UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
MAT 235 Y — CALCULUS II
FALL–WINTER 2007–2008
ASSIGNMENT #2. DUE ON OCTOBER 25.
PROBLEMS AND SOLUTIONS

1. Spheres

a) Find the distance between the two spheres S_1 and S_2:

$$S_1 : (x+3)^2 + (y+7)^2 + (z-1)^2 = 9, \qquad S_2 : x^2 + (y+5)^2 + (z-5)^2 = 4.$$

Solution:

To find the distance between the two spheres, find first the distance between the centers: The centers are $C_1(-3,-7,1)$ and $C_2(0,-5,5)$, so that $|C_1C_2| = \sqrt{(0+3)^2 + (-5+7)^2 + (5-1)^2} = \sqrt{9+4+16} = \sqrt{29}$. Then substract the radii: $R_1 = 3$, $R_2 = 2$, so $|C_1C_2| - R_1 - R_2 = \sqrt{29} - 3 - 2$. This is a positive number thus the spheres don't intersect and one is not included in the other, hence:

$$D = \sqrt{29} - 5.$$

b) Find the distance between the sphere S_1 and the plane P:

$$P : \; x + 2y - 2z = 5.$$

Solution:

First find the distance from the center of the sphere S_1 to the plane P: You can use formula **9**, page 801 in the textbook: The distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is:

$$D = \frac{|ax_1 + bx_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Here, $a = 1$, $b = 2$, $c = -2$ and $d = -5$, so the distance between C_1 and P is:

$$D = \frac{|-3 - 14 - 2 - 5|}{\sqrt{1 + 2^2 + 2^2}} = \frac{24}{\sqrt{9}} = 8.$$

Then substract the radius of the sphere: $8 - 3 = 5$. This is a positive number thus the sphere don't intersect the plane, hence:

$$d = 5.$$

2. Ellipse

a) Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

What is the center and the radius of the osculating circle at the x-intercepts and y-intercepts?

Solution:

Consider a (vector) parametrization of the ellipse:

$$\mathbf{r}(t) = a \cos t \, \mathbf{i} + b \sin t \, \mathbf{j}.$$

The radius of the osculating circle is the reciprocal of the curvature. To compute the curvature we use the formula:

$$\kappa(t) = \frac{|\, \mathbf{r}'(t) \times \mathbf{r}''(t) \,|}{|\, \mathbf{r}'(t) \,|^3}$$

(formula **10** page 833 in the textbook). So we need to compute $\mathbf{r}'$, $\mathbf{r}''$, a cross-product and magnitudes:

$$\begin{aligned}\mathbf{r}'(t) &= -a\sin t\,\mathbf{i} + b\cos t\,\mathbf{j}\\ \mathbf{r}''(t) &= -a\cos t\,\mathbf{i} - b\sin t\,\mathbf{j}\\ \mathbf{r}'(t)\times\mathbf{r}''(t) &= \left(ab\sin^2 t + ab\cos^2 t\right)\mathbf{k}\\ &= ab\,\mathbf{k}\\ |\mathbf{r}'(t)| &= \sqrt{a^2\sin^2 t + b^2\cos^2 t},\end{aligned}$$

hence

$$\kappa(t) = \frac{ab}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}},$$

and the radius of the osculating circle at $\mathbf{r}(t)$ is:

$$R(t) = \sqrt{ab}\left(\frac{a}{b}\sin^2 t + \frac{b}{a}\cos^2 t\right)^{3/2}.$$

The x-intercepts are the points $A(a,0)$ and $A'(-a,0)$, corresponding to $t=0$ and $t=\pi$ respectively; the y-intercepts are the points $B(0,b)$ and $B'(0,-b)$, corresponding to $t=\pi/2$ and $t=-\pi/2$ respectively. Thus:

- The radius of the osculating circle at the x-intercepts is b^2/a;
- The radius of the osculating circle at the y-intercepts is a^2/b.

