary ellipsoids  $\{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ ?

**Solution.** This is a more “open-ended” question, so anything that you write down (that is actually true!) is okay. You may observe that the argument above breaks down in the general

case, and indeed it is not possible to find two such foci when the cross sections along one of the axes are not circles.

## 5. Knowing how to set prices

The owners of two stores, Abel's Widgets and Gadgets and Bart's (highly sophisticated) Doohickeys and Thingamajigs, are both excited about the pending release of Mad Scientist Inc.'s new *X-120* brand widgets. If the press releases and industry hype surrounding the *X-120* is any indication, they'll sell like hotcakes. Abel and Bart are competing and each wants to maximize his revenues. They're very diligent people and have done a lot of market research on both their own markets *and* their competitor's markets. In particular, they each know the following:

- Initially, 1000 people are planning to buy the *X-120* from Bart and 800 from Abel.
- However, some of the people buying from Bart will be persuaded to buy instead from Abel if Abel offers a lower price, and vice versa. For each \$1 increase in the price difference, one more person will switch from the store with a higher price to the store with a lower price.
- They each get some foot traffic by their respective stores, so impulse buyers will buy from them. If (hypothetically) Abel set his price at \$100, then 700 impulse buyers would buy from him. Bart's store is in an area with lower foot traffic, so if he set his price at \$100, then only 500 impulse buyers would buy from him.
- The number of impulse buyers falls linearly with the price. For each \$1 increase in price, both Abel and Bart receive 5 fewer impulse buyers.

The question is: How should Abel and Bart set their prices? We'll proceed in stages.

First, some notation: denote by  $p_A$  the price that Abel charges and  $p_B$  the price that Bart charges. Let  $\pi_A$  be the revenue attained by Abel; it is a function of both  $p_A$  (obviously) and  $p_B$  (since that affects the number of customers Abel gets). Similarly, let  $\pi_B$  be the revenue attained by Bart.

- (a) Write down the defining equations for  $\pi_A(p_A, p_B)$  and  $\pi_B(p_A, p_B)$ . What are (within reason) the domains of  $\pi_A$  and  $\pi_B$ ?

**Solution.** Let's first do this with variables.

- Let  $b_A$  (resp.  $b_B$ ) be the initial number of buyers for Abel (resp. Bart).
- Let  $k$  be the number of people who switch from Bart to Abel (or vice versa) for a one dollar increase in the price difference.

- Let  $c_A$  be the number of impulse buyers from Abel (resp. Bart) if they set their price to one hundred dollars.
- Let  $\alpha$  be the number of additional impulse buyers that Abel or Bart will get if they lower their price by one dollar.

Then we have

$$\begin{aligned}\pi_A(p_A, p_B) &= p_A(b_A - k(p_A - p_B) + c_A - \alpha(p_A - 100)), \\ \pi_B(p_A, p_B) &= p_B(b_B - k(p_A - p_B) + c_B - \alpha(p_B - 100)).\end{aligned}$$

Filling in the given numbers, we have

$$\begin{aligned}\pi_A(p_A, p_B) &= p_A(800 - (p_A - p_B) + 700 - 5(p_A - 100)) \\ &= -6p_A^2 + (p_B + 2000)p_A, \\ \pi_B(p_A, p_B) &= p_B(1000 - (p_A - p_B) + 500 - 5(p_B - 100)). \\ &= -6p_B^2 + (p_A + 2000)p_B.\end{aligned}$$

Since it makes no sense to charge negative prices (Sounds like a sleazy advert campaign: “We’ll *pay you* to take our merchandise!!!!”), we have to say (without additional information) that the domains of  $\pi_A$  and  $\pi_B$  are  $\{(p_A, p_B) \mid p_A, p_B \geq 0\}$ .

- (b) Both Abel and Bart are pretty intelligent people. Each wants to maximize his revenues and each knows that the other wants to do the same. Find values of  $p_A$  and  $p_B$  such that neither Abel nor Bart can increase his revenues by unilaterally deviating from those prices. This is called a *Nash equilibrium*<sup>1</sup> *Hint: Think about maximizing  $\pi_A$  and  $\pi_B$ .*

**Solution.** The trick is to (locally) maximize  $\pi_A$  with respect to  $p_A$  and  $\pi_B$  with respect to  $p_B$ . It is then clear that the condition of the Nash equilibrium holds. Clearly  $\pi_A$  and  $\pi_B$  are defined and continuous throughout the domain where  $p_A$  and  $p_B$  are positive. Also, clearly, the extreme values  $p_A = 0$  and  $p_B = 0$  do not constitute maxima, since for those values  $\pi_A$  and  $\pi_B$ , respectively, are

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<sup>1</sup>The “Nash” is John Nash — the one from the movie “A Beautiful Mind”. His discovery and proof of existence of Nash equilibria is what got him the Nobel prize in economics.

zero! So that leaves us with the conditions

$$\begin{aligned}\frac{\partial}{\partial p_A}\pi_A &= 0, \\ \frac{\partial}{\partial p_B}\pi_B &= 0.\end{aligned}$$

Let us compute

$$\begin{aligned}\frac{\partial}{\partial p_A}\pi_A &= -12p_A + p_B + 2000, \\ \frac{\partial}{\partial p_B}\pi_B &= -12p_B + p_A + 2000.\end{aligned}$$

This gives us a system of linear equations. A solution  $(p_A, p_B)$  to this system gives the required Nash equilibrium. Solving the first equation for  $p_B$  and substituting into the second, we have

$$0 = -12(12p_A - 2000) + p_A + 2000 = -143p_A + 26000.$$

Thus  $p_A \approx \$182.82$ . Substituting that value back into the first equation, we find that  $p_B \approx \$182.82$ .

- (c) The product release date is approaching and Mad Scientist Inc. puts out a press release indicating pricing. The price of the *X-120* brand widget is set at \$175 for volumes of 100 or more. Does this change matters? If so, how?

**Solution.** Both Abel and Bart's prices are above \$175, so there is no problem.

*Note:* Originally it was intended that there be a problem with Bart's price, but that got squashed due to a typo in the problem set. Oh well.