To centers of the osculating circles at the x- and y-intercepts are:

- Center of the osculating circle at A: $C_A\big((a^2-b^2)/a, 0\big)$;
- Center of the osculating circle at A': $C_{A'}\big((b^2-a^2)/a, 0\big)$;
- Center of the osculating circle at B: $C_B\big(0, (b^2-a^2)/b\big)$;
- Center of the osculating circle at B': $C_{A'}\big(0, (a^2-b^2)/b\big)$.

b) Draw the ellipse and its foci for $a=1$, $b=2$. Draw the osculating circles at the points of intersection with the x and y axes.

Solution:

For $a=1$ and $b=2$ we have:

- Radius of the osculating circle at the x-intercepts is 4;
- Radius of the osculating circle at the y-intercepts is 1/2.
- Center of the osculating circle at A: $C_A(-3,0)$;
- Center of the osculating circle at A': $C_{A'}(3,0)$;
- Center of the osculating circle at B: $C_B(0,3/2)$;
- Center of the osculating circle at B': $C_{B'}(0,-3/2)$.

The foci have coordinates $(0,\pm c)$ where $c^2 = b^2 - a^2 = 4-1 = 3$. Thus: $F\big(0,\sqrt{3}\big)$ and $F'\big(0,-\sqrt{3}\big)$. See last page for figure. The two dots on the y-axis are the foci F and F'.

3. Cross-product

a) Let $\mathbf{v}$ be a unit vector in $\mathbb{R}^3$. Let $\mathbf{a}$ be a vector, orthogonal to $\mathbf{v}$. Explain why

$$\mathbf{v}\times(\mathbf{v}\times\mathbf{a}) = -\mathbf{a}.$$

Solution:

Formula 6 of Theorem **8** page 790 in the textbook states that:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Thus:

$$\mathbf{v}\times(\mathbf{v}\times\mathbf{a}) = (\mathbf{v}\cdot\mathbf{a})\mathbf{v} - (\mathbf{v}\cdot\mathbf{v})\mathbf{a}.$$

Since $\mathbf{a}$ is orthogonal to $\mathbf{v}$, $\mathbf{a}\cdot\mathbf{v}=0$ and since $\mathbf{v}$ is a unit vector, $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 = 1$. Hence:

$$\mathbf{v}\times(\mathbf{v}\times\mathbf{a}) = -\mathbf{a}.$$

Alternatively you can use properties of the cross-product and the right-hand rule.

b) Let $\mathbf{v}$ be a unit vector in $\mathbb{R}^3$ and $\mathbf{u}$ any vector in $\mathbb{R}^3$. Explain why

$$\mathbf{v} \times (\mathbf{v} \times (\mathbf{v} \times (\mathbf{v} \times \mathbf{u}))) = -\mathbf{v} \times (\mathbf{v} \times \mathbf{u}).$$

Solution:

Set $\mathbf{a} = \mathbf{v} \times (\mathbf{v} \times \mathbf{u})$. By property of the cross-product, $\mathbf{a}$ is orthogonal to $\mathbf{v}$, thus we can apply the result of question a):

$$\mathbf{v} \times (\mathbf{v} \times (\mathbf{v} \times (\mathbf{v} \times \mathbf{u}))) = \mathbf{v} \times (\mathbf{v} \times \mathbf{a}) = -\mathbf{a} = -\mathbf{v} \times (\mathbf{v} \times \mathbf{u})$$

4. Let $\mathbf{r}(t)$ be a curve in $\mathbb{R}^3$. Denote by $\mathbf{v}(t)$ its velocity. Denote by $\theta(t)$ the angle between $\mathbf{r}(t)$ and $\mathbf{v}(t)$.
a) Give an expression for $\frac{d}{dt}|\mathbf{r}(t)|$ in terms of $\mathbf{v}(t)$ and $\theta(t)$ only.

Solution:

Define the real-valued function f as $f(t) = \mathbf{r}(t) \cdot \mathbf{r}(t)$. Clearly, $|\mathbf{r}(t)| = \sqrt{f(t)}$, and

$$f'(t) = 2\,\mathbf{r}'(t) \cdot \mathbf{r}(t) = 2\,\mathbf{v}(t) \cdot \mathbf{r}(t) = 2\,|\mathbf{v}(t)|\,|\mathbf{r}(t)|\,\cos\theta(t).$$

Thus:

$$\begin{aligned} \frac{d}{dt}|\mathbf{r}(t)| &= \frac{d}{dt}\sqrt{f(t)} \\ &= \frac{f'(t)}{2\sqrt{f(t)}} \\ &= \frac{|\mathbf{v}(t)|\,|\mathbf{r}(t)|\,\cos\theta(t)}{|\mathbf{r}(t)|} \\ &= |\mathbf{v}(t)|\,\cos\theta(t). \end{aligned}$$

b) If $\theta(t) = \frac{\pi}{2}$ for all t, what can you conclude from question a)?

Solution:

If $\theta(t) = \pi/2$ for all t, then $\frac{d}{dt}|\mathbf{r}(t)| = 0$ for all t, thus $|\mathbf{r}(t)|$ is constant, thus the curve defined by $\mathbf{r}(t)$ remains on a sphere centered at the origin.

5. Consider the curve defined by the vector function

$$\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j} + \frac{\sqrt{3}}{2}t^2\,\mathbf{k}.$$

a) Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

Solution:

By definition, $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$. Now,

$$\mathbf{r}'(t) = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + \sqrt{3}t\,\mathbf{k},$$

$$|\mathbf{r}'(t)| = \sqrt{4t^2} = 2|t|.$$

Hence:

$$\mathbf{T}(t) = \begin{cases} \frac{1}{2}\cos t\,\mathbf{i} + \frac{1}{2}\sin t\,\mathbf{j} + \frac{\sqrt{3}}{2}\,\mathbf{k} & \text{if } t > 0 \\ \text{undefined} & \text{if } t = 0 \\ -\frac{1}{2}\cos t\,\mathbf{i} - \frac{1}{2}\sin t\,\mathbf{j} - \frac{\sqrt{3}}{2}\,\mathbf{k} & \text{if } t < 0. \end{cases}$$

By definition, $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. Now,

$$\mathbf{T}'(t) = \begin{cases} -\frac{1}{2}\sin t\,\mathbf{i} + \frac{1}{2}\cos t\,\mathbf{j} & \text{if } t > 0 \\ \text{undefined} & \text{if } t = 0 \\ \frac{1}{2}\sin t\,\mathbf{i} - \frac{1}{2}\cos t\,\mathbf{j} & \text{if } t < 0 \end{cases}$$

$$|\mathbf{T}'(t)| = \begin{cases} \frac{1}{2} & \text{if } t \neq 0 \\ \text{undefined} & \text{if } t = 0. \end{cases}$$

Hence:

$$\mathbf{N}(t) = \begin{cases} -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} & \text{if } t > 0 \\ \text{undefined} & \text{if } t = 0 \\ \sin t\,\mathbf{i} - \cos t\,\mathbf{j} & \text{if } t < 0 \end{cases}$$

b) Find the curvature κ.

Solution:

For the curvature we use Formula **9** page 832 in the textbook:

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Thus, for all $t \neq 0$:

$$\kappa(t) = \frac{1}{4|t|}.$$

c) Determine the tangential and normal components of the acceleration; express the acceleration vector $\mathbf{a}(t)$ in terms of $\mathbf{T}$ and $\mathbf{N}$.

Solution:

For the decomposition $\mathbf{a}(t) = a_T(t)\,\mathbf{T}(t) + a_N(t)\,\mathbf{N}(t)$ we use Formula **8** page 843 in the textbook:

$$a_T(t) = v(t) \quad \text{and} \quad a_N(t) = \kappa(t)v(t)^2,$$

where $v(t)$ is the speed: $v(t) = |\mathbf{r}'(t)|$. Hence the tangential component of the acceleration is:

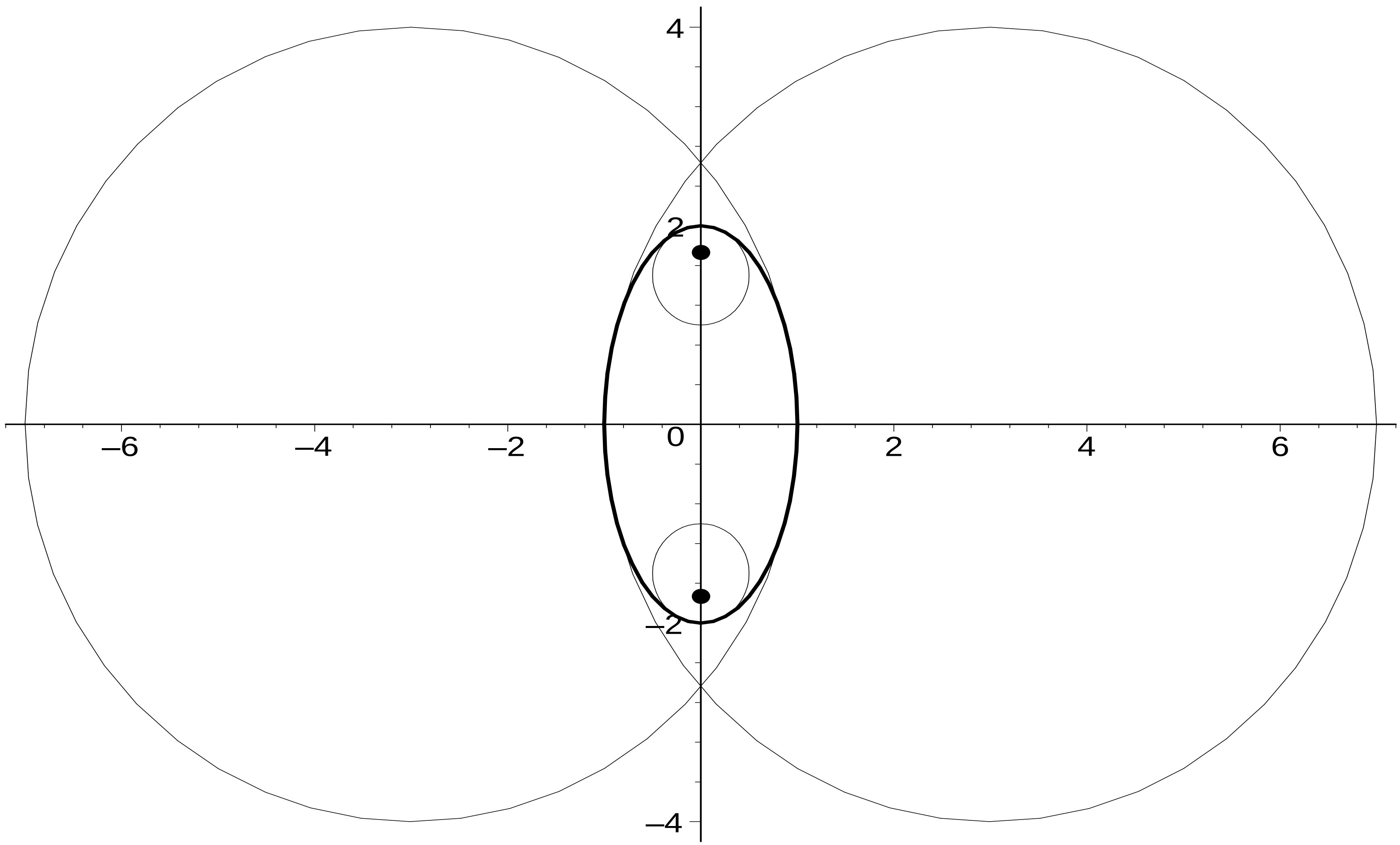
$$a_T(t) = \big(2|t|\big)' = \begin{cases} 2 & \text{if } t > 0 \\ \text{undefined} & \text{if } t = 0 \\ -2 & \text{if } t < 0. \end{cases}$$

For all $t \neq 0$, the normal component of the acceleration is:

$$a_N(t) = \frac{4t^2}{4|t|} = |t|.$$

The acceleration $\mathbf{a}(t)$ is thus expressed as:

$$\mathbf{a}(t) = \begin{cases} 2\,\mathbf{T}(t) + t\,\mathbf{N}(t) & \text{if } t > 0 \\ \text{undefined} & \text{if } t = 0 \\ -2\,\mathbf{T}(t) - t\,\mathbf{N}(t) & \text{if } t < 0. \end{cases}$$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